Asymptotic Error in Euler's Method with a Constant Step Size

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Outline

Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.
  - dynamical systems have a direct impact on human development.

$\Rightarrow$ The importance of studying:

- synchronization
- behavior
- stability

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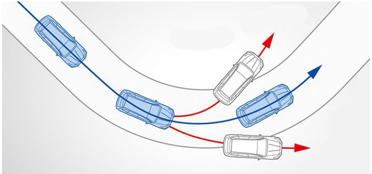
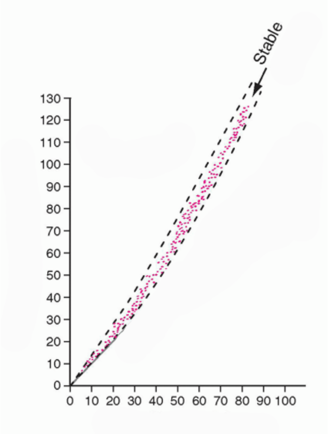


Electronic Stability Control (ESC)

Solar System

Stability

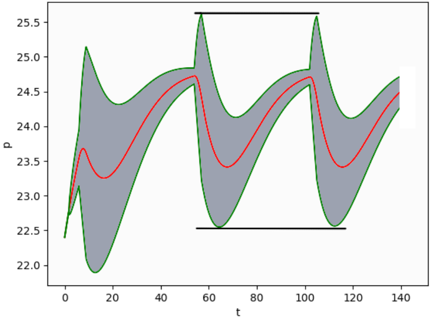
- A dynamical system is stable, if small perturbations to the solution lead to a new solution that stays close to the original solution forever.
- A stable system produces a bounded output for a given bounded input.



Stability

An invariant

- The bounded output of some periodic stable system can be considered as an invariant from certain t.
- An invariant is an unchanged object after operations applied to it.



Invariant

Euler's method and error bounds

Let us consider the differential system:

$$\dot{x}(t) = g(x(t)),$$

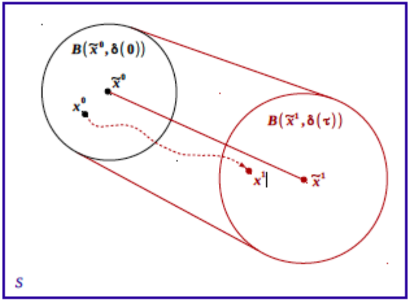
with states $x(t) \in \mathbb{R}^n$ and x_0 a given initial condition.

- $\tilde{x}(t; x_0)$ denotes Euler's approximate value of $x(t)$ (defined by $\tilde{x}(t; x_0) = x_0 + t\, g(x_0)$ for $t \in [0, h]$, where h is the integration time-step).

Proposition

[LCDVCF17] Consider the solution $x(t; y_0)$ of $\frac{dx}{dt} = g(x)$ with initial condition y_0 and the approximate Euler solution $\tilde{x}(t; x_0)$ with initial condition x_0. For all $y_0 \in B(x_0, \varepsilon)$, we have:

$$\|x(t; y_0) - \tilde{x}(t; x_0)\| \leq \delta_\varepsilon(t).$$



[LCDVCF17] A. Le Coënt et al., "Control synthesis of nonlinear sampled switched systems using Euler's method," in SNR, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI:

Definition

$\delta_\varepsilon(t)$ is defined as follows for $t \in [0, \tau]$:
if $\lambda < 0$:

$$\delta_\varepsilon(t) = \left(\varepsilon^2 e^{\lambda t} + \frac{C^2}{\lambda^2} \left(t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} \left(1 - e^{\lambda t} \right) \right) \right)^{\frac{1}{2}}$$

if $\lambda = 0$:

$$\delta_\varepsilon(t) = \left(\varepsilon^2 e^t + C^2(-t^2 - 2t + 2(e^t - 1)) \right)^{\frac{1}{2}}$$

if $\lambda > 0$:

$$\delta_\varepsilon(t) = \left(\varepsilon^2 e^{3\lambda t} + \frac{C^2}{3\lambda^2} \left(-t^2 - \frac{2t}{3\lambda} + \frac{2}{9\lambda^2} \left(e^{3\lambda t} - 1 \right) \right) \right)^{\frac{1}{2}}$$

where C and λ are real constants specific to function f, defined as follows:

$$C = \sup_{y \in \mathcal{S}} L \|g(y)\|,$$

Definition

L denotes the Lipschitz constant for g, and λ is the "one-sided Lipschitz constant" (or "logarithmic Lipschitz constant" [AS14]) associated to g, i. e., the minimal constant such that, for all $y_1, y_2 \in \mathcal{S}$:

$$\langle g(y_1) - g(y_2), y_1 - y_2 \rangle \leq \lambda \|y_1 - y_2\|^2, \qquad (H0)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors of $\mathcal{S}$.

The constant λ can be computed using a nonlinear optimization solver (e. g., CPLEX [Cpl09]) or using the Jacobian matrix of g.

[AS14] Z. Aminzare and E. D. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in **53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014**, 2014, pp. 3835–3847.

[Cpl09] I. I. Cplex, "V12. 1: User's manual for cplex," **International Business Machines Corporation**, vol. 46, no. 53, p. 157, 2009.

Function strongly monotone and co-coercive

Definition

- A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone if there exists $m > 0$ such that, for all $x, y \in \mathbb{R}^n$:
$$(g(x) - g(y))^T (x - y) \geq m\|x - y\|^2$$
- A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is co-coercive if there exists a positive constant a such that for all $x, y \in \mathbb{R}^n$:
$$(g(y) - g(x))^T (y - x) \geq a\|g(y) - g(x)\|^2$$

Gradient descent algorithm

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, the gradient descent algorithm generates a sequence $\{x_k\}_{k\in\mathbb{N}}$ described as:

$$x_{k+1} = x_k - h\nabla f(x_k)$$

where $h > 0$ is a constant step size. This algorithm is generally used to resolve optimization problems of the form $min_{x\in\mathbb{R}^n} f(x)$ for a function f.

