What is a Categorical Model of the Differential and the Resource λ-Calculi?

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Abstract. The *differential λ-calculus* is a paradigmatic functional programming language endowed with a syntactical differentiation operator that allows to apply a program to an argument in a linear way. One of the main features of this language is that it is *resource conscious* and gives the programmer suitable primitives to handle explicitly the resources used by a program during its execution. The differential operator also allows to write the full Taylor expansion of a program. Through this expansion every program can be decomposed into an infinite sum (representing non-deterministic choice) of 'simpler' programs that are strictly linear.

The aim of this paper is to develop an abstract 'model theory' for the untyped differential λ-calculus. In particular, we investigate what should be a general categorical definition of denotational model for this calculus. Starting from the work of Blute, Cockett and Seely on differential categories we provide the notion of *Cartesian closed differential category* and we prove that *linear reflexive objects* living in such categories constitute sound and complete models of the untyped differential λ-calculus. We also give sufficient conditions for Cartesian closed differential categories to model the Taylor expansion. This entails that every model living in such categories equates all programs having the same full Taylor expansion.

We then provide a concrete example of a Cartesian closed differential category modeling the Taylor expansion, namely the category **MRel** of sets and relations from finite multisets to sets. We prove that the extensional model $\mathscr{D}$ of λ-calculus we have recently built in **MRel** is linear, and therefore it is also an extensional model of the untyped differential λ-calculus. In the same category we build a non-extensional model $\mathcal{E}$ and we prove that it is however extensional on its differential part.

Finally, we study the relationship between the differential λ-calculus and the *resource calculus*, a functional programming language combining the ideas behind the differential λ-calculus with those behind Boudol's λ-calculus with multiplicities. We define two translation maps between these two calculi and we study the properties of these translations. In particular, from this analysis it follows that the two calculi share the same notion of model. Therefore the resource calculus can be interpreted by translation into every linear reflexive object living in a Cartesian closed differential category.

Keywords: differential λ-calculus, differential λ-theories, resource calculus, resource λ-theories, differential categories, categorical models, soundness, Taylor expansion.

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Introduction

Among the variety of computational formalisms that have been studied in the literature, the λ-calculus [2] plays an important role as a bridge between logic and computer science. The λ-calculus was originally introduced by Church [15,16] as a foundation for mathematics, where functions – instead of sets – were primitive. This system turned out to be consistent and successful as a tool for formalizing all computable functions. However, the λ-calculus is not resource sensitive since a λ-term can erase its arguments or duplicate them an arbitrary large number of times. This becomes problematic when one wants to deal with programs that are executed in environments with bounded resources (like PDA's) or in presence of depletable arguments (like quantum data that cannot be duplicated for physical reasons). In these contexts we want to be able to express the fact that a program *actually consumes* its arguments. Such an idea of 'resource consumption' is central in Girard's quantitative semantics [26]. This semantics establishes an analogy between linearity in the sense of computer science (programs using arguments exactly once) and algebraic linearity (commutation of sums and products with scalars), giving a new mathematically very appealing interpretation of resource consumption. Drawing on these insights, Ehrhard and Regnier [21] designed a resource sensitive paradigmatic programming language called *the differential λ-calculus*.

The differential λ-calculus is a conservative (see [21, Prop. 19]) extension of the untyped λ-calculus with differential and linear constructions. In this language, there are two different operators that can be used to apply a program to its argument: the usual application and a *linear application*. This last one defines a syntactic derivative operator $\mathsf{D}s \cdot t$ which is an excellent candidate to increase control over programs executed in environments with bounded resources. Indeed, the evaluation of $\mathsf{D}s \cdot t$ (the derivative of the program s on the argument t) has a precise operational meaning: it captures the fact that t is available for s "exactly once". The corresponding meta-operation of substitution, that replaces exactly one (linear) occurrence of x in s by t, is called "differential substitution" and is denoted by $\frac{\partial s}{\partial x} \cdot t$. It is worth noting that when s contains several occurrences of x, one has to choose which occurrence should be replaced and there are several possible choices. When s does not contain any occurrence of x then the differential substitution cannot be performed and the result is 0 (corresponding to an empty program). Thus, the differential substitution forces the presence of non-determinism in the system, which is represented by a formal sum having 0 as neutral element. Therefore, the differential λ-calculus constitutes a useful framework for studying the notions of linearity and non-determinism, and the relation between them.

Taylor expansion. As expected, iterated differentiation yields a natural notion of linear approximation of the ordinary application of a program to its argument. Indeed, the syntactic derivative operator allows to write all the derivatives of a λ-term M, thus it also allows (in presence of countable sums) to define its *full Taylor expansion* M^*. In general, M^* will be an infinite formal linear combination of simple terms (with coefficients in a field), and should satisfy, when M is a usual application NQ:

$$(NQ)^* = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathsf{D}^n N \cdot (\underbrace{Q, \ldots, Q}_{n \text{ times}}))0$$

where $\frac{1}{n!}$ is a numerical coefficient and $\mathsf{D}^n N \cdot (Q, \ldots, Q)$ stands for iterated linear application of N to n copies of Q. The precise operational meaning of the Taylor expansion has been extensively studied in [21,22,24]. The crucial fact of such an expansion is that it gives a *quantitative* account to the β-reduction of λ-calculus (in the sense of Böhm tree computation). Formal connections between Taylor expansions and Böhm trees of usual λ-terms have been presented in [22], using a decorated version of Krivine's machine.

The resource calculus, which is a revisitation of Boudol's λ-calculus with multiplicities [6,7], constitutes an alternative approach to the problem of modeling resource consumption within a functional programming language. In this calculus there is only one operator of application, while the arguments can be either linear or reusable and come in finite multisets called 'bags'. Linear arguments must be used exactly once, while reusable ones can be used *ad libitum.* Also in this setting the evaluation of a function applied to a bag of arguments may give rise to different possible choices, corresponding to the different possibilities of distributing the arguments between the occurrences of the formal parameter.

The main differences between Boudol's calculus and the resource calculus are that the former is affine, is equipped with explicit substitution and has a lazy operational semantics, while the latter is linear and is a true extension of the classical λ-calculus. The current formalization of resource calculus has been proposed by Tranquilli in [44] with the aim of defining a Curry-Howard correspondence with differential nets [23].

The resource calculus has been recently studied from a syntactic point of view by Pagani and Tranquilli [39] for confluence results, by Pagani and the author [34] for separability results and by Pagani and Ronchi della Rocca [38] for results about may and must solvability. Algebraic notions of models for the *strictly linear fragment* of resource calculus have been proposed by Carraro, Ehrhard and Salibra in [14]. In the present paper we mainly focus on the study of the differential λ-calculus, but we will also draw conclusions for the resource calculus.

Denotational semantics. Although the differential λ-calculus is born from semantical considerations (i.e., the deep analysis of coherent spaces performed by Ehrhard and Regnier) the investigations on its denotational semantics are at the very beginning. It is known that finiteness spaces [19] and the relational semantics of linear logic [26] are examples of models of the *simply typed* differential λ-calculus, thus having a very limited expressive power. Concerning the *untyped* differential λ-calculus, it is just known *in the folklore* that the relational model $\mathscr{D}$ introduced in [10] in the relational semantics constitutes a concrete example of model[2]. This picture is reminiscent of the beginning of denotational semantics of λ-calculus, when Scott's $\mathscr{D}_\infty$ was the unique concrete example of model of λ-calculus but no general definition of model was known. Only when an abstract model theory for this calculus has been developed the researchers have been able to provide rich semantics (like the continuous [43], stable [3] and strongly stable semantics [9]) and general methods for building huge classes of models in these semantics.

Categorical notion of model. The aim of the present paper is to provide a general categorical notion of model of the untyped differential λ-calculus. Our starting point will be the work of Blute, Cockett and Seely on (Cartesian) differential categories [4,5]. In these categories a derivative operator $D(-)$ on morphisms is equationally axiomatized; the derivative of a morphism $f : A \to B$ will be a morphism $D(f) : A \times A \to B$, linear in its first component. The authors have then proved that these categories are sound and complete to model suitable term calculi. However, it turns out that the properties of differential categories are too weak for modeling the full differential λ-calculus. For this reason, we will introduce the more powerful notion of *Cartesian closed differential category.* In such categories it is possible to define an operator

$$\frac{f : C \times A \to B \quad g : C \to A}{f \star g : C \times A \to B} \ (\star)$$

[2]This follows from [23] where it is shown that the differential λ-calculus can be translated into differential proofnets, plus [46] where it is proved that $\mathscr{D}$ is a model of such proofnets.

that can be seen as a categorical counterpart of the differential substitution. Intuitively, the morphism $f \star g$ is obtained by force-feeding the second argument A of f with *one copy* of the result of g. The type is not modified because $f \star g$ may still depend on A.

The operator $\star$ allows us to interpret the differential λ-calculus in every *linear reflexive object* $\mathscr{U}$ living in a Cartesian closed differential category $\mathbf{C}$. For a reflexive object $\mathscr{U} = (U, \mathcal{A}, \lambda)$ "to be linear" amounts to ask that the morphisms $\mathcal{A}$ and λ performing the retraction $(U \Rightarrow U) \lhd U$ are linear. We will prove that this categorical notion of model is *sound*; this means that the induced equational theory $\mathrm{Th}(\mathscr{U})$ is actually a *differential λ-theory*. We will also investigate what conditions the category $\mathbf{C}$ should satisfy in order to *model the Taylor expansion*. This entails that all differential programs having the same Taylor expansion are equated in every model living in $\mathbf{C}$.

A question that arises naturally when a notion of model of a certain calculus is introduced is whether it is *equationally complete*, that is whether all equational theories of that calculus can be represented. For instance, in the case of the untyped λ-calculus, Scott and Koymans proved that for every λ-theory $\mathcal{T}$ there is a reflexive object $\mathscr{U}$ in a Cartesian closed category $\mathbf{C}$ such that $\mathrm{Th}(\mathscr{U}) = \mathcal{T}$. We will prove that the notion of linear reflexive object in a Cartesian closed differential category is equationally complete for the differential λ-calculus, provided that we only consider theories satisfying suitable properties. The first property is that in these theories the sum is considered as idempotent, this amounts to say that we only know whether a term appears in a result, not how many times it appears; the second is that these theories are "extensional on linear applications", which means that $\mathsf{D}s \cdot t$ must have a functional behaviour. It turns out that these properties are quite natural in the sense that they are satisfied by all models which have arisen so far.

Relational semantics. In [10] we have built, in collaboration with Bucciarelli and Ehrhard, an extensional model $\mathscr{D}$ of λ-calculus living in the category **MRel** of sets and "relations from finite multisets to sets". This model can be seen as a relational analogue of Scott's $\mathscr{D}_\infty$ [20]. By virtue of its logical nature, $\mathscr{D}$ can be used to model several systems, beyond the untyped λ-calculus. For instance, in [11] the authors have proved that it constitutes an adequate model of a λ-calculus extended with non-deterministic choice and parallel composition, while in [46] Vaux has shown that it is a model of differential proof-nets.

In the present paper we study $\mathscr{D}$ as a model of the untyped differential λ-calculus. Indeed (as expected) the category **MRel** turns out to be an instance of the definition of Cartesian closed differential category, and the relational model $\mathscr{D}$ is easily checked to be linear. We will then study the equational theory induced by $\mathscr{D}$ and prove that it equates all terms having the same Taylor expansion. This property follows from the fact that **MRel** models the Taylor expansion. As a simple consequence we get that the relational semantics is *hugely incomplete* — there is a continuum of equational theories that are not representable by models living in **MRel**.

In the same category, we will also build a model $\mathscr{E}$ which can be seen as a relational analogue of Engeler's graph model [25]. The model $\mathscr{E}$ provides an example of a non-extensional model, which is however extensional on linear applications.

Translations. Finally, we study the inter-relationships existing between the differential λ-calculus and the resource calculus. Actually there is a common belief in the scientific community stating that the two calculi are morally the same, and the choice of studying one language rather than the other one is more a matter of taste than a substantial difference. We will give a formal meaning to this belief by defining a translation map $(\cdot)^r$ from the differential λ-calculus to the resource calculus, and another map $(\cdot)^d$ in the other direction. We will prove that these translations are 'faithful' in the sense that equivalent programs of differential λ-calculus are mapped into equivalent resource programs, and *vice versa*. This shows that the two calculi share the same notion of denotational model; in particular the resource calculus can be interpreted by translation in every linear reflexive object living in a Cartesian closed differential category.

Outline.

Section 1 contains the preliminary notions and notations needed in the rest of the paper. In Section 2 we present the syntax and the axioms of the differential λ-calculus, and we define the associated equational theories. In Section 3 we introduce the notion of Cartesian closed differential category. Section 4 is devoted to show that linear reflexive objects in such categories are sound and complete models of the differential λ-calculus. In Section 5 we build two relational models $\mathscr{D}$ and $\mathscr{E}$ and provide a partial characterization of their equational theories. In Section 6 we define the resource calculus and we study its relationship with the differential λ-calculus. Finally, in Section 7 we discuss the related works, we present our conclusions and we propose some further lines of research.

1. Preliminaries

To keep this article self-contained we summarize some definitions and results that will be used in the sequel. Our main reference for category theory is [1].

1.1. Sets and Multisets

We denote by $\mathbf{N}$ the set of natural numbers. Given $n \in \mathbf{N}$ we write $\mathfrak{S}_n$ for the set of all permutations (bijective maps) of the set $\{1, \dots, n\}$.

Let A be a set. We denote by $\mathcal{P}(A)$ the powerset of A. A *multiset* m over A can be defined as an unordered list $m = [a_1, a_2, \dots]$ with repetitions such that $a_i \in S$ for all indices i. A multiset m is called *finite* if it is a finite list; we denote by $[]$ the empty multiset. Given two multisets $m_1 = [a_1, a_2, \dots]$ and $m_2 = [b_1, b_2, \dots]$ the *multi-union* of m_1, m_2 is defined by $m_1 \uplus m_2 = [a_1, b_1, a_2, b_2, \dots]$.

Finally, we write $\mathcal{M}_f(A)$ for the set of all finite multisets over A.

1.2. Cartesian (Closed) Categories

Let $\mathbf{C}$ be a *Cartesian category* and A, B, C be arbitrary objects of $\mathbf{C}$. We write $\mathbf{C}(A, B)$ for the homset of morphisms from A to B; when there is no chance of confusion we write $f : A \to B$ instead of $f \in \mathbf{C}(A, B)$. We usually denote by $A \times B$ the *categorical product* of A and B, by $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ the associated *projections* and, given a pair of arrows $f : C \to A$ and $g : C \to B$, by $\langle f, g\rangle : C \to A \times B$ the unique arrow such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. We write $f \times g$ for the *product map of f and g* which is defined by $f \times g = \langle f \circ \pi_1, g \circ \pi_2\rangle$.

If the category $\mathbf{C}$ is *Cartesian closed* we write $A \Rightarrow B$ for the *exponential object* and $\mathrm{ev}_{AB} : (A \Rightarrow B) \times A \to B$ for the *evaluation morphism*. Moreover, for any object C and arrow $f : C \times A \to B$, $\Lambda(f) : C \to (A \Rightarrow B)$ stands for the (unique) morphism such that $\mathrm{ev}_{AB} \circ (\Lambda(f) \times \mathrm{Id}_A) = f$. Finally, $\mathbb{1}$ denotes the terminal object and $!_A$ the only morphism in $\mathbf{C}(A, \mathbb{1})$.

