ORBITADOR: A tool to analyze the stability of periodical dynamical systems

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Motivation

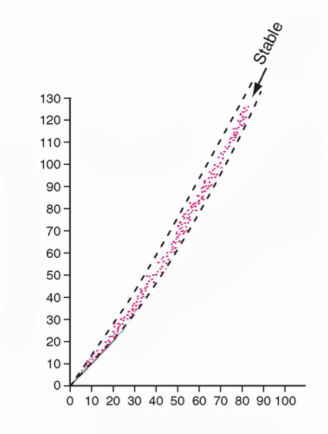
- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.

$\Rightarrow$ The importance of studying:

- synchronization
- behavior
- stability

Stability

- A dynamical system is stable, if small perturbations to the solution lead to a new solution that stays close to the original solution forever.
- A stable system produces a bounded output for a given bounded input.



Stability

Euler's method and error bounds

Let us consider the differential system:

$$\frac{dx(t)}{dt} = f(x(t)),$$

with states $x(t) \in \mathbb{R}^n$ and x_0 a given initial condition.

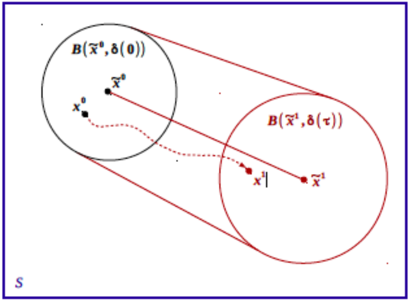
- $\tilde{x}(t; y_0)$ denotes Euler's approximate value of $x(t)$ (defined by $\tilde{x}(t; y_0) = y_0 + t \times f(y_0)$ for $t \in [0, \tau]$, where τ is the integration time-step).

Euler's method and error bounds

Proposition

[LCDVCF17] Consider the solution $x(t; x_0)$ *of* $\frac{dx}{dt} = f(x)$ *with initial condition* x_0 *and the approximate Euler solution* $\tilde{x}(t; x_0)$ *with initial condition* x_0*. For all* $x_0 \in B(x_0, \varepsilon)$*, we have:*

$$\|x(t; x_0) - \tilde{x}(t; x_0)\| \leq \delta_\varepsilon(t).$$



Systems with bounded uncertainty

A differential system with bounded uncertainty is of the form

$$\frac{dx(t)}{dt} = f(x(t), w(t)),$$

with $t \in \mathbb{R}^n_{\geq 0}$, states $x(t) \in \mathbb{R}^n$, and uncertainty $w(t) \in \mathcal{W} \subset \mathbb{R}^n$ ($\mathcal{W}$ is compact, i. e., closed and bounded).

- We suppose (see [LCADSC+17]) that there exist constants $\lambda \in \mathbb{R}$ and $\gamma \in \mathbb{R}_{\geq 0}$ such that, for all $y_1, y_2 \in \mathcal{S}$ and $w_1, w_2 \in \mathcal{W}$:

$$\langle f(y_1, w_1) - f(y_2, w_2), y_1 - y_2 \rangle \leq \lambda \|y_1 - y_2\|^2 + \gamma \|y_1 - y_2\| \|w_1 - w_2\| \quad (H1).$$

- Instead of computing λ and γ globally for $\mathcal{S}$, it is advantageous to compute them *locally* depending on the subregion of $\mathcal{S}$ occupied by the system state during a considered interval of time.

[LCADSC+17] A. Le Coënt et al., "Distributed control synthesis using Euler's method," in **Proc. of International Workshop on Reachability Problems (RP'17)**, ser. Lecture Notes in Computer Science, vol. 247, Springer, 2017, pp. 118–131.