Error bound in Euler's method

Let us consider the sequence $\{\mu_k\}_{k\geq 0}$ where μ_k is defined recursively, for $k \geq 1$ as:

$$\mu_k = \delta_{\mu_{k-1}}(h)$$

Also, for all $k \geq 0$ and $t \in [0, h]$:

$$\delta_{\mu_0}(kh + t) = \delta_{\mu_k}(t)$$

Theorem

Consider the system $\dot{x}(t) = g(x(t))$, with $g : \mathbb{R}^n \to \mathbb{R}^n$. Let $x(t)$ the solution of this system at time t, (y_k) the (explicit) Euler discretization of $\dot{x}(t)$ and $\mu_0 := \|y_0 - x_0\|$. Then, for all $t = kh$:

$$\|y_k - x(t)\| \leq \delta\mu_0(t)$$

Co-coercivity

Theorem

Consider the system $\dot{x}(t) = g(x(t))$, with $g : \mathbb{R}^n \to \mathbb{R}^n$ L-Lipschitz continuous. Let $x(t)$ the solution of this system at time t and (y_k) the Euler discretization of $\dot{x}(t)$. Suppose:

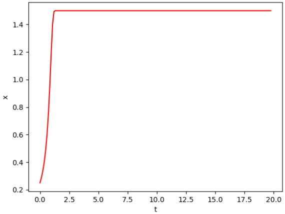
1. $h < 2/L$,
2. $-g$ co-coercive with constant $1/L$,
3. g of OSL constant $\lambda < 0$ (i.e., $-g$ strongly monotone),
4. $g(x^*) = 0$ for some $x^* \in \mathbb{R}^n$ (existence of a stationary point).

Then we have:

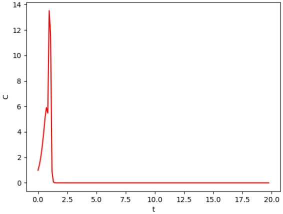
- x^* is the unique stationary point of $\mathbb{R}^n$,
- $y_k \to x^*$ and $x(kh) \to x^*$ as $k \to \infty$ with rate $O(1/k)$ for the averaged iterates.

Example

Consider the differential equation $\dot{x} = g(x)$ with $g(x) = -4x^3 + 6x^2$, and its Euler discretization with $y_0 = 0.25$ and $h = 0.12$.
Using ORBITADOR[Jer21], we calculate $L \leq 12$, where L is the Lipschitz constant of g.



Evolution of y_k (which converges to $x^* = 1.5$)

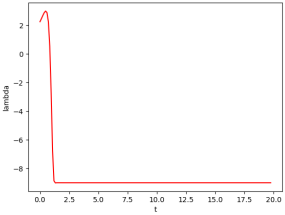


Evolution of $C = L\|g(y)\|$

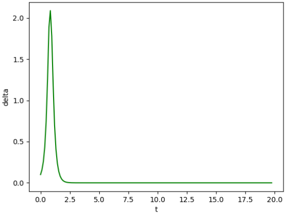
[Jer21] J. Jerray, "Orbitador: A tool to analyze the stability of periodical dynamical systems," in ARCH, (Jul. 9, 2021), G. Frehse and M. Althoff, Eds., ser. EPiC Series in Computing, vol. 80, Brussels, Belgium: EasyChair, 2021, pp. 176–183. DOI: 10.29007/k6xm.

Let $\mathbb{D} = [1.25, 1.75]$. For $\mu_0 = 0.1$ and $h = 0.12 < 2/L$, ORBITADOR shows that:

- $\lambda < 0$ on $\mathbb{D}$,
- $B(y_k, \delta_{\mu_0}(kh)) \subseteq \mathbb{D}$ for all $k \geq 12$, and
- $-g$ co-coercive of constant $1/L$ on $\mathbb{D}$.



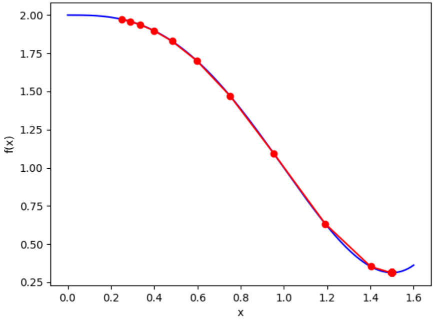
Evolution of λ



Evolution of δ_{μ_0} (which converges to 0)

Besides, $x^* = 1.5 \in \mathbb{D}$ is a stationary point ($g(y^*) = 0$), we check that:

- $\delta_{\mu_0}(kh) \to 0$.
- x^* is the unique stationary point of $\mathbb{D}$, $y_k \in x^*$ and $x(kh) \to x^*$ as $k \to \infty$.
- $C = L\|g(y_k)\| \to 0$.



Graph $(y_k, f(y_k))$

Note that x^* is the minimizer of the non-convex function $f(x) = x^4 - 2x^3 + 2$ (with $-\nabla f(x) = g(x)$).

Conclusion

Conclusion

- We have shown that under certain properties of g called “strong monotonicity” and “co-coercivity”, the discretization error converges to 0.
- This contribution can highlights the relationship between the convergence of continuous differential equations and their discretization.

Bibliography

[AS14] Z. Aminzare and E. D. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in **53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014**, 2014, pp. 3835–3847.

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Thank you for your attention!