We recall that in every Cartesian closed category the following equalities hold:

$$\text{(pair)} \qquad \langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle \qquad \Lambda(f) \circ g = \Lambda(f \circ (g \times \mathrm{Id})) \qquad \text{(Curry)}$$

$$\text{(beta-cat)} \qquad \mathrm{ev} \circ \langle \Lambda(f), g\rangle = f \circ \langle \mathrm{Id}, g\rangle \qquad \Lambda(\mathrm{ev}) = \mathrm{Id} \qquad \text{(Id-Curry)}$$

Moreover, we can define the *uncurry* operator $\Lambda^-(-) = \mathrm{ev} \circ (- \times \mathrm{Id})$. From (beta-cat), (Curry) and (Id-Curry) it follows that $\Lambda(\Lambda^-(f)) = f$ and $\Lambda^-(\Lambda(g)) = g$.

2. The Differential Lambda Calculus

In this section we recall the definition of the *differential λ-calculus* [21], together with some standard properties of the language. We also define the associated equational theories, namely, the *differential λ-theories*. The syntax we use in the present paper is freely inspired by [45].

2.1. Differential Lambda Terms

The set Λ^d of *differential λ-terms* and the set Λ^s of *simple terms* are defined by mutual induction as follows:

$$\Lambda^d : \quad S,T,U,V \ ::= \quad s \mid 0 \mid s+T \qquad\qquad \Lambda^s : \quad s,t,u,v \ ::= \quad x \mid \lambda x.s \mid sT \mid \mathsf{D}s\cdot t$$

The differential λ-term $\mathsf{D}s\cdot t$ represents the *linear application* of s to t. Intuitively, this means that s is provided with exactly one copy of t. Notice that sums may appear also in simple terms as right components of ordinary applications. Although the rule $s+t=s$ will not be valid in our axiomatization, the sum should still be thought of as a version of non-deterministic choice where all actual choice operations are postponed.

Convention 2.1 *We consider differential λ-terms up to α-conversion, and up to associativity and commutativity of the sum. The term 0 is the neutral element of the sum, thus we also add the equation $S+0=S$.*

As a matter of notation we write $\lambda x_1 \dots x_n.s$ for $\lambda x_1.(\cdots(\lambda x_n.s)\cdots)$ and $sT_1\cdots T_k$ for $(\cdots(sT_1)\cdots)T_k$. Moreover, we set $\mathsf{D}^1 s\cdot(t_1) = \mathsf{D}s\cdot t_1$ and $\mathsf{D}^{n+1}s\cdot(t,t_1,\dots,t_n) = \mathsf{D}^n(\mathsf{D}s\cdot t)\cdot(t_1,\dots,t_n)$. When writing $\mathsf{D}^n s\cdot(t_1,\dots,t_n)$ we suppose $n>0$.

Definition 2.2 *The* permutative equality *on differential λ-terms imposes that* $\mathsf{D}^n s\cdot(t_1,\dots,t_n) = \mathsf{D}^n s\cdot(t_{\sigma(1)},\dots,t_{\sigma(n)})$ *for all permutations* $\sigma\in\mathfrak{S}_n$.

Hereafter, we will consider differential λ-terms also up to permutative equality. This is needed, for instance, for proving the Schwarz Theorem (see Subsection 2.2) and hence to speak of a differential operator. Concerning specific λ-terms we set:

$$\mathbf{I}\equiv\lambda x.x \qquad \mathbf{1}\equiv\lambda xy.xy \qquad \Delta\equiv\lambda x.xx \qquad \Omega\equiv\Delta\Delta \qquad \mathbf{Y}\equiv\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$
$$\mathbf{s}\equiv\lambda nxy.nx(xy) \qquad \underline{n}\equiv\lambda sx.s^n(x),\ \text{for every natural number } n\in\mathbf{N}$$

where $\equiv$ stands for syntactical equality up to the above mentioned equivalences on differential λ-terms. Note that $\mathbf{I}$ is the identity, $\mathbf{Y}$ is Curry's fixpoint combinator, $\underline{n}$ the n-th Church numeral and $\mathbf{s}$ implements the successor function on Church numerals. The term Ω denotes the usual paradigmatic unsolvable λ-term.

Definition 2.3 *Let S be a differential λ-term. The set* $\mathrm{FV}(S)$ *of* free variables *of S is defined inductively as follows:*

- $\mathrm{FV}(x)=\{x\}$,
- $\mathrm{FV}(\lambda x.s)=\mathrm{FV}(s)-\{x\}$,
- $\mathrm{FV}(sT)=\mathrm{FV}(s)\cup\mathrm{FV}(T)$,
- $\mathrm{FV}(\mathsf{D}s\cdot t)=\mathrm{FV}(s)\cup\mathrm{FV}(t)$,
- $\mathrm{FV}(0)=\emptyset$,

- $\mathrm{FV}(s+S) = \mathrm{FV}(s) \cup \mathrm{FV}(S)$.

Given differential λ-terms $S_1, \dots, S_k$ *we set* $\mathrm{FV}(S_1, \dots, S_k) = \mathrm{FV}(S_1) \cup \dots \cup \mathrm{FV}(S_k)$.

We now introduce some notations on differential λ-terms that will be particularly useful to define the substitution operators in the next subsection.

Notation 2.4 *We will often use the following abbreviations (notice that these are just syntactic sugar, not real terms):*

- $\lambda x.(\sum_{i=1}^{k} s_i) = \sum_{i=1}^{k} \lambda x.s_i$,
- $(\sum_{i=1}^{k} s_i)T = \sum_{i=1}^{k} s_i T$,
- $\mathsf{D}(\sum_{i=1}^{k} s_i) \cdot (\sum_{j=1}^{n} t_j) = \sum_{i,j} \mathsf{D} s_i \cdot t_j$.

Intuitively, these equalities make sense since the lambda abstraction is linear, the usual application is linear in its left component, and the linear application is a bilinear operator. Notice however that $S(\Sigma_{i=1}^{k} t_i) \neq \Sigma_{i=1}^{k} S t_i$.

Observe that in the particular case of empty sums, we get $\lambda x.0 = 0$, $0T = 0$, $\mathsf{D}0 \cdot t = 0$, $\mathsf{D}s \cdot 0 = 0$. Thus 0 annihilates any term, except when it occurs on the right component of an ordinary application.

2.2. Two Kinds of Substitution

We introduce two kinds of meta-operations of substitution on differential λ-terms: the usual capture-free substitution and the differential substitution. Both definitions strongly use the abbreviations introduced in Notation 2.4.

Definition 2.5 *Let* S, T *be differential λ-terms and* x *be a variable. The* capture-free substitution *of* T *for* x *in* S*, denoted by* $S\{T/x\}$*, is defined by induction on* S *as follows:*

- $y\{T/x\} = \begin{cases} T & \text{if } x = y, \\ y & \text{otherwise,} \end{cases}$
- $(\lambda y.s)\{T/x\} = \lambda y.s\{T/x\}$*, where we suppose by α-conversion that* $x \neq y$ *and* $y \notin \mathrm{FV}(T)$,
- $(sU)\{T/x\} = (s\{T/x\})(U\{T/x\})$,
- $(\mathsf{D}^n s \cdot (u_1, \dots, u_n))\{T/x\} = \mathsf{D}^n (s\{T/x\}) \cdot (u_1\{T/x\}, \dots, u_n\{T/x\})$,
- $0\{T/x\} = 0$,
- $(s+S)\{T/x\} = s\{T/x\} + S\{T/x\}$.

Thus, $S\{T/x\}$ is the result of substituting T for all free occurrences of x in S, subject to the usual proviso about renaming bound variables in S to avoid capture of free variables in T. On the other hand, the differential substitution $\frac{\partial S}{\partial x} \cdot T$ defined below denotes the result of substituting T (still avoiding capture of variables) for *exactly one* – non-deterministically chosen – linear occurrence of x in S. If such an occurrence is not present in S then the result will be 0.

Definition 2.6 *Let* S, T *be differential λ-terms and* x *be a variable. The* differential substitution *of* T *for* x *in* S*, denoted by* $\frac{\partial S}{\partial x} \cdot T$*, is defined by induction on* S *as follows:*

- $\frac{\partial y}{\partial x} \cdot T = \begin{cases} T & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases}$

- $\frac{\partial}{\partial x}(sU) \cdot T = (\frac{\partial s}{\partial x} \cdot T)U + (\mathsf{D}\, s \cdot (\frac{\partial U}{\partial x} \cdot T))U$,
- $\frac{\partial}{\partial x}(\lambda y.s) \cdot T = \lambda y.\frac{\partial s}{\partial x} \cdot T$, *where we suppose by α-conversion that* $x \neq y$ *and* $y \notin \mathrm{FV}(T)$,
- $\frac{\partial}{\partial x}(\mathsf{D}^n\, s \cdot (u_1, \ldots, u_n)) \cdot T = \mathsf{D}^n\, (\frac{\partial s}{\partial x} \cdot T) \cdot (u_1, \ldots, u_n) + \sum_{i=1}^{n} \mathsf{D}^n\, s \cdot (u_1, \ldots, \frac{\partial u_i}{\partial x} \cdot T, \ldots, u_n)$,
- $\frac{\partial 0}{\partial x} \cdot T = 0$,
- $\frac{\partial}{\partial x}(s + U) \cdot T = \frac{\partial s}{\partial x} \cdot T + \frac{\partial U}{\partial x} \cdot T$.

The definition states that the differential substitution distributes over linear constructions. We now spend some words on the case of the usual application sU because it is the most complex one. The result of $\frac{\partial (sU)}{\partial x} \cdot T$ is the sum of two terms since the differential substitution can non-deterministically be applied either to s or to U. In the first case, we can safely apply it to s since the usual application is linear in its left argument, so we obtain $(\frac{\partial s}{\partial x} \cdot T)U$. In the other case we cannot apply it directly to U because the standard application is *not* linear in its right argument. We thus follow two steps: (i) we replace sU by $(\mathsf{D}\, s \cdot U)U$; (ii) we apply the differential substitution to the linear copy of U.

Intuitively, this works because U is morally available infinitely many times in sU, so when the differential substitution goes on U we 'extract' a linear copy of U, that receives the substitution, and we keep the other infinitely many unchanged. This will be much more evident in the definition of the analogous operation for the resource calculus (cf. Definition 6.3).

Example 2.7 *Recall that the simple terms* Δ *and* $\mathbf{I}$ *have been defined at page 7.*

1. $\frac{\partial \Delta}{\partial x} \cdot \mathbf{I} = 0$, *since* x *does not occur free in* Δ,
2. $\frac{\partial x}{\partial x} \cdot \mathbf{I} = \mathbf{I}$,
3. $\frac{\partial (xx)}{\partial x} \cdot \mathbf{I} = \mathbf{I}x + (\mathsf{D}\, x \cdot \mathbf{I})x$,
4. $\frac{\partial}{\partial x}(\frac{\partial (xx)}{\partial x} \cdot \mathbf{I}) \cdot \Delta = (\mathsf{D}\, \mathbf{I} \cdot \Delta)x + (\mathsf{D}\, \Delta \cdot \mathbf{I})x + (\mathsf{D}\, (\mathsf{D}\, x \cdot \mathbf{I}) \cdot \Delta)x$,
5. $((\mathsf{D}\, x \cdot x)x)\{\mathbf{I}/x\} = (\mathsf{D}\, \mathbf{I} \cdot \mathbf{I})\mathbf{I}$.

The differential substitution $\frac{\partial S}{\partial x} \cdot T$ can be thought of as the differential of S with respect to the variable x, linearly applied to T. This may be inferred from the rule for linear application, which relates to the rule for composition of the differential. Moreover, it is easy to check that if $x \notin \mathrm{FV}(S)$ (i.e., S is constant with respect to x) then $\frac{\partial S}{\partial x} \cdot T = 0$. This intuition is also reinforced by the validity of the Schwarz Theorem.

Theorem 2.8 *(Schwarz Theorem) Let* S, T, U *be differential λ-terms. Let* x *and* y *be variables such that* $x \notin \mathrm{FV}(U)$. *Then we have:*

$$\frac{\partial}{\partial y}\left(\frac{\partial S}{\partial x} \cdot T\right) \cdot U = \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial y} \cdot U\right) \cdot T + \frac{\partial S}{\partial x} \cdot \left(\frac{\partial T}{\partial y} \cdot U\right).$$

In particular, when $y \notin \mathrm{FV}(T)$, *then the second summand is 0 and the two differential substitutions commute.*

Proof. The proof is by structural induction on S. Here we just check the case $S \equiv vV$.

$$\begin{array}{rcl} \frac{\partial}{\partial y}(\frac{\partial vV}{\partial x}\cdot T)\cdot U & = & \frac{\partial}{\partial y}((\frac{\partial v}{\partial x}\cdot T)V + (\mathsf{D}v\cdot(\frac{\partial V}{\partial x}\cdot T))V)\cdot U \\ & = & (\frac{\partial}{\partial y}(\frac{\partial v}{\partial x}\cdot T)\cdot U)V + (\mathsf{D}(\frac{\partial v}{\partial x}\cdot T)\cdot(\frac{\partial V}{\partial y}\cdot U))V \\ & & \quad + (\mathsf{D}(\frac{\partial v}{\partial y}\cdot U)\cdot(\frac{\partial V}{\partial x}\cdot T))V + (\mathsf{D}v\cdot(\frac{\partial}{\partial y}(\frac{\partial V}{\partial x}\cdot T)\cdot U))V \\ & & \quad + (\mathsf{D}(\mathsf{D}v\cdot(\frac{\partial V}{\partial x}\cdot T)\cdot(\frac{\partial V}{\partial y}\cdot U)))V \end{array}$$

By applying the induction hypothesis (and the permutative equality) we get:

$$\begin{array}{rcl} \frac{\partial}{\partial y}(\frac{\partial vV}{\partial x}\cdot T)\cdot U & = & (\frac{\partial}{\partial x}(\frac{\partial v}{\partial y}\cdot U)\cdot T)V + (\mathsf{D}(\frac{\partial v}{\partial y}\cdot U)\cdot(\frac{\partial V}{\partial x}\cdot T))V + (\mathsf{D}(\frac{\partial v}{\partial x}\cdot T)\cdot(\frac{\partial V}{\partial y}\cdot U))V \\ & & \quad + (\mathsf{D}v\cdot(\frac{\partial}{\partial x}(\frac{\partial V}{\partial y}\cdot U)\cdot T))V + (\mathsf{D}(\mathsf{D}v\cdot(\frac{\partial V}{\partial y}\cdot U))\cdot(\frac{\partial V}{\partial x}\cdot T))V \\ & & \quad + (\frac{\partial v}{\partial x}\cdot(\frac{\partial T}{\partial y}\cdot U))V + (\mathsf{D}v\cdot(\frac{\partial V}{\partial x}\cdot(\frac{\partial T}{\partial y}\cdot U)))V \\ & = & \frac{\partial}{\partial x}((\frac{\partial v}{\partial y}\cdot U)V + (\mathsf{D}v\cdot(\frac{\partial V}{\partial y}\cdot U))V)\cdot T \\ & & \quad + (\frac{\partial v}{\partial x}\cdot(\frac{\partial T}{\partial y}\cdot U))V + (\mathsf{D}v\cdot(\frac{\partial V}{\partial x}\cdot(\frac{\partial T}{\partial y}\cdot U)))V \\ & = & \frac{\partial}{\partial x}(\frac{\partial vV}{\partial y}\cdot U)\cdot T + \frac{\partial vV}{\partial x}\cdot(\frac{\partial T}{\partial y}\cdot U). \end{array}$$

■

For the sake of readability, it will be sometimes useful to adopt the following notation for multiple differential substitutions.