Correctness

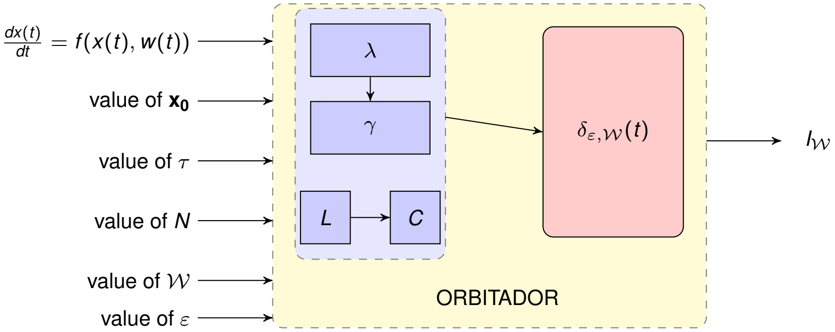
Proposition

Suppose that there exist $T > 0$ (with $T = k\tau$ for some k) and $i \in \mathbb{N}$ such that: $B_{\mathcal{W}}((i+1)T) \subseteq B_{\mathcal{W}}(iT)$. Then $I_{\mathcal{W}} \equiv \bigcup_{t \in [iT,(i+1)T]} B_{\mathcal{W}}(t)$ is a compact invariant set containing all the solutions of $\Sigma_{\mathcal{W}}$ with initial condition in B_0.

$\Rightarrow$ Then there exists a closed orbit (limit cycle or fixed-point) for the unperturbed system Σ which is contained in $I_{\mathcal{W}}$ (see [JF21]).

[JF21] J. Jerray and L. Fribourg, "Determination of limit cycles using stroboscopic set-valued maps," in **7th IFAC Conference on Analysis and Design of Hybrid Systems (ADHS'21), July 7-9, 2021, Brussels, Belgium**, 2021.

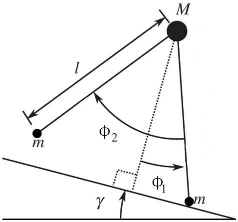
ORBITADOR



ORBITADOR's structure

Biped System

We consider a simple model of biped walker [McG90], seen as a hybrid oscillator.



Biped walker

[McG90] T. McGeer, "Passive dynamic walking," **The International Journal of Robotics Research**, vol. 9, no. 2, pp. 62–82, 1990. DOI: 10.1177/027836499000900206. [Online]. Available: https://doi.org/10.1177/027836499000900206.

Biped System

We consider a simple model of biped walker [McG90], seen as a hybrid oscillator. The model has a continuous state variable $\boldsymbol{x}(t) = (\phi_1(t), \dot{\phi}_1(t), \phi_2(t), \dot{\phi}_2(t))^\top$. The dynamics is described by $\frac{\mathbf{d}\boldsymbol{x}(\mathbf{t})}{\mathbf{dt}} = \boldsymbol{f}(\boldsymbol{x}) + w$ with $w \in \mathcal{W} \subset \mathbb{R}^4$:

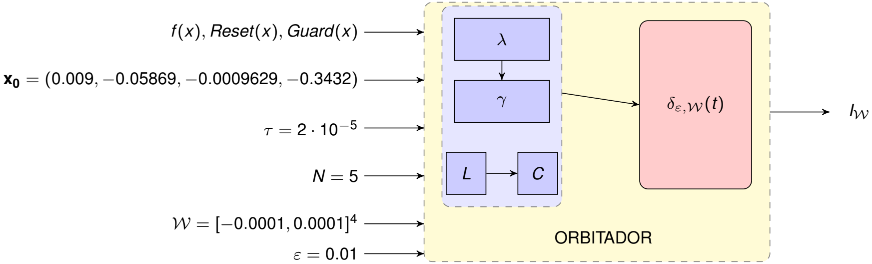
$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} \dot{\phi}_1 \\ \sin(\phi_1 - \gamma) \\ \dot{\phi}_2 \\ \sin(\phi_1 - \gamma) + \dot{\phi}_1^2 \sin\phi_2 - \cos(\phi_1 - \gamma)\sin\phi_2 \end{pmatrix}$$

$$Reset(\boldsymbol{x}) = \begin{pmatrix} -\phi_1 \\ \dot{\phi}_1 \sin(2\phi_1) \\ -2\phi_1 \\ \dot{\phi}_1 \cos 2\phi_1 (1 - \cos 2\phi_1) \end{pmatrix}$$

$$Guard(\boldsymbol{x}) \equiv (2\phi_1 - \phi_2 = 0 \ \wedge \phi_2 < -\delta).$$

