ORBITADOR: A tool to analyze the stability of periodical dynamical systems

Jawher Jerray \(^1\)

\(^1\) Université Sorbonne Paris Nord, LIPN, CNRS, UMR 7030, F-93430, Villetaneuse, France

Friday 9\(^{th}\) July, 2021

ARCH 2021
Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.

⇒ The importance of studying:
- synchronization
- behavior
- stability
Stability

- A dynamical system is **stable**, if small perturbations to the solution lead to a new solution that stays **close** to the original solution forever.
- A **stable** system produces a **bounded output** for a given **bounded input**.
Euler’s method and error bounds

Let us consider the differential system:

$$\frac{dx(t)}{dt} = f(x(t)),$$

with states $x(t) \in \mathbb{R}^n$ and $x_0$ a given initial condition.

- $\tilde{x}(t; y_0)$ denotes Euler’s approximate value of $x(t)$ (defined by $\tilde{x}(t; y_0) = y_0 + t \times f(y_0)$ for $t \in [0, \tau]$, where $\tau$ is the integration time-step).
Euler’s method and error bounds

Proposition

[LCVDF17] Consider the solution $x(t; x_0)$ of $\frac{dx}{dt} = f(x)$ with initial condition $x_0$ and the approximate Euler solution $\tilde{x}(t; x_0)$ with initial condition $x_0$. For all $x_0 \in B(x_0, \varepsilon)$, we have:

$$\|x(t; x_0) - \tilde{x}(t; x_0)\| \leq \delta_\varepsilon(t).$$
Systems with bounded uncertainty

A differential system with bounded uncertainty is of the form

\[ \frac{dx(t)}{dt} = f(x(t), w(t)), \]

with \( t \in \mathbb{R}_{\geq 0} \), states \( x(t) \in \mathbb{R}^n \), and uncertainty \( w(t) \in \mathcal{W} \subset \mathbb{R}^n \) (\( \mathcal{W} \) is compact, i.e., closed and bounded).

- We suppose (see [LCADSC+17]) that there exist constants \( \lambda \in \mathbb{R} \) and \( \gamma \in \mathbb{R}_{\geq 0} \) such that, for all \( y_1, y_2 \in \mathcal{S} \) and \( w_1, w_2 \in \mathcal{W} \):

\[ \langle f(y_1, w_1) - f(y_2, w_2), y_1 - y_2 \rangle \leq \lambda \| y_1 - y_2 \|^2 + \gamma \| y_1 - y_2 \| \| w_1 - w_2 \| \quad (H1). \]

- Instead of computing \( \lambda \) and \( \gamma \) globally for \( \mathcal{S} \), it is advantageous to compute them locally depending on the subregion of \( \mathcal{S} \) occupied by the system state during a considered interval of time.

Correctness

Proposition

Suppose that there exist $T > 0$ (with $T = k\tau$ for some $k$) and $i \in \mathbb{N}$ such that: $B_W((i+1)T) \subseteq B_W(iT)$. Then $I_W \equiv \bigcup_{t \in [iT,(i+1)T]} B_W(t)$ is a compact invariant set containing all the solutions of $\Sigma_W$ with initial condition in $B_0$.

$\Rightarrow$ Then there exists a closed orbit (limit cycle or fixed-point) for the unperturbed system $\Sigma$ which is contained in $I_W$ (see [JF21]).

\[
\frac{dx(t)}{dt} = f(x(t), w(t))
\]

**ORBITADOR**: A tool to analyze the stability of periodical dynamical systems

---

**Motivation**

**Method**

**ORBITADOR**

**Biped example**

**Conclusion and Perspectives**

**References**

**Jawher Jerray (LIPN)**

**ORBIDATORE: ORBITADOR's structure**

- Value of \( x_0 \)
- Value of \( \tau \)
- Value of \( N \)
- Value of \( \mathcal{W} \)
- Value of \( \varepsilon \)

\( l_{\mathcal{W}} \)
We consider a simple model of biped walker [McG90], seen as a hybrid oscillator.

---

Biped System

We consider a simple model of biped walker [McG90], seen as a hybrid oscillator. The model has a continuous state variable \( \mathbf{x}(t) = (\phi_1(t), \dot{\phi}_1(t), \phi_2(t), \dot{\phi}_2(t))^\top \). The dynamics is described by \( \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}) + \mathbf{w} \) with \( \mathbf{w} \in \mathcal{W} \subset \mathbb{R}^4 \):

\[
\mathbf{f}(\mathbf{x}) = \begin{pmatrix}
\dot{\phi}_1 \\
\sin(\phi_1 - \gamma) \\
\dot{\phi}_2 \\
\sin(\phi_1 - \gamma) + \phi_1^2 \sin \phi_2 - \cos(\phi_1 - \gamma) \sin \phi_2
\end{pmatrix}
\]

\[
\text{Reset}(\mathbf{x}) = \begin{pmatrix}
-\phi_1 \\
\phi_1 \sin(2\phi_1) \\
-2\phi_1 \\
\phi_1 \cos 2\phi_1 (1 - \cos 2\phi_1)
\end{pmatrix}
\]

\[
\text{Guard}(\mathbf{x}) \equiv (2\phi_1 - \phi_2 = 0 \land \phi_2 < -\delta).
\]

Biped System with uncertainty

\[ f(x), \text{Reset}(x), \text{Guard}(x) \]

\[ x_0 = (0.009, -0.05869, -0.0009629, -0.3432) \]

\[ \tau = 2 \cdot 10^{-5} \]

\[ N = 5 \]

\[ \mathcal{W} = [-0.0001, 0.0001]^4 \]

\[ \varepsilon = 0.01 \]

Importing Biped example to ORBITADOR
Biped System with uncertainty

Biped system with uncertainty \( w \in \mathcal{W} = [-0.0001, 0.0001]^4 \), initial radius \( \varepsilon = 0.001 \), approximate period \( T = 3.8826 \) s and time-step \( \tau = 2 \cdot 10^{-5} \).
Biped System with uncertainty

Biped system with uncertainty $w \in \mathcal{W} = [-0.0001, 0.0001]^4$, initial radius $\varepsilon = 0.001$, approximate period $T = 3.8826$ s and time-step $\tau = 2 \cdot 10^{-5}$.

- ORBITADOR finds: $B_{\mathcal{W}}((i_0 + 1) T) \subset B_{\mathcal{W}}(i_0 T)$ for $i_0 = 4$.
- The system converges towards an attractive limit cycle contained in $l_{\mathcal{W}} \equiv \bigcup_{t \in [4T, 5T]} B_{\mathcal{W}}(t)$. 
Conclusion

- We presented ORBITADOR a tool that guarantees the existence of limit cycles and constructs invariant sets around them.
- ORBITADOR uses a very general criterion of inclusion of one set in another.

Perspectives

- Adapt the method to solve the convergence to a limit cycle for complex systems.
- Upgrade ORBITADOR so that it computes dynamical systems with control.
<table>
<thead>
<tr>
<th>References</th>
<th>Title</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>[LCDVCF17]</td>
<td>Control synthesis of nonlinear sampled switched systems using Euler’s method</td>
<td>A. Le Coënt, F. De Vuyst, L. Chamoin, and L. Fribourg</td>
</tr>
<tr>
<td>[McG90]</td>
<td>Passive dynamic walking</td>
<td>T. McGeer</td>
</tr>
</tbody>
</table>

**Jawher Jerray (LIPN)**

**ORBITADOR: A tool to analyze the stability of periodical dynamical systems**