Notation 2.9 *We set*

$$\frac{\partial^n S}{\partial x_{1,\dots,x_n}}\cdot(t_1,\dots,t_n) = \frac{\partial}{\partial x_n}\Big(\cdots\frac{\partial S}{\partial x_1}\cdot t_1\cdots\Big)\cdot t_n$$

where $x_i \notin \mathrm{FV}(t_1,\dots,t_n)$ *for all* $1 \le i \le n$.

Remark 2.10 *From Theorem 2.8 we have:*

$$\frac{\partial^n S}{\partial x_{1,\dots,x_n}}\cdot(t_1,\dots,t_n) = \frac{\partial^n S}{\partial x_{\sigma(1),\dots,x_{\sigma(n)}}}\cdot(t_{\sigma(1)},\dots,t_{\sigma(n)}),\ \textit{for all } \sigma\in\mathfrak{S}_n.$$

2.3. Differential Lambda Theories

In this subsection we introduce the axioms associated with the differential λ-calculus and we define the equational theories of this calculus, namely, the *differential λ-theories*.

The axioms of the *differential λ-calculus* are the following (for all $s,t\in\Lambda^s$ and $T\in\Lambda^d$):

$$(\beta)\quad (\lambda x.s)T = s\{T/x\}$$

$$(\beta_D)\quad \mathsf{D}(\lambda x.s)\cdot t = \lambda x.\frac{\partial s}{\partial x}\cdot t.$$

Once oriented from left to right, the (β)-conversion expresses the way of calculating a function $\lambda x.s$ classically applied to an argument T, while the (β_D)-conversion the way of evaluating a function $\lambda x.s$ *linearly* applied to a simple argument t.

Notice that in the result of a linear application the λx does not disappear. This is needed since the simple term s may still contain free occurrences of x. The only way to get rid of the outer lambda abstraction in the term $\lambda x.s$ is to apply it classically to a term T, and then use the (β)-rule; when $x\notin\mathrm{FV}(s)$ a standard choice for T is 0.

The differential λ-calculus is an intensional language — there are syntactically different programs having the same extensional behaviour. We will be sometimes interested in the extensional version of this calculus which is obtained by adding the following axiom (for every $s\in\Lambda^s$):

$$(\eta)\quad \lambda x.sx = s,\ \text{where } x\notin\mathrm{FV}(s)$$

In the differential λ-calculus we have another extensionality axiom, strictly weaker than (η), that can be safely added to the system, namely (for every $s, t \in \Lambda^s$):

$$(\eta_\partial) \quad \lambda x.(\mathsf{D}\, s \cdot t)x = \mathsf{D}\, s \cdot t, \text{ where } x \notin \mathrm{FV}(s,t)$$

The axiom (η_∂) states that the calculus is extensional only in its differential part, that is in presence of the linear application. Intuitively, this means that $\mathsf{D}\, s \cdot t$ must have a functional behaviour, which is always true in a simply typed setting where $s : A \to B$, $t : A$ and $\mathsf{D}\, s \cdot t : A \to B$. Interestingly enough there are very natural models of untyped differential λ-calculus that satisfy (η_∂) but do not satisfy (η). We refer to Subsection 5.3.1 for an example.

A λ^d*-relation* $\mathcal{T}$ is any set of equations between differential λ-terms (which can be thought of as a relation on $\Lambda^d \times \Lambda^d$).

A λ^d-relation $\mathcal{T}$ is called:

- an *equivalence* if it is closed under the following rules (for all $S, T, U \in \Lambda^d$):

$$\frac{}{S = S}\ \text{reflexivity} \qquad \frac{T = S}{S = T}\ \text{symmetry} \qquad \frac{S = T \quad T = U}{S = U}\ \text{transitivity}$$

- *compatible* if it is closed under the following rules (for all $S, T, U, V, S_i, T_i \in \Lambda^d$):

$$\frac{S = T}{\lambda x.S = \lambda x.T}\ \text{lambda} \qquad \frac{S = T \quad U = V}{ST = UV}\ \text{app} \qquad \frac{S = T \quad U = V}{\mathsf{D}\, S \cdot U = \mathsf{D}\, T \cdot V}\ \text{Lapp}$$

$$\frac{S_i = T_i \quad \text{for all } 1 \le i \le n}{\sum_{i=1}^n S_i = \sum_{i=1}^n T_i}\ \text{sum}$$

As a matter of notation, we will write $\mathcal{T} \vdash S = T$ or $S =_{\mathcal{T}} T$ for $S = T \in \mathcal{T}$.

Definition 2.11 *A* differential λ-theory *is any compatible* λ^d*-relation* $\mathcal{T}$ *which is an equivalence relation and includes* (β) *and* (β_D)*. A differential λ-theory* $\mathcal{T}$ *is called* differentially extensional *if it contains* (η_∂) *and* extensional *if it also contains* (η)*. We say that* $\mathcal{T}$ satisfies sum idempotency *whenever* $\mathcal{T} \vdash s + s = s$*.*

The differential λ-theories are naturally ordered by set-theoretical inclusion. We denote by $\lambda\beta^d$ the minimum differential λ-theory, by $\lambda\beta\eta^d_\partial$ the minimum differentially extensional differential λ-theory, and by $\lambda\beta\eta^d$ the minimum extensional differential λ-theory.

We present here some easy examples of equalities between differential λ-terms in $\lambda\beta^d, \lambda\beta\eta^d_\partial$ and $\lambda\beta\eta^d$ in order to help the reader to get familiar with the operations in the calculus.

Example 2.12 *Recall that* $\Delta \equiv \lambda x.xx$*. Then we have:*

1. $\lambda\beta^d \vdash (\mathsf{D}\,\Delta \cdot y)z = yz + (\mathsf{D}\, z \cdot y)z,$
2. $\lambda\beta^d \vdash (\mathsf{D}^2\,\Delta \cdot (x,y))0 = (\mathsf{D}\, x \cdot y)0 + (\mathsf{D}\, y \cdot x)0,$
3. $\lambda\beta^d \vdash \mathsf{D}^3\,\Delta \cdot (x,y,z) = \lambda r.(\mathsf{D}^2\, x \cdot (y,z) + \mathsf{D}^2\, y \cdot (x,z) + \mathsf{D}^2\, z \cdot (x,y) + \mathsf{D}^3\, r \cdot (x,y,z))r,$
4. $\lambda\beta\eta^d_\partial \vdash \mathsf{D}\,(\lambda z.xz) \cdot y = \lambda z.(\mathsf{D}\, x \cdot y)z = \mathsf{D}\, x \cdot y,$
5. $\lambda\beta\eta^d \vdash \mathsf{D}\,\Delta \cdot z = \lambda x.zx + \lambda x.(\mathsf{D}\, x \cdot z)x = z + \lambda x.(\mathsf{D}\, x \cdot z)x.$

Note that in this calculus (as in the usual λ-calculus extended with non-deterministic choice [18]) a single simple term can generate an infinite sum of terms, like in the example below.

Example 2.13 *Recall (from page 7) that* $\mathbf{Y}$ *is Curry's fixpoint combinator,* $\underline{n}$ *is the* n*-th Church numeral and* $\mathbf{s}$ *denotes the successor.*

1. $\lambda\beta^d \vdash \mathbf{Y}(x+y) = x(\mathbf{Y}(x+y)) + y(\mathbf{Y}(x+y))$ *for all variables* x, y,
2. $\lambda\beta^d \vdash \mathbf{Y}((\lambda z.\underline{0}) + \mathbf{s}) = \underline{0}\ +\ \mathbf{s}(\mathbf{Y}((\lambda z.\underline{0}) + \mathbf{s})) = \underline{0}\ +\ \underline{1}\ +\ \mathbf{s}(\mathbf{s}(\mathbf{Y}((\lambda z.\underline{0}) + \mathbf{s}))) = \cdots$

2.4. A Theory of Taylor Expansion

One of the most interesting consequences of adding a syntactical differential operator to the λ-calculus is that, in presence of infinite sums, this allows to define the Taylor expansion of a program. Such an expansion is classically defined in the literature only for ordinary λ-terms [21,22,24]. In this subsection we generalize this notion to differential λ-terms. To avoid the annoying problem of handling coefficients we consider an idempotent sum.

Definition 2.14 *Given a differential* λ*-term* S *we define its* (full) Taylor expansion S^* *by induction on* S *as follows:*

- $x^* = x$,
- $(\lambda x.s)^* = \lambda x.s^*$,
- $(\mathsf{D}^k s \cdot (t_1, \dots, t_k))^* = \mathsf{D}^k s^* \cdot (t_1^*, \dots, t_k^*)$,
- $(sT)^* = \Sigma_{k\in\mathbf{N}}(\mathsf{D}^k s^* \cdot (T^*, \dots, T^*))0$,
- $(s+T)^* = s^* + T^*$.

Thus, the "target language" of the Taylor expansion is much simpler than the full differential λ-calculus. For instance, the general application of the λ-calculus is not needed anymore, we will only need iterated linear applications and ordinary applications to 0. We will however need countable sums, that are not present in general in the differential λ-calculus. Hereafter, the target calculus of the Taylor expansion will be denoted by Λ^d_∞.

We will write $\vec{S}$ to denote sequences of differential λ-terms $S_1, \dots, S_k$ (with $k \geq 0$).

Remark 2.15 *Every term* $S \in \Lambda^d_\infty$ *can be written as a (possibly infinite) sum of terms of shape:*

$$\lambda\vec{y}.(\mathsf{D}^{n_1}(\cdots(\mathsf{D}^{n_k} s \cdot (\vec{t}_k))\vec{0}) \cdots \cdot (\vec{t}_1))\vec{0}$$

where $\vec{t}_i$ *is a sequence of simple terms of length* $n_i \in \mathbf{N}$ *(for* $1 \leq i \leq k$*) and the simple term* s *is either a variable or a lambda abstraction.*

We now try to clarify what does it mean that two differential λ-terms S and T "have the same Taylor expansion". Indeed we may have that $S^* = \Sigma_{i\in I} s_i$ and $T^* = \Sigma_{j\in J} t_j$ where I, J are countable sets. In this case one could be tempted to define $S^* = T^*$ by asking for the existence of a bijective correspondence between I and J such that each s_i is $\lambda\beta^d$-equivalent to some t_j. However, in the general case, this definition does not capture the equivalence between infinite sums that we have in mind. For instance, $S^* = T^*$ might hold because there are partitions $\{I_k\}_{k\in K}$ and $\{J_k\}_{k\in K}$ of I and J, respectively, such that for every $k \in K$ the sets I_k, J_k are finite and $\Sigma_{i\in I_k} s_i =_{\lambda\beta^d} \Sigma_{i\in J_k} s_j$. The naïf definition works well when all summands of the two sums we are equating are 'in normal form'. Since the Λ^d_∞ calculus (morally) enjoys strongly normalization, we can define the normal form of every $S \in \Lambda^d_\infty$ as follows.

Definition 2.16 *Given $S \in \Lambda^d_\infty$, we define the* normal form of S *as follows.*

- *If* $S \equiv \sum_{i\in I} s_i$ *we set* $\mathrm{NF}(S) = \sum_{i\in I} \mathrm{NF}(s_i)$.
- *If* $S \equiv \lambda\vec{y}.(\mathsf{D}^{n_1}(\cdots(\mathsf{D}^{n_k} x \cdot (\vec{t}_k))\vec{0}) \cdots \cdot (\vec{t}_1))\vec{0}$ *then:*

$$\mathrm{NF}(S) = \lambda\vec{y}.(\mathsf{D}^{n_1}(\cdots(\mathsf{D}^{n_k} x \cdot (\mathrm{NF}(\vec{t}_k)))\vec{0}) \cdots \cdot (\mathrm{NF}(\vec{t}_1)))\vec{0}.$$

- *If* $S \equiv \lambda\vec{y}.(\mathsf{D}^{n_1}(\cdots(\mathsf{D}^{n_k}(\lambda x.s) \cdot (\vec{t}_k))\vec{0}) \cdots \cdot (\vec{t}_1))\vec{0}$ *with* $n_k > 0$ *then:*

$$\mathrm{NF}(S) = \mathrm{NF}\big(\lambda\vec{y}.(\mathsf{D}^{n_1}(\cdots(\mathsf{D}^{n_{k-1}}((\lambda x.\tfrac{\partial^{n_k} s}{\partial x,\ldots,x} \cdot (\vec{t}_k))\vec{0}) \cdot (\vec{t}_{k-1}))\vec{0}) \cdots \cdot (\vec{t}_1))\vec{0}\big).$$

- *If* $S \equiv \lambda\vec{y}.(\mathsf{D}^{n_1}(\cdots(\mathsf{D}^{n_k}((\lambda x.s)0\vec{0}) \cdot (\vec{t}_k))\vec{0}) \cdots \cdot (\vec{t}_1))\vec{0}$ *then:*

$$\mathrm{NF}(S) = \mathrm{NF}\big(\lambda\vec{y}.(\mathsf{D}^{n_1}(\cdots(\mathsf{D}^{n_k}((s\{0/x\})\vec{0}) \cdot (\vec{t}_k))\vec{0}) \cdots \cdot (\vec{t}_1))\vec{0}\big).$$

By Remark 2.15 the definition above covers all possible cases.

We are now able to define the differential λ-theory generated by equating all differential λ-terms having the same Taylor expansion.

Definition 2.17 *Given $S, T \in \Lambda^d$ we say that* $\mathrm{NF}(S^*) = \mathrm{NF}(T^*)$ *whenever* $\mathrm{NF}(S^*) = \sum_{i\in I} s_i$, $\mathrm{NF}(T^*) = \sum_{j\in J} t_j$ *and there is an isomorphism $\iota : I \to J$ such that $\lambda\beta^d \vdash s_i = t_{\iota(i)}$. We set*

$$\mathcal{E} = \{(S,T) \in \Lambda^d \times \Lambda^d \mid \mathrm{NF}(S^*) = \mathrm{NF}(T^*)\}.$$

It is not difficult to check that $\mathcal{E}$ is actually a differential λ-theory.

Two usual λ-terms s, t have the same Böhm tree [2, Ch. 10] if, and only if, $\mathcal{E} \vdash s = t$ holds. The 'if' part of this equivalence is fairly straightforward, whereas the 'only if' part is proved in [22]. Thus, the theory $\mathcal{E}$ can be seen as an extension of the theory of Böhm trees in the context of differential λ-calculus.

3. A Differential Model Theory

In this section we will provide the categorical framework in which the models of the differential λ-calculus live, namely, the *Cartesian closed differential categories*[3]. The material presented in Subsection 3.1 is mainly borrowed from [5].

3.1. Cartesian Differential Categories

Differential λ-terms will be interpreted as morphisms in a suitable category $\mathbf{C}$. Since in the syntax we have sums of terms, we need a sum on the morphisms of $\mathbf{C}$ satisfying the equations introduced in Notation 2.4. For this reason, we will focus our attention on left-additive categories.

A category $\mathbf{C}$ is *left-additive* whenever each homset has a structure of commutative monoid $(\mathbf{C}(A,B), +_{AB}, 0_{AB})$ and $(g+h)\circ f = (g\circ f) + (h\circ f)$ and $0 \circ f = 0$.

