Guaranteed phase synchronization of hybrid oscillators using symbolic Euler’s method

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Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.
  - dynamical systems have a direct impact on human development.

⇒ The importance of studying:
- stability compared to the initial conditions
- behavior
- synchronization
Motivation

Dynamical systems:
- in which a function describes the time dependence of a point in a geometrical space.
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⇒ The importance of studying:
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Heart rate variability (HRV)

Solar System
Synchronization

- Adjustment of rhythms of active oscillatory objects due to their weak interaction
- Coordination of multiple events.

Two oscillators in phase after a lapse of time
How to highlight the synchronization of dynamical system formally?

- Challenge of describing such systems because their equations are non-linear.
- To study non-linear systems, we often visualize them in a space of configurations (position and speed).
We consider a system composed of 2 subsystems governed by a system of differential equations (ODEs) of the form $\dot{x}(t) = f(x(t))$. The system of ODEs is thus of the form:

$$\begin{align*}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t)) \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t))
\end{align*}$$

(1)

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$, where $m$ is the dimension of the state space of each subsystem.
Synchronization using a reachability method

Given a point of $x_i(t)$ of $S_i \equiv (a_i, b_i, e_i)$ at time $t$, we can define its phase $\phi[x_i(t)]$ by:

$$\phi[x_i(t)] = \frac{\text{ord}(x_i(t)) - \text{ord}(a_i)}{\text{ord}(b_i) - \text{ord}(a_i)}$$

where $a_i, b_i$ are the end points of its main diagonal, $e_i$ the size of its horizontal base and $\text{ord}(a_i)$ (resp. $\text{ord}(b_i)$) denotes the ordinate of $a_i$ (resp. $b_i$).
Synchronization using a reachability method

Scheme of $S_1$ (left) and $S_2$ (right) at $t = 0$ (top) and for some $t \in [kT, (k + 1)T]$ (bottom)
We use the symbolically Euler’s method [LCDVCF17,Fri17].

We consider a subset $B = B_1 \times B_2$, where $B_i \subseteq \mathbb{R}^m$ ($i = 1, 2$) is a ball of the form $B(c_i, r)$ with $c_i$ is the centre and $r$ is the radius.

In order to compute the set of solutions starting at $B^0$. We define for $t \geq 0$:

$$B^{\text{euler}}(t) = B(c_1(t), r(t)) \times B(c_2(t), r(t)),$$

where $(c_1(t), c_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$ is the approximated value of solution $x(t)$ of $\dot{x} = f(x)$ with initial condition $x(0) = (c_1^0, c_2^0)$ given by Euler’s explicit method, and $r(t) \approx r^0 e^{\lambda t}$ is the expanded radius using the one-sided Lipschitz constant $\lambda$ [Söd06].
Symbolic reachability using Euler’s method

Proposition

Given a covering \( \{B_j\}_{j \in J_i} \) of \( S_i \) \((i = 1, 2)\).
If for all \((j_1, j_2) \in J_1 \times J_2\), \( \text{PROC}_1(B_{j_1} \times B_{j_2}) \) succeeds.
Then, for all initial condition \((x_1^0, x_2^0) \in S\), there exists \( t \in [kT, (k + 1)T) \) such that \((x_1(t), x_2(t)) \in S\). Besides:

\[
|\text{phase}(x_1(t)) - \text{phase}(x_2(t))| \leq \epsilon + \min(e_1/f_1, e_2/f_2)
\]

where \( f_i = |\text{ord}(b_i) - \text{ord}(a_i)| \).
The passive biped model [McG90], seen as a hybrid oscillator, exhibits indeed a stable limit-cycle oscillation for appropriate parameter values that corresponds to periodic movements of the legs [SKN17].


Biped example

The model has a continuous state variable $\mathbf{x}(t) = (\phi_1(t), \dot{\phi}_1(t), \phi_2(t), \dot{\phi}_2(t))^\top$. The dynamics is described by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with:

$$
\mathbf{f}(\mathbf{x}) = \begin{pmatrix}
\dot{\phi}_1 \\
\sin(\phi_1 - \gamma) \\
\dot{\phi}_2 \\
\sin(\phi_1 - \gamma) + \phi_1^2 \sin \phi_2 - \cos(\phi_1 - \gamma) \sin \phi_2
\end{pmatrix}
$$

(2)

$$
\text{Reset}(\mathbf{x}) = \begin{pmatrix}
-\phi_1 \\
\phi_1 \sin(2\phi_1) \\
-2\phi_1 \\
\phi_1 \cos 2\phi_1 (1 - \cos 2\phi_1)
\end{pmatrix}
$$

(3)

$$
\text{Guard}(\mathbf{x}) \equiv (2\phi_1 - \phi_2 = 0 \land \phi_2 < -\delta).
$$

(4)

with $\delta = 0.1$ and $\gamma = 0.009$. 
Biped example

Biped: A cyclic trajectory for plan $\phi_1$ (left) and $\phi_2$ (right); the green zone indicates the contractive area ($\lambda < 0$) and the red zone the expansive one ($\lambda > 0$)

- The time-step used in Euler’s method is $\tau = 2 \cdot 10^{-5}$.
- The period of the system is $T = 776440\tau$.
- The radius expansion factor after one period is $E = 2.63$.
- The number of periods considered for synchronization is $k = 30$. 
Biped: Synchronization of 10 (pairs of) balls, located initially on the parallelogram perimeters, after $k = 30$ periods (without radius expansion for clarity).
Conclusion and Perspectives

Conclusion

- We presented a symbolic reachability method to prove phase synchronization of oscillators.
- A *finite* number of points, displaced from their original position on a synchronization orbit, return after some time into a close neighborhood of the orbit.

Perspectives

- Adapt the *phase reduction* to solve systems with higher state space dimension.
- Replace the symbolic Euler's method by any other symbolic reachability procedure to cover larger sets $S$. 


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