Call-by-Value, Again!

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Programming Language Theory

λ-calculus

$$\begin{array}{lrcl} \text{terms :} & \Lambda : M, N & ::= & x \mid \lambda x.M \mid (MN) \\ \beta\text{-reduction :} & (\lambda x.M)N & \mapsto_\beta & M\{N/x\} \end{array}$$

Approximation Theory

Böhm Trees

- Approximants

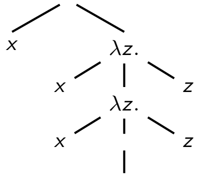
$$A, A_i ::= \bot \mid \lambda\vec{x}.yA_1 \cdots A_n$$

- Approximants of a λ-term M

$$\mathcal{A}(M) = \{A \mid M \to^* N, A \sqsubseteq_\bot N\}$$

- Böhm tree of M

$\mathcal{BT}(Yx)$ with $Yx =_\beta x(\lambda z.Yxz)$



Denotational Models

- Filter Models

≃ Intersection type systems

$$\alpha, \beta ::= a \mid \alpha \wedge \beta \mid \alpha \to \beta$$

- Interpretation of a λ-term M

$$[\![M]\!] = \{\alpha \mid \exists\Gamma, \Gamma \vdash M : \alpha\}$$

- Approximation Theorem

$$\Gamma \vdash M : \alpha$$
$$\Longleftrightarrow$$
$$\exists A \in \mathcal{A}(M) \,.\, \Gamma \vdash A : \alpha$$

Operational Properties of Programs

A term M is:

- Normalizing: if $M \to_\beta^* V$ for some V in NF.
- Head normalizing: if $M \to_\beta^* \lambda x_1 \dots x_n.xM_1 \cdots M_l$.
- Looping: if M is not head-normalizing.
  Exemple : $\omega_3 = \lambda x.xxx$
  $\Omega_3 = \omega_3\omega_3 \to_\beta \Omega_3\omega_3 \to_\beta \Omega_3\omega_3\omega_3 \dots$

Solvability:

M is solvable if $\exists x_1, \dots x_n, \exists M_1, \dots, M_k$ such that

$$(\lambda x_1 \dots x_n.M)M_1 \cdots M_k \to_\beta^* I$$

(I = the identity, a completely defined result)

Characterizations of Solvability

M is solvable exactly when...

Characterizations

- Logical: In a suitable intersection type assignment system

$$\exists \Gamma, \alpha \,.\, \Gamma \vdash M : \alpha, \text{ with } \alpha \text{ “proper”}$$

- Semantical:

$$\mathcal{BT}(M) \neq \bot$$

- Operational :

$$M \text{ is head normalizing}$$

Λ_{CBV}: Call-by-Value λ-calculus

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$$\lambda\text{-terms}: \quad \begin{array}{lrcl} \text{values}: & \mathrm{Val}: V, U & ::= & x \mid \lambda x.M \\ \text{terms}: & \Lambda: M, N & ::= & (MN) \mid V \end{array}$$

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β_v-reduction:

$$(\lambda x.M)V \mapsto_{\beta_v} M\{V/x\}$$

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And σ-rules:

$$\begin{array}{llll} (\lambda x.M)NN' & \mapsto_{\sigma_1} & (\lambda x.MN')N & \textit{with } x \notin \mathrm{fv}(N') \\ V((\lambda x.M)N) & \mapsto_{\sigma_3} & (\lambda x.VM)N & \textit{with } x \notin \mathrm{fv}(V) \end{array}$$

CbV Approximants

$\bot$ represents an undefined value.

$$\frac{V \in \mathrm{Val}}{\bot \sqsubseteq_\bot V}$$

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Approximants

$$\begin{array}{llll} (\mathcal{A}) & A & ::= & H \mid R \\ & H & ::= & \bot \mid x \mid \lambda x.A \mid xHA_1 \cdots A_n \\ & R & ::= & (\lambda x.A)(yHA_1 \cdots A_n) \end{array}$$