Definition 3.1 *A morphism f in $\mathbf{C}$ is said to be* additive *if, in addition, it satisfies $f\circ(g+h) = (f\circ g) + (f\circ h)$ and $f \circ 0 = 0$.*

[3] These categories have been first introduced in [12] (where they were called *differential λ-categories*) and proposed as models of the simply typed differential λ-calculus and simply typed resource calculus.

A category is *Cartesian left-additive* if it is a left-additive category with products such that all projections and pairings of additive maps are additive.

Definition 3.2 *A* Cartesian differential category *is a Cartesian left-additive category having an operator* $D(-)$ *that maps every morphism* $f : A \to B$ *into a morphism* $D(f) : A \times A \to B$ *and satisfies the following axioms:*

D1. $D(f + g) = D(f) + D(g)$ *and* $D(0) = 0$,

D2. $D(f) \circ \langle h + k, v\rangle = D(f) \circ \langle h, v\rangle + D(f) \circ \langle k, v\rangle$ *and* $D(f) \circ \langle 0, v\rangle = 0$,

D3. $D(\mathrm{Id}) = \pi_1$, $D(\pi_1) = \pi_1 \circ \pi_1$ *and* $D(\pi_2) = \pi_2 \circ \pi_1$,

D4. $D(\langle f, g\rangle) = \langle D(f), D(g)\rangle$,

D5. $D(f \circ g) = D(f) \circ \langle D(g), g \circ \pi_2\rangle$,

D6. $D(D(f)) \circ \langle\langle g, 0\rangle, \langle h, k\rangle\rangle = D(f) \circ \langle g, k\rangle$,

D7. $D(D(f)) \circ \langle\langle 0, h\rangle, \langle g, k\rangle\rangle = D(D(f)) \circ \langle\langle 0, g\rangle, \langle h, k\rangle\rangle$.

We try to provide some intuitions on these axioms. (D1) says that the operator $D(-)$ is linear; (D2) says that $D(-)$ is additive in its first coordinate; (D3) and (D4) ask that $D(-)$ behaves coherently with the product structure; (D5) is the usual chain rule; (D6) requires that $D(f)$ is linear in its first component. (D7) states the independence of the order of "partial differentiation".

Remark 3.3 *In a Cartesian differential category we obtain partial derivatives from the full ones by "zeroing out" the components on which the differentiation is not required. For example, suppose that we want to define the partial derivative* $D_1(f)$ *of* $f : C \times A \to B$ *on its first component; then, it is sufficient to set* $D_1(f) = D(f) \circ (\langle \mathrm{Id}_C, 0_A\rangle \times \mathrm{Id}_{C\times A}) : C \times (C \times A) \to B$.

Similarly, we define $D_2(f) = D(f) \circ (\langle 0_C, \mathrm{Id}_A\rangle \times \mathrm{Id}_{C\times A}) : A \times (C \times A) \to B$, *the partial derivative of* f *on its second component.*

This remark follows since every differential $D(f)$ can be reconstructed from its partial derivatives as follows:

$$\begin{aligned} D(f) = &\ D(f) \circ \langle\langle \pi_1 \circ \pi_1, \pi_2 \circ \pi_1\rangle, \pi_2\rangle \\ = &\ D(f) \circ \langle\langle \pi_1 \circ \pi_1, 0\rangle, \pi_2\rangle + D(f) \circ \langle\langle 0, \pi_2 \circ \pi_1\rangle, \pi_2\rangle \\ = &\ D(f) \circ (\langle \mathrm{Id}, 0\rangle \times \mathrm{Id}) \circ (\pi_1 \times \mathrm{Id}) + D(f) \circ (\langle 0, \mathrm{Id}\rangle \times \mathrm{Id}) \circ (\pi_2 \times \mathrm{Id}) \\ = &\ D_1(f) \circ (\pi_1 \times \mathrm{Id}) + D_2(f) \circ (\pi_2 \times \mathrm{Id}). \end{aligned}$$

3.2. Linear Morphisms

In Cartesian differential categories we are able to express the fact that a morphism is 'linear' by asking that its differential is constant.

Definition 3.4 *In a Cartesian differential category, a morphism* $f : A \to B$ *is called* linear *if* $D(f) = f \circ \pi_1$.

Lemma 3.5 *Every linear morphism* $f : A \to B$ *is additive.*

Proof. By definition of linear morphism we have $D(f) = f \circ \pi_1$. For all $g, h : C \to A$ we have

$$f \circ (g + h) = f \circ \pi_1 \circ \langle g + h, g\rangle = D(f) \circ \langle g + h, g\rangle = \\ D(f) \circ \langle g, g\rangle + D(f) \circ \langle h, g\rangle = f \circ \pi_1 \circ \langle g, g\rangle + f \circ \pi_1 \circ \langle h, g\rangle = f \circ g + f \circ h$$

Moreover $f \circ 0 = f \circ \pi_1 \circ \langle 0, 0\rangle = D(f) \circ \langle 0, 0\rangle = 0$. We conclude that f is additive. ■

Lemma 3.6 *The composition of two linear morphisms is linear.*

Proof. Let f, g be two linear maps. We have to prove that $D(f \circ g) = f \circ g \circ \pi_1$. By *(D5)* we have $D(f \circ g) = D(f) \circ \langle D(g), g \circ \pi_2\rangle$. Since f, g are linear we have $D(f) \circ \langle D(g), g \circ \pi_2\rangle = f \circ \pi_1 \circ \langle g \circ \pi_1, g \circ \pi_2\rangle = f \circ g \circ \pi_1$. ■

Thus, in fact, every Cartesian differential category has a subcategory of linear maps.

3.3. Cartesian Closed Differential Categories

Cartesian differential categories are not enough to interpret the differential λ-calculus, since the differential operator does not behave automatically well with respect to the Cartesian closed structure. For this reason we now introduce the notion of *Cartesian closed differential category.*

Definition 3.7 *A category is* Cartesian closed left-additive *if it is a Cartesian left-additive category which is Cartesian closed and satisfies:*

$$(+\text{-curry}) \quad \Lambda(f + g) = \Lambda(f) + \Lambda(g) \qquad\qquad \Lambda(0) = 0 \quad (0\text{-curry})$$

From these properties of $\Lambda(-)$ we can easily prove that the evaluation morphism is additive in its left component.

Lemma 3.8 *In every Cartesian closed left-additive category the following axioms hold (for all $f, g : C \to (A \Rightarrow B)$ and $h : C \to A$):*

$$(+\text{-eval}) \quad \mathrm{ev} \circ \langle f + g, h\rangle = \mathrm{ev} \circ \langle f, h\rangle + \mathrm{ev} \circ \langle g, h\rangle \qquad\qquad \mathrm{ev} \circ \langle 0, h\rangle = 0 \quad (0\text{-eval})$$

Proof. Let $f' = \Lambda^-(f)$ and $g' = \Lambda^-(g)$. Then we have:

$$\begin{array}{lll} \mathrm{ev} \circ \langle f + g, h\rangle & = \mathrm{ev} \circ ((\Lambda(f') + \Lambda(g')) \times \mathrm{Id}) \circ \langle \mathrm{Id}, h\rangle & \text{by def. of } f', g' \\ & = \Lambda^-((\Lambda(f') + \Lambda(g')) \circ \langle \mathrm{Id}, h\rangle & \text{by def. of } \Lambda^- \\ & = \Lambda^-(\Lambda(f' + g')) \circ \langle \mathrm{Id}, h\rangle & \text{by (+-curry)} \\ & = (f' + g') \circ \langle \mathrm{Id}, h\rangle & \text{by def. of } \Lambda^- \\ & = f' \circ \langle \mathrm{Id}, h\rangle + g' \circ \langle \mathrm{Id}, h\rangle & \text{by left-additivity} \\ & = \Lambda^-(f) \circ \langle \mathrm{Id}, h\rangle + \Lambda^-(g) \circ \langle \mathrm{Id}, h\rangle & \text{by def. of } f', g' \\ & = \mathrm{ev} \circ (f \times \mathrm{Id}) \circ \langle \mathrm{Id}, h\rangle + \mathrm{ev} \circ (g \times \mathrm{Id}) \circ \langle \mathrm{Id}, h\rangle & \text{by def. of } \Lambda^- \\ & = \mathrm{ev} \circ \langle f, h\rangle + \mathrm{ev} \circ \langle g, h\rangle & \end{array}$$

Moreover $\mathrm{ev} \circ \langle 0, g\rangle = \mathrm{ev} \circ \langle \Lambda(0), g\rangle = 0 \circ \langle \mathrm{Id}, g\rangle = 0$. ■

Definition 3.9 *A* Cartesian closed differential category *is a Cartesian differential category which is Cartesian closed left-additive and such that, for all $f : C \times A \to B$:*

$$(\text{D-curry}) \quad D(\Lambda(f)) = \Lambda(D(f) \circ \langle \pi_1 \times 0_A, \pi_2 \times \mathrm{Id}_A\rangle).$$

Indeed, in a Cartesian closed differential category we have two ways to derivate $f : C \times A \to B$ in its first component: we can use the trick of Remark 3.3, or we can 'hide' the component A by currying f and then derive $\Lambda(f)$. Intuitively, (D-curry) requires that these two methods are equivalent.

Lemma 3.10 *In every Cartesian closed differential category the following axiom holds (for all $h : C \to (A \Rightarrow B)$ and $g : C \to A$):*

$$\text{(D-eval)}\ D(\mathrm{ev} \circ \langle h, g\rangle) = \mathrm{ev} \circ \langle D(h), g \circ \pi_2\rangle + D(\Lambda^-(h)) \circ \langle\langle 0_C, D(g)\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle$$

Proof. Let $h' = \Lambda^-(h) : C \times A \to B$. Then we have:

$$\begin{array}{ll}
D(\mathrm{ev} \circ \langle h, g\rangle) = & \text{by def. of } h' \\
D(\mathrm{ev} \circ \langle \Lambda(h'), g\rangle) = & \text{by (beta-cat)} \\
D(h' \circ \langle \mathrm{Id}_C, g\rangle) = & \text{by (D5)} \\
D(h') \circ \langle D(\langle \mathrm{Id}_C, g\rangle), \langle \mathrm{Id}_C, g\rangle \circ \pi_2\rangle = & \text{by (D4) and (D3)} \\
D(h') \circ \langle\langle \pi_1, D(g)\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle = & \text{since pairing is additive} \\
D(h') \circ \langle\langle \pi_1, 0_A\rangle + \langle 0_C, D(g)\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle = & \text{by (D2)} \\
D(h') \circ \langle\langle \pi_1, 0_A\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle + D(h') \circ \langle\langle 0_C, D(g)\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle = & \\
D(h') \circ \langle \pi_1 \times 0_A, \pi_2 \times \mathrm{Id}_A\rangle \circ \langle \mathrm{Id}_{C\times C}, g \circ \pi_2\rangle & \\
\quad + D(h') \circ \langle\langle 0_C, D(g)\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle = & \text{by (beta-cat)} \\
\mathrm{ev} \circ \langle \Lambda(D(h') \circ \langle \pi_1 \times 0_A, \pi_2 \times \mathrm{Id}_A\rangle), g \circ \pi_2\rangle & \\
\quad + D(h') \circ \langle\langle 0_C, D(g)\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle = & \text{by (D-curry)} \\
\mathrm{ev} \circ \langle D(\Lambda(h')), g \circ \pi_2\rangle + D(\Lambda^-(\Lambda(h'))) \circ \langle\langle 0_C, D(g)\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle = & \text{by def. of } h' \\
\mathrm{ev} \circ \langle D(h), g \circ \pi_2\rangle + D(\Lambda^-(h)) \circ \langle\langle 0_C, D(g)\rangle, \langle \pi_2, g \circ \pi_2\rangle\rangle &
\end{array}$$

■

The axiom *(D-eval)* can be seen as a chain rule for denotations of differential λ-terms (cf. Lemma 3.18(i), below).

In Cartesian closed differential categories we are able to define a binary operator $\star$ on morphisms, that can be seen as the semantic counterpart of differential substitution. The idea behind $f \star g$ is to derive the map $f : A \to B$ and then apply the argument $g : A$ in its linear component. However differential λ-terms are interpreted *in a certain context*, thus we need to handle the context C and consider maps $f : C \times A \to A$ and $g : C \to A$.

Definition 3.11 *The operator*

$$\frac{f : C \times A \to B \quad g : C \to A}{f \star g : C \times A \to B}\ (\star)$$

is defined by $f \star g = D(f) \circ \langle\langle 0_C^{C\times A}, g \circ \pi_1\rangle, \mathrm{Id}_{C\times A}\rangle$.

The morphism $f \star g$ is obtained by differentiating f in its second component (partial differentiation), and applying g in that component. The precise correspondence between $\star$ and the differential substitution is given in Theorem 4.10.

Remark 3.12 *Actually the operators $D(-)$ and $\star$ are mutually definable. To define $D(-)$ in terms of $\star$ just set $D(f) = (f \circ \pi_2) \star \mathrm{Id}$. To check that this definition is meaningful we show that it holds in every Cartesian differential category: indeed, by Definition 3.11, $(f \circ \pi_2) \star \mathrm{Id} = D(f \circ \pi_2) \circ \langle\langle 0, \pi_1\rangle, \mathrm{Id}\rangle = D(f) \circ \langle \pi_2 \circ \pi_1, \pi_2 \circ \pi_2\rangle \circ \langle\langle 0, \pi_1\rangle, \mathrm{Id}\rangle = D(f)$. Thus it would be possible* *to formulate the whole theory of Cartesian closed differential categories by axiomatizing the behaviour of $\star$ instead of that of $D(-)$. In this work we prefer to use $D(-)$ because it is a more basic operation, already studied in the literature, and the complexities of the two approaches are comparable.*

It is possible to characterize linear morphisms in terms of the operator $\star$ as follows.

Lemma 3.13 *A morphism $f : A \to B$ is linear iff for all $g : C \to A$:*

$$(f \circ \pi_2) \star g = (f \circ g) \circ \pi_1 : C \times A \to B$$

Proof. $(\Rightarrow)$ Suppose that f is linear. By definition of $\star$ we have that $(f \circ \pi_2) \star g = D(f \circ \pi_2) \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{C\times A}\rangle$. By applying *(D5)* and *(D3)*, this is equal to $D(f) \circ \langle \pi_2 \circ \pi_1, \pi_2 \circ \pi_2\rangle \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{C\times A}\rangle = D(f) \circ \langle g \circ \pi_1, \pi_2\rangle$. Since f is linear we have $D(f) = f \circ \pi_1$, thus $D(f) \circ \langle g \circ \pi_1, \pi_2\rangle = f \circ g \circ \pi_1$.

$(\Leftarrow)$ Suppose $(f \circ \pi_2) \star g = (f \circ g) \circ \pi_1$ for all $g : C \to A$. In particular, this is true for $C = A$ and $g = \mathrm{Id}_A$. Thus we have $(f \circ \pi_2) \star \mathrm{Id}_A = f \circ \pi_1$. We conclude since:

$$\begin{aligned}
(f \circ \pi_2) \star \mathrm{Id}_A = \;& D(f \circ \pi_2) \circ \langle\langle 0_A, \pi_1\rangle, \mathrm{Id}_{A\times A}\rangle && \text{by def. of } \star\\
=\;& D(f) \circ \langle \pi_2 \circ \pi_1, \pi_2 \circ \pi_2\rangle \circ \langle\langle 0_A, \pi_1\rangle, \mathrm{Id}_{A\times A}\rangle && \text{by (D5)+(D3)}\\
=\;& D(f) \circ \langle \pi_1, \pi_2\rangle = D(f)
\end{aligned}$$

■

The operator $\star$ enjoys the following commutation property.

