Guaranteed phase synchronization of hybrid oscillators using symbolic Euler's method

Jawher Jerray [1] Laurent Fribourg[2] Étienne André[3]

[1] Université Sorbonne Paris Nord, LIPN, CNRS, UMR 7030, F-93430, Villetaneuse, France
and [2] Université Paris-Saclay, LSV, CNRS, ENS Paris-Saclay
and [3] Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France

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Outline

Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state (causality).
  - dynamical systems are everywhere.
  - dynamical systems have a direct impact on human development.

$\Rightarrow$ The importance of studying:

- stability compared to the initial conditions
- behavior
- synchronization

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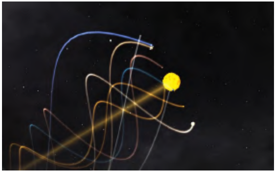
A flock of birds

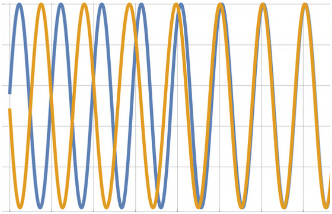
Schooling fish



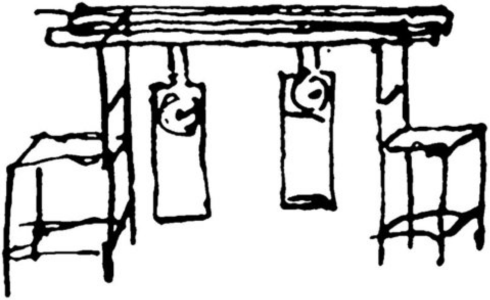
Solar System

Synchronization

- Coordination of multiple events.
- Done within an acceptably brief period of time.
- The example of two suspended mechanical clocks done by Huygens.



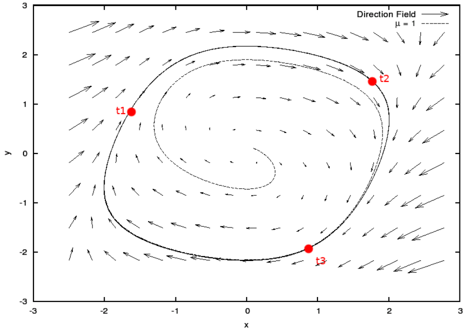
Two oscillators in phase after a lapse of time

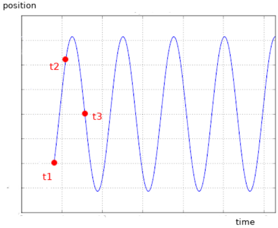


Original drawing of Christian Huygens in which he observed synchronization

how to highlight the synchronization of dynamical system formally?

- Challenge of describing such systems because their equations are non-linear.
- To study non-linear systems, we often visualize them in a space of configurations (position and speed).





Synchronization using a reachability method

We consider a system composed of 2 subsystems governed by a system of differential equations (ODEs) of the form $\dot{x}(t) = f(x(t))$. The system of ODEs is thus of the form:

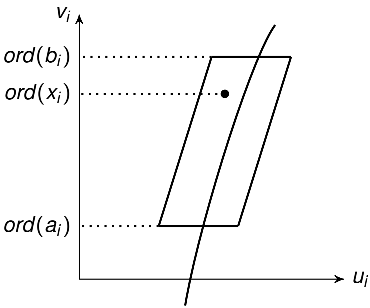
$$\begin{cases} \dot{x_1}(t) = f_1(x_1(t), x_2(t)) \\ \dot{x_2}(t) = f_2(x_1(t), x_2(t)) \end{cases} \tag{1}$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$, where m is the dimension of the state space of each subsystem.

The set S_i is thus characterized by a triple (a_i, b_i, e_i) where a_i and b_i are the end points of its main diagonal, and e_i the size of its horizontal base. We assume that the parallelogram S_i is "long", i.e.:

(H) The width e_i of S_i is "small" w.r.t. $f_i = |ord(b_i) - ord(a_i)|$.