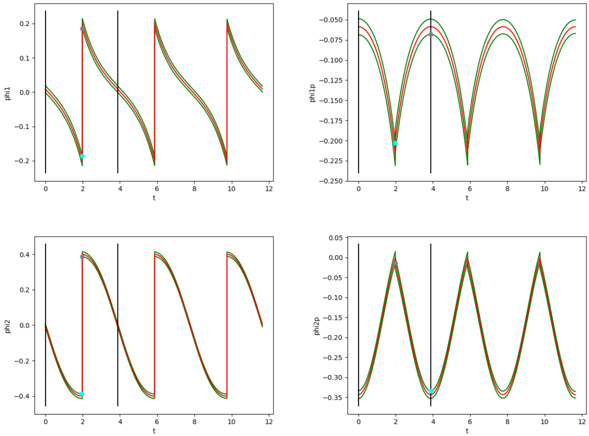
[McG90] T. McGeer, "Passive dynamic walking," **The International Journal of Robotics Research**, vol. 9, no. 2, pp. 62–82, 1990. DOI: 10.1177/027836499000900206. [Online]. Available: https://doi.org/10.1177/027836499000900206.

Biped System with uncertainty



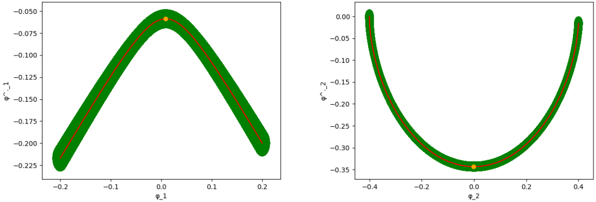
Importing Biped example to ORBITADOR

Biped System with uncertainty



Biped system with uncertainty $w \in \mathcal{W} = [-0.0001, 0.0001]^4$, initial radius $\varepsilon = 0.001$, approximate period $T = 3.8826s$ and time-step $\tau = 2 \cdot 10^{-5}$.

Biped System with uncertainty



Biped system with uncertainty $w \in \mathcal{W} = [-0.0001, 0.0001]^4$, initial radius $\varepsilon = 0.001$, approximate period $T = 3.8826s$ and time-step $\tau = 2 \cdot 10^{-5}$.

- ORBITADOR finds: $B_{\mathcal{W}}((i_0+1)T) \subset B_{\mathcal{W}}(i_0 T)$ for $i_0 = 4$.
- The system converges towards an attractive limit cycle contained in $I_{\mathcal{W}} \equiv \bigcup_{t \in [4T, 5T]} B_{\mathcal{W}}(t)$.

Conclusion and Perspectives

Conclusion

- We presented ORBITADOR a tool that guarantees the existence of limit cycles and constructs invariant sets around them.
- ORBITADOR uses a very general criterion of inclusion of one set in another.

Perspectives

- Adapt the method to solve the convergence to a limit cycle for complex systems.
- Upgrade ORBITADOR so that it computes dynamical systems with control.

[JF21] J. Jerray and L. Fribourg, “Determination of limit cycles using stroboscopic set-valued maps,” in **7th IFAC Conference on Analysis and Design of Hybrid Systems (ADHS’21), July 7-9, 2021, Brussels, Belgium**, 2021.

[LCADSC+17] A. Le Coënt, J. Alexandre Dit Sandretto, A. Chapoutot, L. Fribourg, F. De Vuyst, and L. Chamoin, “Distributed control synthesis using Euler’s method,” in **Proc. of International Workshop on Reachability Problems (RP’17)**, ser. Lecture Notes in Computer Science, vol. 247, Springer, 2017, pp. 118–131.

[LCDVCF17] A. Le Coënt, F. De Vuyst, L. Chamoin, and L. Fribourg, “Control synthesis of nonlinear sampled switched systems using Euler’s method,” in **SNR**, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI: 10.4204/EPTCS.247.2.

[McG90] T. McGeer, “Passive dynamic walking,” **The International Journal of Robotics Research**, vol. 9, no. 2, pp. 62–82, 1990. DOI: 10.1177/027836499000900206. [Online]. Available: https://doi.org/10.1177/027836499000900206.