Lemma 3.14 *Let $f : C \times A \to B$ and $g, h : C \to A$. Then $(f \star g) \star h = (f \star h) \star g$.*

Proof. We set $\varphi_g = \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{C\times A}\rangle$ and $\varphi_h = \langle\langle 0_C, h \circ \pi_1\rangle, \mathrm{Id}_{C\times A}\rangle$. We have:

$$\begin{array}{ll}
(f \star g) \star h = D(D(f) \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{C\times A}\rangle) \circ \varphi_h = & \text{by (D5)}\\
D(D(f)) \circ \langle D(\langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}\rangle), \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}\rangle \circ \pi_2\rangle \circ \varphi_h = & \text{by (D4)}\\
D(D(f)) \circ \langle\langle\langle 0_C, D(g \circ \pi_1)\rangle, \pi_1\rangle, \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}\rangle \circ \pi_2\rangle \circ \varphi_h = & \text{by (D5)}\\
D(D(f)) \circ \langle\langle\langle 0_C, D(g) \circ \langle \pi_1 \circ \pi_1, \pi_1 \circ \pi_2\rangle\rangle, \pi_1\rangle, \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}\rangle \circ \pi_2\rangle \circ \varphi_h = & \\
D(D(f)) \circ \langle\langle\langle 0_C, D(g) \circ \langle 0_C, \pi_1\rangle\rangle, \langle 0_C, h \circ \pi_1\rangle\rangle, \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}\rangle\rangle = & \text{by (D2)}\\
D(D(f)) \circ \langle\langle 0_{C\times A}, \langle 0_C, h \circ \pi_1\rangle\rangle, \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}\rangle\rangle = & \text{by (D7)}\\
D(D(f)) \circ \langle\langle 0_{C\times A}, \langle 0_C, g \circ \pi_1\rangle\rangle, \langle\langle 0_C, h \circ \pi_1\rangle, \mathrm{Id}\rangle\rangle = & \text{by (D2)}\\
D(D(f)) \circ \langle\langle\langle 0_C, D(h) \circ \langle 0_C, \pi_1\rangle\rangle, \langle 0_C, g \circ \pi_1\rangle\rangle, \langle\langle 0_C, h \circ \pi_1\rangle, \mathrm{Id}\rangle\rangle = & \\
D(D(f)) \circ \langle\langle\langle 0_C, D(h) \circ \langle \pi_1 \circ \pi_1, \pi_1 \circ \pi_2\rangle\rangle, \pi_1\rangle, \langle\langle 0_C, h \circ \pi_1\rangle, \mathrm{Id}\rangle \circ \pi_2\rangle \circ \varphi_g = & \text{by (D5)}\\
D(D(f)) \circ \langle\langle\langle 0_C, D(h \circ \pi_1)\rangle, \pi_1\rangle, \langle\langle 0_C, h \circ \pi_1\rangle, \mathrm{Id}\rangle \circ \pi_2\rangle \circ \varphi_g = & \text{by (D4)}\\
D(D(f)) \circ \langle D(\langle\langle 0_C, h \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle), \langle\langle 0_C, h \circ \pi_1\rangle, \mathrm{Id}\rangle \circ \pi_2\rangle \circ \varphi_g = & \text{by (D5)}\\
D(D(f) \circ \langle\langle 0_C, h \circ \pi_1\rangle, \mathrm{Id}\rangle) \circ \varphi_g = (f \star h) \star g &
\end{array}$$

■

Definition 3.15 *Let* $\mathrm{sw}_{ABC} = \langle\langle \pi_1 \circ \pi_1, \pi_2\rangle, \pi_2 \circ \pi_1\rangle : (A \times B) \times C \to (A \times C) \times B$.

Remark 3.16 $\mathrm{sw} \circ \mathrm{sw} = \mathrm{Id}_{(A\times B)\times C}$, $\mathrm{sw} \circ \langle\langle f, g\rangle, h\rangle = \langle\langle f, h\rangle, g\rangle$ *and* $D(\mathrm{sw}) = \mathrm{sw} \circ \pi_1$.

The following two technical lemmas will be used in Subsection 4.3 to show the soundness of the categorical models of the differential λ-calculus. The interested reader can find the whole proofs in the technical Appendix A.

Lemma 3.17 *Let* $f : (C \times A) \times D \to B$ *and* $g : C \to A$, $h : C \to B'$. *Then:*

(i) $\pi_2 \star g = g \circ \pi_1$,

(ii) $(h \circ \pi_1) \star g = 0$,

(iii) $\Lambda(f) \star g = \Lambda(((f \circ \mathrm{sw}) \star (g \circ \pi_1)) \circ \mathrm{sw})$.

Proof. (Outline) (i) follows by applying *(D3)*. (ii) follows by applying *(D2)*, *(D3)* and *(D5)*. (iii) follows by *(Curry)*, *(D-curry)* and *(D2)*, *(D3)*, *(D5)*. ■

Lemma 3.18 *Let* $f : C \times A \to (D \Rightarrow B)$ *and* $g : C \to A$, $h : C \times A \to D$. *Then:*

(i) $(\mathrm{ev} \circ \langle f, h\rangle) \star g = \mathrm{ev} \circ \langle f \star g + \Lambda(\Lambda^-(f) \star (h \star g)), h\rangle$,

(ii) $\Lambda(\Lambda^-(f) \star h) \star g = \Lambda(\Lambda^-(f \star g) \star h) + \Lambda(\Lambda^-(f) \star (h \star g))$,

(iii) $\Lambda(\Lambda^-(f) \star h) \circ \langle \mathrm{Id}_C, g\rangle = \Lambda(\Lambda^-(f \circ \langle \mathrm{Id}_C, g\rangle) \star (h \circ \langle \mathrm{Id}_C, g\rangle))$.

Proof. (Outline) *(i)* follows by applying *(D-eval)* and *(beta-cat)*.

(ii) This equation can be simplified by using the axioms of Cartesian closed left-additive categories. Indeed, the right side can be written as $\Lambda((\Lambda^-(f \star g) \star h) + \Lambda^-(f) \star (h \star g))$. By taking a morphism f' such that $f = \Lambda(f')$ and by applying Lemma 3.17(iii) the item (ii) becomes equivalent to $((f' \star h) \circ \mathrm{sw}) \star (g \circ \pi_1) \circ \mathrm{sw} = (((f' \circ \mathrm{sw}) \star (g \circ \pi_1)) \circ \mathrm{sw}) \star h + f' \star (h \star g)$. This follows by *(Curry)* and *(D2-7)*.

(iii) follows by *(Curry)* and *(D2-5)*. ■

4. Categorical Models of the Differential Lambda Calculus

In [12] we have proved that Cartesian closed differential categories constitute *sound* models of the simply typed differential λ-calculus. In this section we will show that all reflexive objects living in these categories and satisfying a linearity condition are sound models of the *untyped* version of this calculus.

4.1. Linear Reflexive Objects in Cartesian Closed Differential Categories

In a category $\mathbf{C}$, an object A is a *retract* of an object B, written $A \lhd B$, if there are morphisms $f : A \to B$ and $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$. When also $f \circ g = \mathrm{Id}_B$ holds we say that A and B are *isomorphic*, written $A \cong B$, and that f, g are *isomorphisms*.

In a Cartesian closed category $\mathbf{C}$ a *reflexive object* $\mathscr{U}$ ought to mean a triple $(U, \mathcal{A}, \lambda)$ where U is an object of $\mathbf{C}$ and $\mathcal{A} : U \to (U \Rightarrow U)$ and $\lambda : (U \Rightarrow U) \to U$ are two morphisms performing the retraction $(U \Rightarrow U) \lhd U$. When $(U \Rightarrow U) \cong U$ we say that $\mathscr{U}$ is *extensional*.

Definition 4.1 *A reflexive object* $\mathscr{U} = (U, \mathcal{A}, \lambda)$ *in a Cartesian closed differential category is* linear *if both* $\mathcal{A}$ *and* λ *are linear morphisms.*

We are now able to provide our definition of model of the untyped differential λ-calculus.

Definition 4.2 *A* categorical model $\mathscr{U}$ of the differential λ-calculus *is a linear reflexive object in a Cartesian closed differential category. The model* $\mathscr{U}$ *is called* differentially extensional *(resp.* extensional*) if its equational theory is.*

It is routine to check that $\mathscr{U}$ is an extensional model (i.e., $\mathrm{Th}(\mathscr{U}) \supseteq \lambda\beta\eta^d$) if and only if it is extensional as a reflexive object (i.e., $(U \Rightarrow U) \cong U$).

The following lemma is useful for proving that a reflexive object in a Cartesian closed differential category is linear.

Lemma 4.3 *Let $\mathscr{U}$ be a reflexive object.*

(i) If $\mathcal{A}$ and $\lambda \circ \mathcal{A}$ are linear then $\mathscr{U}$ is linear.

(ii) If $\mathscr{U}$ is extensional and either $\mathcal{A}$ or λ is linear then $\mathscr{U}$ is linear.

Proof. (i) Suppose $\mathcal{A}$ and $\lambda\circ\mathcal{A}$ are linear morphisms. We now show that also λ is linear. Indeed we have:

$$\begin{array}{rll} D(\lambda) = & D(\lambda)\circ(\mathcal{A}\times\mathcal{A})\circ(\lambda\times\lambda) = D(\lambda)\circ\langle\mathcal{A}\circ\pi_1,\mathcal{A}\circ\pi_2\rangle\circ(\lambda\times\lambda) = & \text{by } \mathcal{A} \text{ linear} \\ = & D(\lambda)\circ\langle D(\mathcal{A}),\mathcal{A}\circ\pi_2\rangle\circ(\lambda\times\lambda) = D(\lambda\circ\mathcal{A})\circ(\lambda\times\lambda) = & \text{by } \lambda\circ\mathcal{A} \text{ linear} \\ = & \lambda\circ\mathcal{A}\circ\pi_1\circ\langle\lambda\circ\pi_1,\lambda\circ\pi_2\rangle = \lambda\circ\mathcal{A}\circ\lambda\circ\pi_1 = \lambda\circ\pi_1. & \end{array}$$

(ii) If $\mathcal{A}$ is linear then it follows directly from (i) since $\lambda\circ\mathcal{A} = \mathrm{Id}_U$ and the identity is linear. If λ is linear, calculations analogous to those made in (i) show that also $\mathcal{A}$ is. ■

Notice that, in general, there may be extensional reflexive objects that are not linear. However, in the concrete example of Cartesian closed differential category we will provide in Section 5 every extensional reflexive object will be linear (see Corollary 5.6).

Lemma 4.4 *Let $\mathscr{U}$ be a linear reflexive object and let $f : U^{n+1} \to (U \Rightarrow U)$, $h : U^{n+1} \to U$ $g : U^n \to U$. Then:*

(i) $\lambda\circ(f\star g) = (\lambda\circ f)\star g$,

(ii) $\mathcal{A}\circ(h\star g) = (\mathcal{A}\circ h)\star g$.

Proof. (i) By definition of $\star$ we have $(\lambda\circ f)\star g = D(\lambda\circ f)\circ\langle\langle 0_{U^n}, g\circ\pi_1\rangle,\mathrm{Id}_{U^{n+1}}\rangle$. By *(D5)* we have $D(\lambda\circ f) = D(\lambda)\circ\langle D(f), f\circ\pi_2\rangle$. Since λ is linear we have $D(\lambda) = \lambda\circ\pi_1$, thus $D(\lambda)\circ\langle D(f), f\circ\pi_2\rangle = \lambda\circ\pi_1\circ\langle D(f), f\circ\pi_2\rangle = \lambda\circ D(f)$. Hence, $D(\lambda\circ f)\circ\langle\langle 0_{U^n}, g\circ\pi_1\rangle,\mathrm{Id}_{U^{n+1}}\rangle = \lambda\circ D(f)\circ\langle\langle 0_{U^n}, g\circ\pi_1\rangle,\mathrm{Id}_{U^{n+1}}\rangle = \lambda\circ(f\star g)$.

(ii) Analogous to (i). ■

4.2. Defining the Interpretation

Let $\vec{x} = x_1,\dots,x_n$ be an ordered sequence of variables without repetitions. We say that $\vec{x}$ is *adequate* for $S_1,\dots,S_k \in \Lambda^d$ if $\mathrm{FV}(S_1,\dots,S_k) \subseteq \{x_1,\dots,x_n\}$. Given an object U we write $U^{\vec{x}}$ for the $\{x_1,\dots,x_n\}$-indexed categorical product of n copies of U (when $n = 0$ we consider $U^{\vec{x}} = \mathbb{1}$). Moreover, we define the i-th projection $\pi_i^{\vec{x}} : U^{\vec{x}} \to U$ by

$$\pi_i^{\vec{x}} = \begin{cases} \pi_2 & \text{if } i = n, \\ \pi_i^{x_1,\dots,x_{n-1}}\circ\pi_1 & \text{otherwise.} \end{cases}$$

Definition 4.5 *Let $\mathscr{U}$ be a categorical model, S be a differential λ-term and $\vec{x} = x_1,\dots,x_n$ be adequate for S. The* interpretation of S in $\mathscr{U}$ (with respect to $\vec{x}$) *will be a morphism $[\![S]\!]_{\vec{x}} : U^{\vec{x}} \to U$ defined by induction as follows:*

- $[\![x_i]\!]_{\vec{x}} = \pi_i^{\vec{x}}$,

- $[\![sT]\!]_{\vec{x}} = \mathrm{ev} \circ \langle \mathcal{A} \circ [\![s]\!]_{\vec{x}}, [\![T]\!]_{\vec{x}} \rangle$,
- $[\![\lambda z.s]\!]_{\vec{x}} = \lambda \circ \Lambda([\![s]\!]_{\vec{x},z})$, *where by α-conversion we suppose that z does not occur in $\vec{x}$,*
- $[\![\mathsf{D}^1 s \cdot (t)]\!]_{\vec{x}} = \lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![s]\!]_{\vec{x}}) \star [\![t]\!]_{\vec{x}})$,
- $[\![\mathsf{D}^{n+1} s \cdot (t_1, \dots, t_n, t_{n+1})]\!]_{\vec{x}} = \lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![\mathsf{D}^n s \cdot (t_1, \dots, t_n)]\!]_{\vec{x}}) \star [\![t_{n+1}]\!]_{\vec{x}})$,
- $[\![0]\!]_{\vec{x}} = 0_U^{U^{\vec{x}}}$,
- $[\![s + S]\!]_{\vec{x}} = [\![s]\!]_{\vec{x}} + [\![S]\!]_{\vec{x}}$.