Approximants of a Term

$$\begin{array}{rcl} \mathcal{A}(M) & = & \{A \in \mathcal{A} \text{ s.t. } \exists N \in \Lambda \,.\, M \to_v^* N \text{ and } A \sqsubseteq_\perp N\} \\ \mathcal{BT}(M) & = & \bigsqcup \mathcal{A}(M) \end{array}$$

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$$\mathcal{BT}(M) = \bigsqcup \mathcal{A}(M)$$

Approximation Theorem :

Let $M \in \Lambda$, $\alpha \in \mathit{Types}$ and Γ be an environment:

$$\Gamma \vdash M : \alpha \iff \exists A \in \mathcal{A}(M)\,.\,\Gamma \vdash A : \alpha$$

Relational Model

Observational Equivalence:

$$M =_{op} N \quad \Longleftrightarrow \quad \forall C : \exists V . [\, C[M] \to^{*} V \Leftrightarrow \exists U, C[N] \to^{*} U \,]$$

Definition :

A model, with an interpretation $[\![\cdot]\!]$ is:

- adequate if $[\![M]\!] = [\![N]\!] \Rightarrow M =_{op} N$
- fully abstract if $[\![M]\!] = [\![N]\!] \Leftrightarrow M =_{op} N$

Type Assignment System

Countable set $A = \{a, b, c, \dots\}$ of *atomic types*.

$$
\begin{array}{llll}
(\mathit{Types}) & \alpha, \beta & ::= & a \mid [\,] \mid \sigma \to \alpha \\
(\mathit{Multi-Types}) & \sigma, \tau, \rho & ::= & [\alpha_1, \dots, \alpha_n] \quad \text{with } \alpha_i \neq [\,]
\end{array}
$$

Inference rules:

$$
\frac{}{x : [\alpha] \vdash x : \alpha}
\qquad
\frac{\Gamma, x : \sigma \vdash M : \alpha}{\Gamma \vdash \lambda x.M : \sigma \to \alpha}
\qquad
\frac{\Gamma_0 \vdash M : \sigma \to \alpha \quad \Gamma_1 \vdash N : \sigma}{\Gamma_0 + \Gamma_1 \vdash MN : \alpha}
$$

$$
\frac{V \in \mathrm{Val}}{\vdash V : [\,]}
\qquad
\frac{\Gamma_1 \vdash M : \alpha_1 \quad \cdots \quad \Gamma_n \vdash M : \alpha_n \quad n > 0}{\sum_{i=1}^{n} \Gamma_i \vdash M : [\alpha_1, \dots, \alpha_n]}
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In the abstraction rule: $x \notin \mathit{dom}(\Gamma)$.

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In the abstraction rule: $x \notin dom(\Gamma)$.

Characterizations of Valuability and Potential Valuability

Definition: M is

- valuable if $\exists V$ such that $M \to_v^* V$
- potentially valuable if $\exists x_1, \dots x_n, \exists M_1, \dots M_l$ such that $(\lambda x_1 \dots x_n.M)M_1 \cdots M_l$ is valuable

Theorem :

Let $M \in \Lambda$, then:

- M is valuable $\iff \exists \Gamma, \Gamma \vdash M : [\,] \iff \bot \in \mathcal{A}(M)$.
- M is potentially valuable $\iff \exists \Gamma, \alpha \,.\, \Gamma \vdash M : \alpha \iff \mathcal{A}(M) \neq \varnothing$.