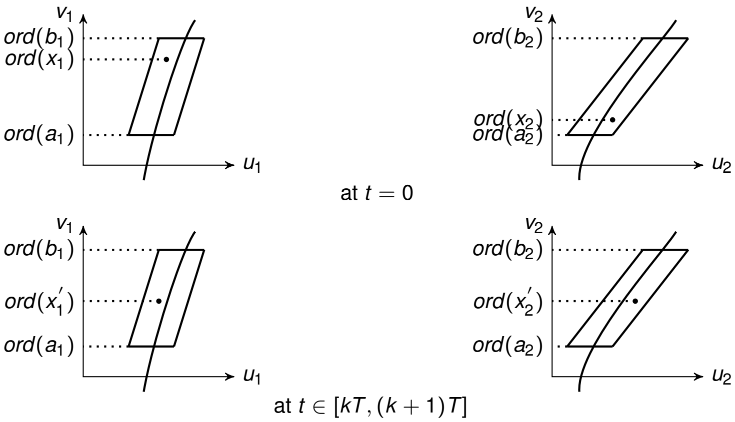
where $ord(a_i)$ (resp. $ord(b_i)$) denotes the ordinate of a_i (resp. b_i).



Given a point of $x_i(s)$ of $S_i \equiv (a_i, b_i, e_i)$ at time s ($i = 1, 2$), we can thus define its *phase* $\phi[x_i(s)]$ (in a "linearized" and "normalized" manner w.r.t. S_i) by:

$$\phi[x_i(s)] = (ord(x_i(s)) - ord(a_i))/(ord(b_i) - ord(a_i))$$

Synchronization using a reachability method



Scheme of S_1 (left) and S_2 (right) at $t = 0$ (top) and for some $t \in [kT, (k+1)T)$ (bottom)

Symbolic reachability using Euler's method

As a symbolic method, we use here the *symbolic Euler's method* [LCDVCF17,Fri17] and we consider a subset under the form of "(double) ball" of the form $B = B_1 \times B_2$, where $B_i \subset \mathbb{R}^m$ $(i = 1, 2)$ is a ball of the form $\mathcal{B}(c_i, r)$ with $c_i \in \mathbb{R}^m$ (*centre*) and r a positive real (*radius*). In order to compute (an overapproximation of) the set of solutions starting at B^0. We define for $t \geq 0$:

$$B^{euler}(t) = \mathcal{B}(c_1(t), r(t)) \times \mathcal{B}(c_2(t), r(t)),$$

where $(c_1(t), c_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$ is the approximated value of solution $x(t)$ of $\dot{x} = f(x)$ with initial condition $x(0) = (c_1^0, c_2^0)$ given by *Euler's explicit method*, and $r(t) \approx r^0 e^{\lambda t}$ is the *expanded* radius using the *one-sided Lipschitz constant* λ.

[LCDVCF17] A. Le Coënt et al., "Control synthesis of nonlinear sampled switched systems using Euler's method," in SNR, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI: 10.4204/EPTCS.247.2.

[Fri17] L. Fribourg, "Euler's method applied to the control of switched systems," in FORMATS, (Sep. 5, 2017–Sep. 7, 2017), ser. LNCS, vol. 10419, Berlin, Germany: Springer, Sep. 2017, pp. 3–21. DOI: 10.1007/978-3-319-65765-3_1. [Online]. Available: https://doi.org/10.1007/978-3-319-65765-3_1.

One-Sided Lipschitz (OSL) constant

Definition

The *one-sided Lipschitz (OSL) constant for* **f** *on D*, denoted by λ, is defined by

$$\lambda := \sup_{y_1 \neq y_2 \in D} \frac{\langle \mathbf{f}(y_1) - \mathbf{f}(y_2), y_1 - y_2 \rangle}{\|y_1 - y_2\|^2},$$

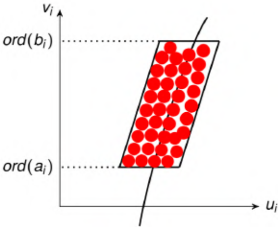
where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors of $\mathbb{R}^n \times \mathbb{R}^n$, and $\|\cdot\|$ the Euclidean norm.