Remark 4.6 *Easy calculations give*

$$[\![\mathsf{D}^n s \cdot (t_1, \dots, t_n)]\!]_{\vec{x}} = \lambda \circ \Lambda((\cdots(\Lambda^-(\mathcal{A} \circ [\![s]\!]_{\vec{x}}) \star [\![t_1]\!]_{\vec{x}}) \cdots) \star [\![t_n]\!]_{\vec{x}}).$$

Lemma 3.14 entails that this interpretation does not depend on the chosen representative of the permutative equivalence class. In other words, we have $[\![\mathsf{D}^n s \cdot (t_1, \dots, t_n)]\!]_{\vec{x}} = [\![\mathsf{D}^n s \cdot (t_{\sigma(1)}, \dots, t_{\sigma(n)})]\!]_{\vec{x}}$ *for every permutation* $\sigma \in \mathfrak{S}_n$*.*

4.3. Soundness

Given a categorical model $\mathscr{U}$ we can define the *equational theory of* $\mathscr{U}$ as follows:

$$\mathrm{Th}(\mathscr{U}) = \{S = T \mid [\![S]\!]_{\vec{x}} = [\![T]\!]_{\vec{x}} \text{ for some } \vec{x} \text{ adequate for } S, T\}.$$

The aim of this section is to prove that the interpretation we have defined is *sound*, i.e., that $\mathrm{Th}(\mathscr{U})$ is a differential λ-theory for every model $\mathscr{U}$.

The following convention allows us to lighten the statements of our theorems.

Convention 4.7 *Hereafter, and until the end of the section, we consider a fixed (but arbitrary) linear reflexive object $\mathscr{U}$ living in a Cartesian closed differential category* **C**. *Moreover, whenever we write $[\![S]\!]_{\vec{x}}$, we suppose that $\vec{x}$ is an adequate sequence for S.*

The proof of the next lemma is easy, and it is left to the reader. Recall that the morphism sw has been introduced in Definition 3.15.

Lemma 4.8 *Let $S \in \Lambda^d$.*

(i) If $z \notin \mathrm{FV}(S)$ then $[\![S]\!]_{\vec{x};z} = [\![S]\!]_{\vec{x}} \circ \pi_1$, where z does not occur in $\vec{x}$,

(ii) $[\![S]\!]_{\vec{x};y;z} = [\![S]\!]_{\vec{x};z;y} \circ \mathrm{sw}$, where z and y do not occur in $\vec{x}$.

Theorem 4.9 *(Classic Substitution Theorem) Let $S, T \in \Lambda^d$, $\vec{x} = x_1, \dots, x_n$ and y not occurring in $\vec{x}$. Then:*

$$[\![S\{T/y\}]\!]_{\vec{x}} = [\![S]\!]_{\vec{x};y} \circ \langle \mathrm{Id}, [\![T]\!]_{\vec{x}} \rangle.$$

Proof. By induction on S. The only interesting case is $S \equiv \mathsf{D}^n s \cdot (u_1, \dots, u_n)$: we treat it by cases on n.

Case $n = 1$. By definition of substitution we have $[\![(\mathsf{D} s \cdot u_1)\{T/y\}]\!]_{\vec{x}} = [\![\mathsf{D} s\{T/y\} \cdot u_1\{T/y\}]\!]_{\vec{x}}$. By definition of $[\![-]\!]$ this is equal to $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![s\{T/y\}]\!]_{\vec{x}}) \star [\![u_1\{T/y\}]\!]_{\vec{x}})$. By induction hypothesis we get $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![s]\!]_{\vec{x};y} \circ \langle \mathrm{Id}, [\![T]\!]_{\vec{x}} \rangle) \star ([\![u_1]\!]_{\vec{x};y} \circ \langle \mathrm{Id}, [\![T]\!]_{\vec{x}} \rangle))$. By applying Lemma 3.18(*iii*) this is equal to $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![s]\!]_{\vec{x};y}) \star [\![u_1]\!]_{\vec{x};y}) \circ \langle \mathrm{Id}, [\![T]\!]_{\vec{x}} \rangle = [\![\mathsf{D} s \cdot u_1]\!]_{\vec{x};y} \circ \langle \mathrm{Id}, [\![T]\!]_{\vec{x}} \rangle$.

Case $n > 1$. By definition of substitution we have $[\![(\mathsf{D}^n s\cdot(u_1,\dots,u_n))\{T/y\}]\!]_{\vec{x}} = [\![(\mathsf{D}^n s\{T/y\}\cdot(u_1\{T/y\},\dots,u_n\{T/y\}))]\!]_{\vec{x}}$. By applying the definition of $[\![-]\!]$ this is equal to $\lambda\circ\Lambda(\Lambda^-(\mathcal{A}\circ[\![\mathsf{D}^{n-1} s\{T/y\}\cdot(u_1\{T/y\},\dots,u_{n-1}\{T/y\})]\!]_{\vec{x}})\star[\![u_n\{T/y\}]\!]_{\vec{x}})$. By definition of substitution this is $\lambda\circ\Lambda(\Lambda^-(\mathcal{A}\circ[\![(\mathsf{D}^{n-1} s\cdot(u_1,\dots,u_{n-1}))\{T/y\}]\!]_{\vec{x}})\star[\![u_n\{T/y\}]\!]_{\vec{x}})$. By applying the induction hypothesis twice we get $\lambda\circ\Lambda(\Lambda^-(\mathcal{A}\circ[\![(\mathsf{D}^{n-1} s\cdot(u_1,\dots,u_{n-1})]\!]_{\vec{x},y}\circ\langle\mathrm{Id},[\![T]\!]_{\vec{x}}\rangle)\star([\![u_n]\!]_{\vec{x},y}\circ\langle\mathrm{Id},[\![T]\!]_{\vec{x}}\rangle))$. By Lemma 3.18($iii$) this is equal to $\lambda\circ\Lambda(\Lambda^-(\mathcal{A}\circ[\![\mathsf{D}^{n-1} s\cdot(u_1,\dots,u_{n-1})]\!]_{\vec{x},y})\star[\![u_n]\!]_{\vec{x},y})\circ\langle\mathrm{Id},[\![T]\!]_{\vec{x}}\rangle = [\![(\mathsf{D}^n s\cdot(u_1,\dots,u_n))]\!]_{\vec{x};y}\circ\langle\mathrm{Id},[\![T]\!]_{\vec{x}}\rangle$. ■

Theorem 4.10 *(Differential Substitution Theorem) Let $S,T\in\Lambda^d$, $\vec{x}=x_1,\dots,x_n$ and y not occurring in $\vec{x}$. Then:*

$$[\![\tfrac{\partial S}{\partial y}\cdot T]\!]_{\vec{x};y} = [\![S]\!]_{\vec{x};y}\star[\![T]\!]_{\vec{x}}.$$

Proof. By structural induction on S.

- case $S\equiv y$. Then $[\![\frac{\partial y}{\partial y}\cdot T]\!]_{\vec{x},y} = [\![T]\!]_{\vec{x},y} = [\![T]\!]_{\vec{x}}\circ\pi_1 = \pi_2\star[\![T]\!]_{\vec{x}} = [\![y]\!]_{\vec{x},y}\star[\![T]\!]_{\vec{x}}$ by Lemma 3.17(i).

- case $S\equiv x_i\neq y$. Then $[\![\frac{\partial x_i}{\partial y}\cdot T]\!]_{\vec{x},y} = [\![0]\!]_{\vec{x},y} = 0$. By Lemma 3.17($ii$) we have $0 = ([\![x_i]\!]_{\vec{x}}\circ\pi_1)\star[\![T]\!]_{\vec{x}} = [\![x_i]\!]_{\vec{x},y}\star[\![T]\!]_{\vec{x}}$.

- case $S\equiv\lambda z.v$. By definition of differential substitution we have that $[\![\frac{\partial\lambda z.v}{\partial y}\cdot T]\!]_{\vec{x},y} = [\![\lambda z.\frac{\partial v}{\partial y}\cdot T]\!]_{\vec{x},y} = \lambda\circ\Lambda([\![\frac{\partial v}{\partial y}\cdot T]\!]_{\vec{x},y,z})$. Applying Lemma 4.8($ii$), this is equal to $\lambda\circ\Lambda([\![\frac{\partial v}{\partial y}\cdot T]\!]_{\vec{x},z,y}\circ\mathrm{sw})$. By induction hypothesis we obtain $\lambda\circ\Lambda(([\![v]\!]_{\vec{x},z,y}\star[\![T]\!]_{\vec{x},z})\circ\mathrm{sw})$. Supposing without loss of generality that $z\notin\mathrm{FV}(T)$ we have, by Lemma 4.8(i), $[\![T]\!]_{\vec{x},z} = [\![T]\!]_{\vec{x}}\circ\pi_1$. Thus, applying Lemma 3.17(iii), we have that

  $$\lambda\circ\Lambda(([\![v]\!]_{\vec{x},z,y}\star([\![T]\!]_{\vec{x}}\circ\pi_1))\circ\mathrm{sw}) = \lambda\circ(\Lambda([\![v]\!]_{\vec{x},z,y}\circ\mathrm{sw})\star[\![T]\!]_{\vec{x}})$$

  which is equal to $\lambda\circ(\Lambda([\![v]\!]_{\vec{x},y,z})\star[\![T]\!]_{\vec{x}})$ by Lemma 4.8(ii). Since $\mathscr{U}$ is linear, we can apply Lemma 4.4(i) and get $\lambda\circ(\Lambda([\![v]\!]_{\vec{x},y,z})\star[\![T]\!]_{\vec{x}}) = (\lambda\circ\Lambda([\![v]\!]_{\vec{x},y,z}))\star[\![T]\!]_{\vec{x}} = [\![\lambda z.v]\!]_{\vec{x},y}\star[\![T]\!]_{\vec{x}}$.

- case $S\equiv sU$. By definition of differential substitution we have that $[\![\frac{\partial sU}{\partial y}\cdot T]\!]_{\vec{x},y} = [\![(\frac{\partial s}{\partial y}\cdot T)U]\!]_{\vec{x},y} + [\![(\mathsf{D} s\cdot(\frac{\partial U}{\partial y}\cdot T))U]\!]_{\vec{x},y}$. Let us consider the two summands componentwise. On the one side we have $[\![(\frac{\partial s}{\partial y}\cdot T)U]\!]_{\vec{x},y} = \mathrm{ev}\circ\langle\mathcal{A}\circ[\![\frac{\partial s}{\partial y}\cdot T]\!]_{\vec{x},y},[\![U]\!]_{\vec{x},y}\rangle$ which is equal, by induction hypothesis, to $\mathrm{ev}\circ\langle\mathcal{A}\circ([\![s]\!]_{\vec{x},y}\star[\![T]\!]_{\vec{x}}),[\![U]\!]_{\vec{x},y}\rangle$. By Lemma 4.4($ii$) this is equal to $\mathrm{ev}\circ\langle(\mathcal{A}\circ[\![s]\!]_{\vec{x},y})\star[\![T]\!]_{\vec{x}},[\![U]\!]_{\vec{x},y}\rangle$.

  On the other side we have (using $\mathcal{A}\circ\lambda = \mathrm{Id}_{U\Rightarrow U}$):

  $$[\![(\mathsf{D} s\cdot(\tfrac{\partial U}{\partial y}\cdot T))U]\!]_{\vec{x},y} = \mathrm{ev}\circ\langle\Lambda(\Lambda^-(\mathcal{A}\circ[\![s]\!]_{\vec{x},y})\star[\![\tfrac{\partial U}{\partial y}\cdot T]\!]_{\vec{x},y}),[\![U]\!]_{\vec{x},y}\rangle,$$

  by induction hypothesis this is equal to

  $$\mathrm{ev}\circ\langle\Lambda(\Lambda^-(\mathcal{A}\circ[\![s]\!]_{\vec{x},y})\star([\![U]\!]_{\vec{x},y}\star[\![T]\!]_{\vec{x}})),[\![T]\!]_{\vec{x},y}\rangle.$$

  By applying Lemma 3.8 we can rewrite the sum of this two summands as follows:

  $$\mathrm{ev}\circ\langle(\mathcal{A}\circ[\![s]\!]_{\vec{x},y})\star[\![T]\!]_{\vec{x}} + \Lambda(\Lambda^-(\mathcal{A}\circ[\![s]\!]_{\vec{x},y})\star([\![U]\!]_{\vec{x},y}\star[\![T]\!]_{\vec{x}})),[\![U]\!]_{\vec{x},y}\rangle.$$

  By Lemma 3.18(i) this is $(\mathrm{ev}\circ\langle\mathcal{A}\circ[\![s]\!]_{\vec{x},y},[\![U]\!]_{\vec{x},y}\rangle)\star[\![T]\!]_{\vec{x}} = [\![sU]\!]_{\vec{x},y}\star[\![T]\!]_{\vec{x}}$.

- case $S \equiv \mathsf{D}^n v \cdot (u_1, \dots, u_n)$. By cases on n.

  Subcase $n = 1$. By definition of differential substitution, we have

$$[\![\frac{\partial}{\partial y}(\mathsf{D}\, v \cdot u_1) \cdot T]\!]_{\vec{x},y} = [\![\mathsf{D}\,(\frac{\partial v}{\partial y} \cdot T) \cdot u_1]\!]_{\vec{x},y} + [\![\mathsf{D}\, v \cdot (\frac{\partial u_1}{\partial y} \cdot T)]\!]_{\vec{x},y}.$$

  Consider the two summands separately. On the one side we have $[\![\mathsf{D}\,(\frac{\partial v}{\partial y} \cdot T) \cdot u_1]\!]_{\vec{x},y} = \lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![\frac{\partial v}{\partial y} \cdot T]\!]_{\vec{x},y}) \star [\![u_1]\!]_{\vec{x},y})$. By the inductive hypothesis this is equal to $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ ([\![v]\!]_{\vec{x},y} \star [\![T]\!]_{\vec{x}})) \star [\![u_1]\!]_{\vec{x},y})$, which is equal to $\lambda \circ \Lambda(\Lambda^-((\mathcal{A} \circ [\![v]\!]_{\vec{x},y}) \star [\![T]\!]_{\vec{x}}) \star [\![u_1]\!]_{\vec{x},y})$ by Lemma 4.4(ii).

  On the other side, we have that $[\![\mathsf{D}\, v \cdot (\frac{\partial u_1}{\partial y} \cdot T)]\!]_{\vec{x},y} = \lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![v]\!]_{\vec{x},y}) \star [\![\frac{\partial u_1}{\partial y} \cdot T]\!]_{\vec{x},y})$. By induction hypothesis this is $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![v]\!]_{\vec{x},y}) \star ([\![u_1]\!]_{\vec{x},y} \star [\![T]\!]_{\vec{x}}))$.

  Since λ is linear, we can apply Lemma 3.13 and write the sum of the two morphisms as:

$$\lambda \circ \big(\Lambda(\Lambda^-((\mathcal{A} \circ [\![v]\!]_{\vec{x},y}) \star [\![T]\!]_{\vec{x}}) \star [\![u_1]\!]_{\vec{x},y}) + \Lambda(\Lambda^-(\mathcal{A} \circ [\![v]\!]_{\vec{x},y}) \star ([\![u_1]\!]_{\vec{x},y} \star [\![T]\!]_{\vec{x}}))\big).$$

  By applying Lemma 3.18(ii), we obtain $\lambda \circ (\Lambda(\Lambda^-(\mathcal{A} \circ [\![v]\!]_{\vec{x},y}) \star [\![u_1]\!]_{\vec{x},y}) \star [\![T]\!]_{\vec{x}})$ which is equal to $[\![\mathsf{D}\, v \cdot u_1]\!]_{\vec{x},y} \star [\![T]\!]_{\vec{x}}$.