More Precise Approximants

Refined Approximants

Subsets $\mathcal{S}, \mathcal{U} \subseteq \mathcal{A}$:

$$
\begin{array}{llll}
(\mathcal{S}) & S & ::= & H' \mid R' \\
 & H' & ::= & x \mid \lambda x.S \mid xHA_1 \cdots A_n \\
 & R' & ::= & (\lambda x.S)(yHA_1 \cdots A_n) \\
 \\
(\mathcal{U}) & U & ::= & \bot \mid \lambda x.U \\
 & & \mid & (\lambda x.U)(yHA_1 \cdots A_n)
\end{array}
$$

Characterizations of Solvability

Trivial type: $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_n \to [\,]$ with $n \geq 0$.
A type not trivial is proper.

Theorem : Characterizations of Solvability

For $M \in \Lambda$, the following are equivalent:

- M is solvable
- $\exists \alpha$ proper, $\exists \Gamma$ such that $\Gamma \vdash M : \alpha$
- $\exists A$ such that $A \in \mathcal{A}(M) \cap \mathcal{S}$

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Corollary : M is unsolvable iff $\mathcal{A}(M) \subseteq \mathcal{U}$.

Semi-Sensible Model

The type assignment system induces a relational model $\mathcal{M}$.

Corollary :

The model $\mathcal{M}$ is not sensible, but semi-sensible.

Decidability of the Inhabitation Problem

The inhabitation problem for system $\mathcal{M}$

For every environment Γ *and type* α *is there a* λ*-term* M *satisfying* $\Gamma \vdash M : \alpha$ *?*

Inhabitation algorithm

$$\frac{}{\bot \in \mathsf{IM}(\varnothing;\, [\,])} \qquad \frac{}{\bot \in \mathsf{IT}(\varnothing;\, [\,])} \qquad \frac{}{x \in \mathsf{IT}(x : [\alpha];\, \alpha)}$$

$$\frac{A \in \mathsf{IT}(\Gamma, x : \sigma;\, \alpha)}{\lambda x.A \in \mathsf{IT}(\Gamma;\, \sigma \to \alpha)} \qquad \frac{A_i \in \mathsf{IT}(\Gamma_i;\, \alpha_i) \quad \uparrow\{A_i\}_{i\in I} \quad A = \bigsqcup_{i\in I} A_i}{A \in \mathsf{IM}(\Sigma_{i\in I}\Gamma_i;\, [\alpha_i]_{i\in I})}$$

$$\frac{A_j \in \mathsf{IM}(\Gamma_j;\, \sigma_j) \quad 0 \le j \le n \quad A_0 \in \mathcal{H}}{xA_0 \cdots A_n \in \mathsf{IT}(\sum_{j=0}^{n} \Gamma_j + x : [\sigma_0 \to \cdots \to \sigma_n \to \alpha];\, \alpha)}$$

$$\frac{A_j \in \mathsf{IM}(\Gamma_j;\, \sum_{i=0}^{m} \tau_j^i) \quad 0 \le j \le n \quad A_0 \in \mathcal{H} \quad A \in \mathsf{IT}(\Gamma_{n+1}, x : [\alpha_i]_{0\le i\le m};\, \alpha)}{(\lambda x.A)(yA_0 \cdots A_n) \in \mathsf{IT}(\sum_{j=0}^{n+1} \Gamma_j + y : [\tau_0^i \to \cdots \to \tau_n^i \to \alpha_i]_{0\le i\le m};\, \alpha)}$$

Figure: Inhabitation algorithm for system $\mathcal{M}$. In the last rule $x \notin \mathit{free\text{-}var}(yA_0 \cdots A_n)$.

Algorithm Properties

The inhabitation algorithm terminates.

Theorem : Soundness and Completeness

- If $A \in \mathrm{IT}(\Gamma; \alpha)$ then, $\forall M \in \Lambda$ such that $A \sqsubseteq_{\perp} M$, we have $\Gamma \vdash M : \alpha$.
- If $\Gamma \vdash M : \alpha$ then $\exists A \in \mathrm{IT}(\Gamma; \alpha)$ such that $A \in \mathcal{A}(M)$.

Conclusion

Future work:

- extend results to other models of the class
- study of categorical construction

Thank you for listening !