Value of λ

- when $\lambda \leq 0$ locally, indicates contractive zone
- when $\lambda \geq 0$ locally, indicates expansive zone

Symbolic reachability using Euler's method

Given S_i ($i = 1, 2$) defined as a parallelogram (a_i, b_i, e_i), in order to show the phenomenon of phase synchronization, we first *cover* S_i with a *finite* set $\{B_{j,i}\}_{j \in J_i}$ of balls $B_{j,i} \subset \mathbb{R}^m$ (i.e., for $i = 1, 2$, $S_i \subset \bigcup_{j \in J_i} B_{j,i}$).



Proposition

Given a covering $\{B_j\}_{j \in J_i}$ of S_i ($i = 1, 2$), if, for all $(j_1, j_2) \in J_1 \times J_2$, $PROC1(B_{j_1} \times B_{j_2})$ succeeds, then, for all initial condition $(x_1^0, x_2^0) \in S$, there exists $t \in [kT, (k+1)T)$ such that $(x_1(t), x_2(t)) \in S$. Besides: $|phase(x_1(t)) - phase(x_2(t))| \leq \epsilon + \min(e_1/f_1, e_2/f_2)$, where e_i is the width of S_i, and $f_i = |ord(b_i) - ord(a_i)|$ its height ($i = 1, 2$).

When $\epsilon \ll \min(e_1/f_1, e_2/f_2)$, the final difference of phase between $x_1(t)$ and $x_2(t)$ is practically upper bounded by $\min(e_1/f_1, e_2/f_2)$. , then:
For any initial point $(x_1^0, x_2^0) \in S$, there exists $t \in [kT, (k+1)T)$ such that $x_1(t)$ and $x_2(t)$ are *almost in phase*.

Brusselator

Brusselator is a theoretical model for a type of autocatalytic reaction. It is a reaction-diffusion system. We consider the 1D Brusselator partial differential equation (PDE), as given in [CP93]. We suppose a state of the form $x(y,t) = (u(y,t), v(y,t))$ where $y \in \Omega = [0, \ell]$ is the spatial location. The PDE is of the form:

$$\begin{cases} \frac{\partial u}{\partial t} = A + u^2 v - (B+1)u + \sigma \nabla^2 u \\ \frac{\partial v}{\partial t} = Bu - u^2 v + \sigma \nabla^2 v \end{cases} \qquad (2)$$

with boundary condition: $u(0,t) = u(\ell,t) = 1$, $v(0,t) = v(\ell,t) = 3$,
and initial condition: $x_0(y) = (u(y,0), v(y,0))$ with $u(y,0) = 1 + sin(2\pi y)$, $v(y,0) = 3$.

[CP93] P. Chartier and B. Philippe, "A parallel shooting technique for solving dissipative ODE's," **Computing**, vol. 51, no. 3, pp. 209–236, 1993, ISSN: 1436-5057. DOI: 10.1007/BF02238534.

Brusselator

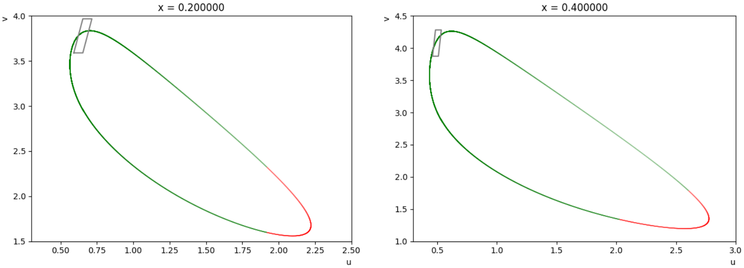
We transform the PDE into a system of ODEs by spatial discretization using a grid of $N+1$ points with $N=4$.
The system of ordinary differential equations for this example is described by