  Subcase $n > 1$. Performing easy calculations we get $[\![\frac{\partial}{\partial y}(\mathsf{D}^n v \cdot (u_1, \dots, u_n)) \cdot T]\!]_{\vec{x};y} = [\![\mathsf{D}\,(\frac{\partial}{\partial y}(\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})) \cdot T) \cdot u_n]\!]_{\vec{x};y} + [\![\mathsf{D}\,(\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})) \cdot (\frac{\partial u_n}{\partial y} \cdot T)]\!]_{\vec{x};y}$. We consider the two summands separately:

  (1) $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![\frac{\partial}{\partial y}(\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})) \cdot T]\!]_{\vec{x},y}) \star [\![u_n]\!]_{\vec{x},y}) =$ by IH
  $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ ([\![\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})]\!]_{\vec{x},y} \star [\![T]\!]_{\vec{x}})) \star [\![u_n]\!]_{\vec{x},y}) =$ by Lemma 4.4(ii)
  $\lambda \circ \Lambda(\Lambda^-((\mathcal{A} \circ [\![\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})]\!]_{\vec{x},y}) \star [\![T]\!]_{\vec{x}}) \star [\![u_n]\!]_{\vec{x},y})$.

  (2) $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})]\!]_{\vec{x},y}) \star [\![\frac{\partial u_n}{\partial y} \cdot T]\!]_{\vec{x},y}) =$ by IH
  $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})]\!]_{\vec{x},y}) \star ([\![u_n]\!]_{\vec{x},y} \star [\![T]\!]_{\vec{x}}))$.

  Since λ is linear, we have that (1) + (2) is equal to

$$\begin{aligned}\lambda\ \circ \big(&\Lambda(\Lambda^-((\mathcal{A} \circ [\![\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})]\!]_{\vec{x},y}) \star [\![T]\!]_{\vec{x}}) \star [\![u_n]\!]_{\vec{x},y}) + \\ &\Lambda(\Lambda^-(\mathcal{A} \circ [\![\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})]\!]_{\vec{x},y}) \star ([\![u_n]\!]_{\vec{x},y} \star [\![T]\!]_{\vec{x}}))\big)\end{aligned}$$

  By Lemma 3.18(ii) we get $\lambda \circ (\Lambda(\Lambda^-(\mathcal{A} \circ [\![\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})]\!]_{\vec{x},y}) \star [\![u_n]\!]_{\vec{x}}) \star [\![T]\!]_{\vec{x}})$. By Lemma 4.4($i$) this is equal to $\lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ [\![\mathsf{D}^{n-1} v \cdot (u_1, \dots, u_{n-1})]\!]_{\vec{x},y}) \star [\![u_n]\!]_{\vec{x}}) \star [\![T]\!]_{\vec{x}}$, *i.e.*, to $[\![\mathsf{D}^n v \cdot (u_1, \dots, u_n)]\!]_{\vec{x};y} \star [\![T]\!]_{\vec{x}}$.

- all other cases (*i.e.,* $S \equiv 0$ and $S \equiv s + U$) are straightforward.

■

We are now able to provide the main result of this section.

Theorem 4.11 *(Soundness) Every linear reflexive object $\mathscr{U}$ in a Cartesian closed differential category* **C** *is a sound model of the differential λ-calculus.*

Proof. It is easy to check that the categorical interpretation is contextual. We now prove that $\mathrm{Th}(\mathscr{U})$ is closed under the rules (β) and (β_D):

- (β) Let $[\![(\lambda y.s)T]\!]_{\vec{x}} = \mathrm{ev} \circ \langle \mathcal{A} \circ \lambda \circ \Lambda([\![s]\!]_{\vec{x},y}), [\![T]\!]_{\vec{x}} \rangle$. Since $\mathcal{A} \circ \lambda = \mathrm{Id}$ this is equal to $\mathrm{ev} \circ \langle \Lambda([\![s]\!]_{\vec{x},y}), [\![T]\!]_{\vec{x}} \rangle$. On the other side we have $[\![s\{T/y\}]\!]_{\vec{x}} = [\![s]\!]_{\vec{x},y} \circ \langle \mathrm{Id}, [\![T]\!]_{\vec{x}} \rangle$ by the Theorem 4.9 and, by (beta-cat), $[\![s]\!]_{\vec{x},y} \circ \langle \mathrm{Id}, [\![T]\!]_{\vec{x}} \rangle = \mathrm{ev} \circ \langle \Lambda([\![s]\!]_{\vec{x},y}), [\![T]\!]_{\vec{x}} \rangle$.
- (β_D) Let $[\![\mathsf{D}\,(\lambda y.s) \cdot t]\!]_{\vec{x}} = \lambda \circ \Lambda(\Lambda^-(\mathcal{A} \circ \lambda \circ \Lambda([\![s]\!]_{\vec{x},y})) \star [\![t]\!]_{\vec{x}})$. Since $\mathcal{A} \circ \lambda = \mathrm{Id}$ this is equal to $\lambda \circ \Lambda(\Lambda^-(\Lambda([\![s]\!]_{\vec{x},y})) \star [\![t]\!]_{\vec{x}}) = \lambda \circ \Lambda([\![s]\!]_{\vec{x},y} \star [\![t]\!]_{\vec{x}})$. By applying Theorem 4.10, this is equal to $\lambda \circ \Lambda([\![\frac{\partial s}{\partial y} \cdot t]\!]_{\vec{x},y}) = [\![\lambda y.\frac{\partial s}{\partial y} \cdot t]\!]_{\vec{x}}$.

We conclude that $\mathrm{Th}(\mathscr{U})$ is a differential λ-theory. ∎

The above theorem shows that linear reflexive objects in Cartesian closed differential categories are sound models of the untyped differential λ-calculus.

Proposition 4.12 *If $\mathscr{U}$ is extensional, then* $\mathrm{Th}(\mathscr{U})$ *is extensional.*

Proof. Like in the case of usual λ-calculus, easy calculations show that $[\![\lambda x.sx]\!]_{\vec{x}} = \lambda \circ \Lambda(\mathrm{ev}) \circ \mathcal{A} \circ [\![s]\!]_{\vec{x}}$ which is equal to $[\![s]\!]_{\vec{x}}$ since $\Lambda(\mathrm{ev}) = \mathrm{Id}$ and $\lambda \circ \mathcal{A} = \mathrm{Id}$. ∎

4.4. Equational Completeness

An important result in the regular λ-calculus is the *equational completeness theorem* proved by Scott in [42] and subsequently refined by Koymans [30]. This theorem states that every λ-theory is the theory of a reflexive object in a Cartesian closed category. In this section we discuss whether the categorical notion of model of the differential λ-calculus presented in Section 4 is complete too. In other words we investigate the question whether for every differential λ-theory $\mathcal{T}$ there is a linear reflexive object $\mathscr{U}_{\mathcal{T}}$ living in a suitable Cartesian closed differential category $\mathbf{C}_{\mathcal{T}}$ such that $\mathrm{Th}(\mathscr{U}_{\mathcal{T}}) = \mathcal{T}$. We will be able to answer positively this question, provided that $\mathcal{T}$ is differentially extensional and satisfies sum idempotency. This restriction is quite reasonable since all known models which have arisen so far do satisfy these properties (see Subsections 5.1.1, 5.3.1, below). However, such conditions arise from some technical choices we have to make — it is at the moment unknown whether different choices might lead to a more general theorem.

Before going further, we outline the proof of the classic Scott-Koymans' result which is achieved in two steps:

(*i*) given a λ-theory $\mathcal{T}$ one proves that the set of λ-terms modulo $\mathcal{T}$ together with the application operator defined between equivalence classes constitutes a *λ-model*[4] $\mathscr{M}_{\mathcal{T}}$ (called *the term model of* $\mathcal{T}$) having as theory exactly $\mathcal{T}$;

(*ii*) by applying to $\mathscr{M}_{\mathcal{T}}$ a construction called *Karoubi envelope* [29] one builds a (very syntactic) Cartesian closed category $\mathbf{C}_{\mathcal{T}}$ in which the identity $\mathbf{I}$ is a reflexive object such that $\mathrm{Th}(\mathbf{I}) = \mathcal{T}$.

Summing up, the idea of the proof is to find suitable λ-terms to encode the structure of the category (pairing, currying, evaluation, and the like) and prove that they actually define a category with such a structure.

In our context the categorical operator $D(-)$ can be easily defined in terms of the linear application. Intuitively the term representing $D(f)$ takes in input a pair and applies the first component linearly and the second in the usual way, in accordance with the categorical axiomatization of $D(-)$. The main problem we need to solve is that the encoding of categorical pairing

[4] A 'λ-model' is a combinatory algebra satisfying the five axioms of Curry and the Meyer-Scott axiom. We refer to [2, Ch. 5] for more details.

$\langle f, g\rangle$ used by Scott is not additive. Indeed, such pairing is defined starting from Church's encoding of the pair in λ-calculus given by $\langle\!\langle f, g\rangle\!\rangle \equiv \lambda x.xfg$ with projections $p_1 = \lambda z.z(\lambda xy.x)$, $p_2 = \lambda z.z(\lambda xy.y)$. Obviously with this definition we have $\langle\!\langle f_1+f_2, g_1+g_2\rangle\!\rangle \neq \langle\!\langle f_1, g_1\rangle\!\rangle + \langle\!\langle f_2, g_2\rangle\!\rangle$ since the sums do not occur in linear position. We will see that the encoding of an additive pairing can be obtained using the linear application and the sum of the differential λ-calculus.

Notation 4.13 *Given a differential λ-theory $\mathcal{T}$ we write $\Lambda^d_{\mathcal{T}}$ for $\Lambda^d/\mathcal{T}$.*

From now on, and until the end of the section, we set $A\circ B \equiv \lambda x.A(Bx)$. We say that $A \in \Lambda^d_{\mathcal{T}}$ is *idempotent* if $A\circ A = A$ and *additive* if $A(x+y) = Ax + Ay$.

Definition 4.14 *Let $\mathcal{T}$ be a differential λ-theory. The category $\mathbf{C}_{\mathcal{T}}$ associated with $\mathcal{T}$ is defined as follows:*

Objects	$\{A \in \Lambda^d_{\mathcal{T}} \mid A$ *is idempotent and additive* $\}$
Arrows	$\mathbf{C}_{\mathcal{T}}(A, B) = \{ f \in \Lambda^d_{\mathcal{T}} \mid B\circ f\circ A = f \}$
Identities	$\mathrm{Id}_A = A$
Composition	$f\circ g$

It is easy to verify that $\mathbf{C}_{\mathcal{T}}$ is indeed a category.

We now encode the ordered pair $\langle\!\langle S, T\rangle\!\rangle$ in the differential λ-calculus as follows. We will use this notion in the definition of categorical pairing.

Definition 4.15 *The encoding of the pair into the differential λ-calculus is given by:*

$$\langle\!\langle S, T\rangle\!\rangle \equiv \lambda y.(S + \mathsf{D}y\cdot T),\ \textit{for some } y \notin \mathrm{FV}(S, T)$$

with projections $p_1 \equiv \lambda x.x0$ and $p_2 \equiv \lambda x.(\mathsf{D}x\cdot \mathbf{I})00$.

It is immediate to verify that $p_i\langle\!\langle S_1, S_2\rangle\!\rangle = S_i$ (for $i = 1, 2$) and that $\langle\!\langle S_1 + S_2, T_1 + T_2\rangle\!\rangle = \langle\!\langle S_1, T_1\rangle\!\rangle + \langle\!\langle S_2, T_2\rangle\!\rangle$. This encoding is inspired by the set-theoretical definition of the ordered pair: the pair of S, T is morally the set containing S, T (the sum being the union) slightly modified to be able to distinguish them. Such a distinction consists in the number of linear resources they can receive (zero for the first component and one for the second).

With this encoding we are able to endow $\mathbf{C}_{\mathcal{T}}$ with a structure of differential Cartesian closed category, under the assumption that the sum is idempotent (like set-theoretical union).

Theorem 4.16 *For all differential λ-theories $\mathcal{T}$ satisfying sum idempotency we have:*

(i) $\mathbf{C}_{\mathcal{T}}$ *is differential Cartesian closed,*

(ii) the triple $(\mathbf{I}, \mathbf{1}, \mathbf{1})$ *is a linear reflexive object.*

Proof. (i) *Terminal object.* This is $\mathbb{1} \equiv \lambda xy.y$; note that $f : A \to \mathbb{1}$ if and only if $f \equiv\, !_A \equiv \lambda xy.y$.

Products. Given two objects A_1, A_2, the object $A_1 \times A_2 \equiv \lambda z.\langle\!\langle A_1(p_1z), A_2(p_2z)\rangle\!\rangle$ is the Cartesian product of A_1, A_2.

Projections	$\pi_1 : A_1 \times A_2 \to A_1,$	$\pi_1^{A_1,A_2} \equiv A_1\circ p_1$
	$\pi_2 : A_1 \times A_2 \to A_2,$	$\pi_2^{A_1,A_2} \equiv A_2\circ p_2$
Pairing	Let $f : A \to B$ and $g : A \to C$	
	then $\langle f, g\rangle \equiv \lambda z.\langle\!\langle fz, gz\rangle\!\rangle$	

Exponents. Given two objects A, B the object $A \Rightarrow B \equiv \lambda z.B \circ z \circ A$ is the exponential object internalizing $\mathbf{C}_{\mathcal{T}}(A, B)$. The evaluation morphism $\mathrm{ev} : (A \Rightarrow B) \times A \to B$ is defined by

$$\mathrm{ev} \equiv \lambda z.B((p_1 z)(A(p_2 z)))$$

while the curry of a morphism $f : A \times B \to C$ is given by

$$\Lambda(f) \equiv \lambda xy.f\langle\!\langle x, y\rangle\!\rangle$$

Differential operator. Given a morphism $f : A \to B$ we define

$$D(f) \equiv \lambda z.B\big((\mathsf{D} f \cdot (A(p_1 z)))(A(p_2 z))\big)$$

Left-additive structure. We interpret the sum in the category as the sum on $\Lambda^d_{\mathcal{T}}$.

The calculations that show that everything works are *very* lengthy but straightforward. As a simple example we prove that categorical pairing is actually additive:

$$\begin{aligned}
\langle f_1 + f_2, g_1 + g_2\rangle &= \lambda z.\langle\!\langle f_1 z + f_2 z, g_1 z + g_2 z\rangle\!\rangle \\
&= \lambda y.(f_1 z + f_2 z) + \lambda y.\mathsf{D} y \cdot (g_1 z + g_2 z) \\
&= \lambda y.f_1 z + \lambda y.f_2 z + \lambda y.\mathsf{D} y \cdot (g_1 z) + \lambda y.\mathsf{D} y \cdot (g_2 z) \\
&= \langle f_1, g_1\rangle + \langle f_2, g_2\rangle
\end{aligned}$$

(ii) Note that $(\mathbf{I} \Rightarrow \mathbf{I}) = \mathbf{1}$. Then $\mathbf{I}$ is a reflexive object since $\mathbf{1} : \mathbf{1} \to \mathbf{I}$, $\mathbf{1} : \mathbf{I} \to \mathbf{1}$ and $\mathbf{1} \circ \mathbf{1} = \mathrm{Id}_{\mathbf{1}}$. Moreover $\mathbf{I}$ is linear as a reflexive object:

$$D(\mathbf{1}) = \lambda z.(\mathsf{D}\mathbf{1} \cdot (p_1 z))(p_2 z) = \lambda z.(\lambda xy.p_1 zy)(p_2 z) = \lambda zy.p_1 zy = \mathbf{1} \circ \pi_1.$$

■

In the above proof the idempotency of the sum is needed, for instance, to prove the axiom (D-curry). It is left for future works to find an encoding of the additive pairing that does not require the idempotency of the sum.