$$\begin{cases} \dot{u}_1 = A + u_1^2 v_1 - (B+1)u_1 + \sigma(u_0 - 2u_1 + u_2) \\ \dot{v}_1 = Bu_1 - u_1^2 v_1 + \sigma(v_0 - 2v_1 + v_2) \\ \dot{u}_2 = A + u_2^2 v_2 - (B+1)u_2 + \sigma(u_1 - 2u_2 + u_3) \\ \dot{v}_2 = Bu_2 - u_2^2 v_2 + \sigma(v_1 - 2v_2 + v_3) \\ \dot{u}_3 = A + u_3^2 v_3 - (B+1)u_3 + \sigma(u_2 - 2u_3 + u_4) \\ \dot{v}_3 = Bu_3 - u_3^2 v_3 + \sigma(v_2 - 2v_3 + v_4) \\ \dot{u}_4 = A + u_4^2 v_4 - (B+1)u_4 + \sigma(u_3 - 2u_4 + u_5) \\ \dot{v}_4 = Bu_4 - u_4^2 v_4 + \sigma(v_3 - 2v_4 + v_5) \end{cases} \tag{3}$$

with $u_0 = u_5 = 1$ and $v_0 = v_5 = 3$.

Brusselator

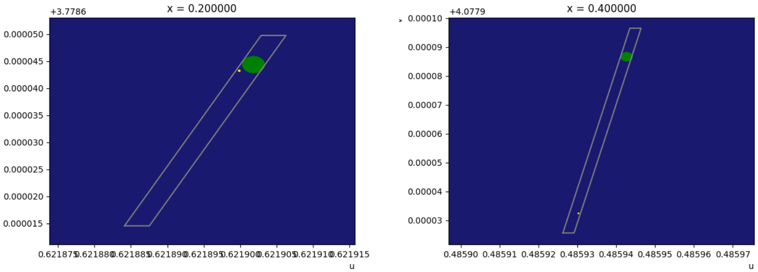
By using symmetry, we can reduce the problem to plans $x = 0.2$ and $x = 0.4$.



Brusselator: A cyclic trajectory for plan $x = 0.2$ (left) and $x = 0.4$ (right); the green zone indicates the contractive area ($\lambda < 0$) and the red zone the expansive one ($\lambda > 0$)

- The time-step used in Euler's method is $\tau = 2 \cdot 10^{-4}$.
- The period of the system is $T = 34564\tau$.
- The expansion factor of the ball radius after one period is $E = 2.12$.
- The number of periods considered for synchronization is $k = 5$ (so the expansion factor after k periods $= 2.12^5 \approx 43$).

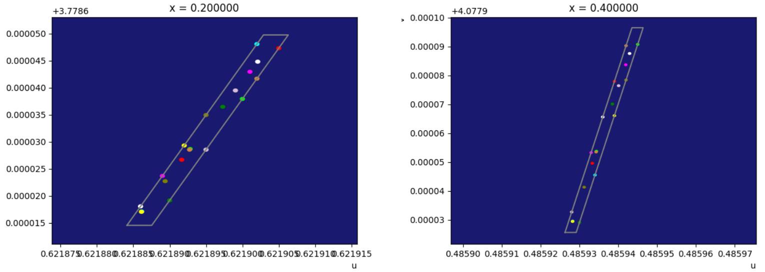
Brusselator



Brusselator: Synchronization of the two components of a ball, located initially near opposite vertices of the parallelograms (yellow), after $k = 5$ periods (green).

- The radius of the initial ball covering S is $= 3.5 \cdot 10^{-8}$.
- After $k = 5$ periods, the radius of the ball image is $1.5 \cdot 10^{-6}$.
- The phase of the initial ball center is 0.82 in plan $x = 0.2$, and 0.09 in plan $x = 0.4$, so the difference of phase $\Delta(phase(centers))$, at $t = 0$, is 0.73.
- The phase of the image ball center is 0.87461 in plan $x = 0.2$, and 0.87463 in plan $x = 0.4$, so the difference of phase $\Delta(phase(centers))$, after $k = 5$ periods, is now $2 \cdot 10^{-5} \approx 0$.

Brusselator



Brusselator: Synchronization of 10 (pairs of) balls, located initially on the parallelogram perimeters, after $k = 5$ periods (without radius expansion for clarity).

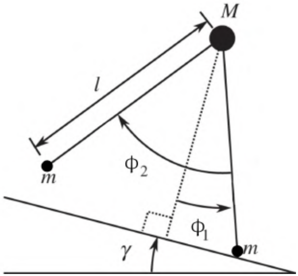
Brusselator

The list of phases of 10 ball centers for the Brusselator example.

Point	phase initial point in u_1	phase initial point in u_2	phase image point in u_1	phase image point in u_2	Δ(*phase*) for initial point	Δ(*phase*) for image point
1	0.13	0.05	0.63224	0.63221	0.08	$2 \cdot 10^{-}5$
2	0.40	0.10	0.72512	0.72511	0.30	$8 \cdot 10^{-}6$
3	0.26	0.39	0.83112	0.83113	0.13	$6 \cdot 10^{-}6$
4	0.95	0.28	0.0383	0.0382	0.67	$9 \cdot 10^{-}5$
5	0.42	0.57	0.0366	0.0365	0.15	$9 \cdot 10^{-}5$
6	0.10	0.56	0.88834	0.88836	0.46	$1 \cdot 10^{-}5$
7	0.58	0.74	0.2103	0.2102	0.16	$7 \cdot 10^{-}5$
8	0.66	0.92	0.3929	0.3928	0.25	$5 \cdot 10^{-}5$
9	0.93	0.74	0.3318	0.3317	0.19	$6 \cdot 10^{-}5$
10	0.77	0.91	0.3890	0.3889	0.14	$5 \cdot 10^{-}5$

Biped

We extend the method of verification of phase synchronization to *hybrid systems*.
We describe here the results of such an extension to the *passive biped model* [McG90], seen as a hybrid oscillator. The passive biped model exhibits indeed a stable limit-cycle oscillation for appropriate parameter values that corresponds to periodic movements of the legs [SKN17].



Biped walker

[McG90] T. McGeer, "Passive dynamic walking," **The International Journal of Robotics Research**, vol. 9, no. 2, pp. 62–82, 1990. DOI: 10.1177/027836499000900206. [Online]. Available: https://doi.org/10.1177/027836499000900206.

[SKN17] S. Shirasaka, W. Kurebayashi, and H. Nakao, "Phase reduction theory for hybrid nonlinear oscillators," **Physical Review E**, vol. 95, 1 Jan. 2017. DOI: 10.1103/PhysRevE.95.012212.

Biped

The model has a continuous state variable $\boldsymbol{x}(t) = (\phi_1(t), \dot{\phi}_1(t), \phi_2(t), \dot{\phi}_2(t))^\top$. The dynamics is described by $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with:

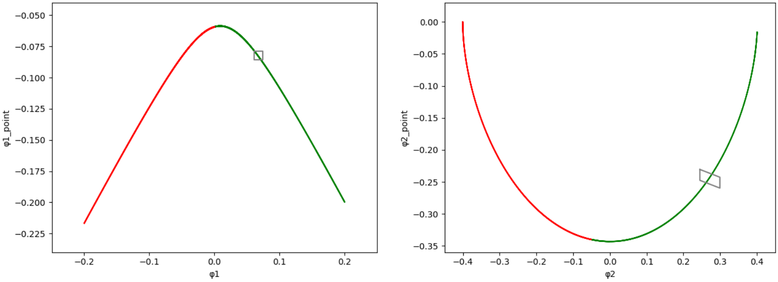
$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} \dot{\phi}_1 \\ \sin(\phi_1 - \gamma) \\ \dot{\phi}_2 \\ \sin(\phi_1 - \gamma) + \dot{\phi}_1^2 \sin\phi_2 - \cos(\phi_1 - \gamma)\sin\phi_2 \end{pmatrix} \tag{4}$$

$$Reset(\boldsymbol{x}) = \begin{pmatrix} -\phi_1 \\ \dot{\phi}_1 \sin(2\phi_1) \\ -2\phi_1 \\ \dot{\phi}_1 \cos 2\phi_1 (1 - \cos 2\phi_1) \end{pmatrix} \tag{5}$$

$$Guard(\boldsymbol{x}) \equiv (2\phi_1 - \phi_2 = 0 \ \wedge \phi_2 < -\delta). \tag{6}$$

with $\delta = 0.1$ and $\gamma = 0.009$.

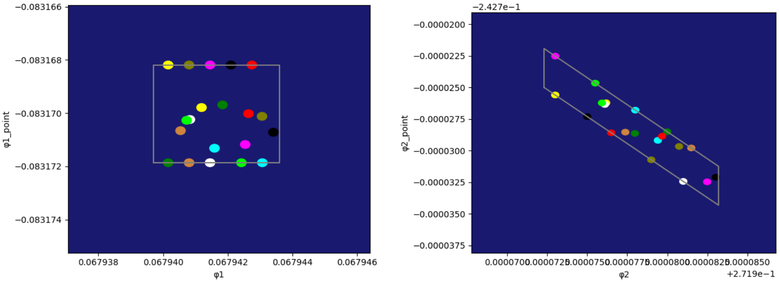
Biped



Biped: A cyclic trajectory for plan ϕ_1 (left) and ϕ_2 (right); the green zone indicates the contractive area ($\lambda < 0$) and the red zone the expansive one ($\lambda > 0$)

- The time-step used in Euler's method is $\tau = 2 \cdot 10^{-5}$.
- The period of the system is $T = 776440\tau$.
- The radius expansion factor after one period is $E = 2.63$.
- The number of periods considered for synchronization is $k = 30$.

Biped



Biped: Synchronization of 10 (pairs of) balls, located initially on the parallelogram perimeters, after $k = 30$ periods (without radius expansion for clarity).

Biped

The list of phases of 10 ball centers in the biped example.

Point	phase initial point in ϕ_1	phase initial point in ϕ_2	phase image point in ϕ_1	phase image point in ϕ_2	Δ(*phase*) **for initial point**	Δ(*phase*) **for image point**
1	0.88	0.29	0.45	0.48	0.59	0.03
2	0.38	0.75	0.05	0.02	0.37	0.03
3	0.55	0.94	0.27	0.07	0.39	0.21
4	0.14	0.48	0.52	0.35	0.34	0.17
5	0.88	0.94	0.62	0.64	0.05	0.03
6	0.55	0.20	0.71	0.65	0.35	0.06
7	0.72	0.39	0.14	0.23	0.33	0.09
8	0.30	0.71	0.74	0.67	0.40	0.07
9	0.22	0.61	0.25	0.32	0.40	0.08
10	0.72	0.16	0.78	0.53	0.56	0.25

Conclusion and Perspectives

Conclusion

- We presented a symbolic reachability method to prove phase synchronization of oscillators.
- The method shows that a *finite* number of points, displaced from their original position on a synchronization orbit, return after some time into a close neighborhood of the orbit.

Perspectives

- Adapt the classical “adjoint” method (or *phase reduction*) rather than the method used here. In order to solve systems with higher state space dimension.
- Replace the symbolic Euler's method used here by any other symbolic reachability procedure to cover larger sets S.

P. Chartier and B. Philippe, "A parallel shooting technique for solving dissipative ODE's," **Computing**, vol. 51, no. 3, pp. 209–236, 1993, ISSN: 1436-5057. DOI: 10.1007/BF02238534.

L. Fribourg, "Euler's method applied to the control of switched systems," in **FORMATS**, (Sep. 5, 2017–Sep. 7, 2017), ser. LNCS, vol. 10419, Berlin, Germany: Springer, Sep. 2017, pp. 3–21. DOI: 10.1007/978-3-319-65765-3_1. [Online]. Available: https://doi.org/10.1007/978-3-319-65765-3_1.

A. Le Coënt, F. De Vuyst, L. Chamoin, and L. Fribourg, "Control synthesis of nonlinear sampled switched systems using Euler's method," in **SNR**, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI: 10.4204/EPTCS.247.2.

T. McGeer, "Passive dynamic walking," **The International Journal of Robotics Research**, vol. 9, no. 2, pp. 62–82, 1990. DOI: 10.1177/027836499000900206. [Online]. Available: https://doi.org/10.1177/027836499000900206.

S. Shirasaka, W. Kurebayashi, and H. Nakao, "Phase reduction theory for hybrid nonlinear oscillators," **Physical Review E**, vol. 95, 1 Jan. 2017. DOI: 10.1103/PhysRevE.95.012212.

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