In order to provide a characterization of the interpretation of a differential λ-term S we need the following definition.

Definition 4.17 *The* full η_∂-expansion $\widehat{S}$ *of a differential λ-term* $S \in \Lambda^d$ *is defined by induction (where* y *is a fresh variable):*

$$\widehat{x} \equiv x, \qquad \widehat{\lambda x.s} \equiv \lambda x.\widehat{s}, \qquad \widehat{sT} \equiv \widehat{s}\widehat{T}, \qquad \widehat{\mathsf{D} s \cdot t} \equiv \lambda y.(\mathsf{D}\widehat{s} \cdot \widehat{t})y, \qquad \widehat{\textstyle\sum_i s_i} \equiv \textstyle\sum_i \widehat{s_i}.$$

Roughly speaking, the term $\widehat{S}$ is obtained from S performing one η_∂-expansion in all its subterms of shape $\mathsf{D} s \cdot t$. The adjective *full* refers to the fact that the η_∂-expansion is done inductively on the structure of S.

Remark 4.18 *Obviously, if* $\mathcal{T}$ *is differentially extensional then* $\mathcal{T} \vdash S = \widehat{S}$ *for all* $S \in \Lambda^d$.

Proposition 4.19 *In the model* $\mathbf{I}$ *living in* $\mathbf{C}_{\mathcal{T}}$ *the following holds (for some* $z \notin \mathrm{FV}(S)$*):*

$$[\![S]\!]_{\vec{x}} = \lambda z.\widehat{S}\{\pi^{\vec{x}}_{x_1} z/x_1\} \cdots \{\pi^{\vec{x}}_{x_n} z/x_n\} : \mathbf{I}^{\vec{x}} \to \mathbf{I}.$$

Proof. By induction on the structure of S. The only two non-trivial cases are the following.
case $S \equiv \lambda y.T$. Then we have:

$$\begin{array}{rcll} [\![\lambda x_{n+1}.T]\!]_{\vec{x}} & = & \mathbf{1}\circ\Lambda([\![T]\!]_{\vec{x},x_{n+1}}) & \text{by definition of } [\![\cdot]\!]_{\vec{x}} \\ & = & \mathbf{1}\circ(\lambda y_1y_2.[\![T]\!]_{\vec{x},x_{n+1}}\langle\!\langle y_1,y_2\rangle\!\rangle) & \text{by definition of } \Lambda(\cdot) \\ & = & \lambda y_1y_2.(\lambda z.\widehat{T}\{\pi^{\vec{x},x_{n+1}}_{x_1}z/x_1\}\cdots\{\pi^{\vec{x},x_{n+1}}_{x_{n+1}}z/x_{n+1}\})\langle\!\langle y_1,y_2\rangle\!\rangle & \text{by induction hypothesis} \\ & = & \lambda y_1y_2.\widehat{T}\{\pi^{\vec{x}}_{x_1}y_1/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}y_1/x_n\}\{y_2/x_{n+1}\} & \text{by } \beta\text{-reduction} \\ & = & \lambda z.(\lambda x_{n+1}.\widehat{T})\{\pi^{\vec{x}}_{x_1}z/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}z/x_n\} & \text{by } \alpha\text{-conversion} \end{array}$$

case $S \equiv \mathsf{D}T\cdot U$. Then easy (but very lengthy) calculations give:

$$\begin{array}{rcl} [\![\mathsf{D}T\cdot U]\!]_{\vec{x}} & = & \lambda zy.(\mathsf{D}([\![T]\!]_{\vec{x}}z)\cdot([\![U]\!]_{\vec{x}}z))y \\ & = & \lambda zy.(\mathsf{D}((\lambda z.T\{\pi^{\vec{x}}_{x_1}z/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}z/x_n\})z)\cdot((\lambda z.S\{\pi^{\vec{x}}_{x_1}z/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}z/x_n\})z))y \\ & = & \lambda zy.(\mathsf{D}(T\{\pi^{\vec{x}}_{x_1}z/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}z/x_n\})\cdot(S\{\pi^{\vec{x}}_{x_1}/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}z/x_n\}))y \end{array}$$

■

Therefore, in the theory of the model $\mathbf{I}$, equations of the form $\mathsf{D}s\cdot t = \lambda y.(\mathsf{D}s\cdot t)y$ might be added. No equation can be added when the theory $\mathcal{T}$ is already differentially extensional.

Theorem 4.20 *[Equational Completeness] Every differentially extensional differential λ-theory $\mathcal{T}$ satisfying sum idempotency is the theory of a linear reflexive object in a differential Cartesian closed category.*

Proof. For all closed terms $S, T \in \Lambda^d$ we have, by Proposition 4.19, $[\![S]\!]_{\vec{x}} = [\![T]\!]_{\vec{x}}$ entails $\mathcal{T} \vdash \lambda z.\widehat{S} = \lambda z.\widehat{T}$ and, by Remark 4.18, $\mathcal{T} \vdash \lambda z.S = \lambda z.T$. Since $\mathcal{T}$ is a differential λ-theory we also have $\mathcal{T} \vdash (\lambda z.S)0 = (\lambda z.T)0$. Since $z \notin \mathrm{FV}(S,T)$ and $\lambda\beta^d \subseteq \mathcal{T}$ we get $\mathcal{T} \vdash S = T$, therefore $\mathrm{Th}(\mathbf{I}) = \mathcal{T}$. ■

In Subsection 6.3.1 we will show that an analogous theorem holds for the resource calculus, without the restriction to theories that are differentially extensional.

Remark 4.21 *Theorem 4.20 does not entail that the theory of every model in a differential Cartesian closed category must satisfy sum-idempotency and the differential extensional axiom. Those are just technical conditions arising from that specific proof of the completeness theorem. However, at the moment, no more general proof is known.*

The completeness theorem constitutes an important result and suggests that the notion of model we chose for the differential λ-calculus is actually correct.

On the other hand, denotational models are usually introduced because they allow to study a calculus by means of more abstract mathematical structures on which a broader range of tools and proof techniques are available. In this respect, the categorical models living in $\mathbf{C}_{\mathcal{T}}$ are not satisfactory because they are very syntactical and proving operational properties of differential λ-terms via these models does not make it any easier than working directly with the syntax.

For this reason it would be interesting to find meaningful classes of models (semantics) that are *complete* in the sense that they allow to represent all differential λ-theories. Already for the usual λ-calculus it is well known that the main semantics, i.e. the continuous, the stable and the strongly stable semantics, are all *hugely incomplete* — there is a continuum of λ-theories that cannot be representable by models living in such semantics [41]. In Section 5.3 we will see that a similar result holds for the relational semantics of the differential λ-calculus (Corollary 5.11).

The problem of finding a complete semantics of the (differential) λ-calculus is open and quite difficult.

4.5. Comparison with the Categorical Models of the Untyped Lambda Calculus

The definition of categorical model of the differential λ-calculus proposed in this paper seems to be a generalization without surprises of the classical definition of model of the λ-calculus, i.e., the notion of reflexive object in a Cartesian closed category. However, while this notion is – by far – the most famous categorical definition of model of λ-calculus, it is not the most general one. Indeed, as pointed out by Martini in [36], in the proof of soundness [2, Prop. 5.5.5] for categorical models there is one axiom of Cartesian closed categories that is never used, namely the axiom (Id-Curry) which is equivalent to ask for the unicity of the operator $\Lambda(-)$ in the category (and this entails $\Lambda(\Lambda^-(f)) = f$).

For this reason Martini proposed reflexive objects living in *weak* Cartesian closed categories as a more general notion of model of λ-calculus. In these categories we have just a retraction (not an isomorphism) between the homsets $\mathbf{C}(C \times A, B) \lhd \mathbf{C}(C, A \Rightarrow B)$. Thus $A \Rightarrow B$ is no longer an object representing *exactly* $\mathbf{C}(A, B)$ — there are different objects that can equally well accomplish the job. Recently, De Carvalho [17] successfully used this notion to build concrete models living in very natural weak Cartesian closed categories inspired from the semantics of linear logic.

In our differential framework this generalization cannot be applied since the proof of soundness relies on the fact that $\Lambda(\Lambda^-(f)) = f$. This is actually needed to give a meaningful interpretation of the linear application $\mathsf{D}s \cdot t$. Hence the definition of categorical model of the differential λ-calculus we presented differs from the corresponding one for the usual λ-calculus more than one could imagine at a first look.

4.6. Modeling the Taylor Expansion

In this subsection we provide sufficient conditions for models living in Cartesian closed differential categories to equate all differential λ-terms having the same Taylor expansion. As an interesting fact, this happens to be a property of the category rather than of the reflexive objects. Therefore, all models living in a category "modeling the Taylor expansion" have an equational theory including $\mathcal{E}$.

Since the definition of the Taylor expansion asks for infinite sums, we need to consider Cartesian closed differential categories $\mathbf{C}$ where it is possible to sum infinitely many morphisms. Formally, we require that for every countable set I and every family $\{f_i\}_{i\in I}$ of morphisms $f_i : A \to B$ we have $\sum_{i\in I} f_i \in \mathbf{C}(A, B)$. In this case we say that $\mathbf{C}$ *has countable sums.* To avoid the tedious problem of handling coefficients we suppose that the sum on the morphisms is idempotent.

Definition 4.22 *A Cartesian closed differential category* models the Taylor Expansion *if it has countable sums and the following axiom holds (for every $f : C \times A \to B$ and $g : C \to A$):*

$$\textit{(Taylor)} \qquad \mathrm{ev} \circ \langle f, g\rangle = \sum_{k\in\mathbf{N}} ((\cdots(\Lambda^-(f) \underbrace{\star g)\cdots) \star g}_{k\ \textit{times}}) \circ \langle \mathrm{Id}, 0\rangle.$$

Recall that the Taylor expansion S^* of a differential λ-term S has been defined in Subsection 2.4. Given a model $\mathscr{U}$ of the differential λ-calculus living in a Cartesian closed differential category having countable sums we can extend the interpretation given in Definition 4.5 to terms in Λ^d_∞ by setting $[\![\Sigma_{i\in I} s_i]\!]_{\vec{x}} = \sum_{i\in I} [\![s_i]\!]_{\vec{x}}$, for every countable set I.

Theorem 4.23 *Let S be a differential λ-term and $\mathscr{U}$ be a model living in a Cartesian closed differential category having countable sums and modeling the Taylor Expansion. Then:*

$$[\![S]\!]_{\vec{x}} = [\![S^*]\!]_{\vec{x}}.$$

Proof. By structural induction on S. The only interesting case is $S \equiv sT$.

$$\begin{array}{rcll}
[\![sT]\!]_{\vec{x}} & = & \mathrm{ev}\circ\langle\mathcal{A}\circ[\![s]\!]_{\vec{x}},[\![T]\!]_{\vec{x}}\rangle & \text{by def. of } [\![-]\!]_{\vec{x}}\\
& = & \sum_{k\in\mathbf{N}}((\cdots(\Lambda^-(\mathcal{A}\circ[\![s]\!]_{\vec{x}})\underbrace{\star[\![T]\!]_{\vec{x}})\cdots)\star[\![T]\!]_{\vec{x}}}_{k\text{ times}})\circ\langle\mathrm{Id},0\rangle & \text{by (Taylor)}\\
& = & \sum_{k\in\mathbf{N}}\mathrm{ev}\circ\langle\Lambda((\cdots(\Lambda^-([\![s]\!]_{\vec{x}}\underbrace{\star[\![T]\!]_{\vec{x}})\cdots)\star[\![T]\!]_{\vec{x}}}_{k\text{ times}}),0\rangle & \text{by (beta-cat)}\\
& = & \sum_{k\in\mathbf{N}}\mathrm{ev}\circ\langle\mathcal{A}\circ\lambda\circ\Lambda((\cdots(\Lambda^-([\![s]\!]_{\vec{x}}\underbrace{\star[\![T]\!]_{\vec{x}})\cdots)\star[\![T]\!]_{\vec{x}}}_{k\text{ times}}),0\rangle & \text{by } \mathcal{A}\circ\lambda=\mathrm{Id}\\
& = & \sum_{k\in\mathbf{N}}\mathrm{ev}\circ\langle\mathcal{A}\circ[\![\mathsf{D}^k s\cdot(T,\ldots,T)]\!]_{\vec{x}},0\rangle & \text{by def. of } [\![-]\!]_{\vec{x}}\\
& = & [\![\Sigma_{k\in\mathbf{N}}(\mathsf{D}^k s\cdot(T,\ldots,T))0]\!]_{\vec{x}} & \text{by def. of } [\![-]\!]_{\vec{x}}\\
& = & [\![(sT)^*]\!]_{\vec{x}} & \text{by def. of } (\cdot)^*
\end{array}$$

■

By adapting the proof of Theorem 4.11 one can prove that $[\![S^*]\!]_{\vec{x}} = [\![\mathrm{NF}(S^*)]\!]_{\vec{x}}$ for every differential λ-term S. From this fact and Theorem 4.23 we get the following result.

Corollary 4.24 *Every model $\mathscr{U}$ living in a Cartesian closed differential category that models the Taylor expansion satisfies $\mathcal{E} \subseteq \mathrm{Th}(\mathscr{U})$.*

5. A Relational Model of the Differential Lambda Calculus

In this section we provide the main example of Cartesian closed differential category known in the literature. What we have in mind is the category **MRel** [26,10], which is the co-Kleisli category of the functor $\mathcal{M}_f(-)$ over the $\star$-autonomous category **Rel** of sets and relations. We will also show that the reflexive object $\mathscr{D}$ living in **MRel** built in [10] to model the usual λ-calculus is linear, and then it constitutes a model of the untyped differential λ-calculus. We will then provide a partial characterization of its equational theory showing that it contains $\lambda\beta\eta^d$ and $\mathcal{E}$ (this follows from the fact that **MRel** models the Taylor expansion).

Remark 5.1 *In [12] we have provided another example of Cartesian closed differential category: the category* **MFin***, which is the co-Kleisli of the functor $\mathcal{M}_f(-)$ over the $\star$-autonomous category of finiteness spaces and finitary relations [19]. In this paper we do not present the category* **MFin** *since it does not contain any reflexive object (see [19,47]) and hence it cannot be used as a semantics of the untyped differential λ-calculus. Other examples of semantics useful for modeling the untyped differential λ-calculus (including semantics that* do not *model the Taylor expansion) will be discussed in Subsection 7.2.*

5.1. Relational Semantics

We recall that the definitions and notations concerning multisets have been introduced in Subsection 1.1. We now provide a direct definition of the category **MRel**:

- The objects of **MRel** are all the sets.
- A morphism from A to B is a relation from $\mathcal{M}_f(A)$ to B; in other words, $\mathbf{MRel}(A,B) = \mathcal{P}(\mathcal{M}_f(A)\times B)$.
- The identity of A is the relation $\mathrm{Id}_A = \{([\alpha],\alpha) \mid \alpha\in A\} \in \mathbf{MRel}(A,A)$.
- The composition of $s\in\mathbf{MRel}(A,B)$ and $t\in\mathbf{MRel}(B,C)$ is defined by:

$$\begin{array}{rcl} t\circ s = \{(m,\gamma) & \mid & \exists k\in\mathbf{N},\ \exists(m_1,\beta_1),\ldots,(m_k,\beta_k)\in s \text{ such that}\\ & & m = m_1\uplus\ldots\uplus m_k \text{ and } ([\beta_1,\ldots,\beta_k],\gamma)\in t\ \}.\end{array}$$

Given two sets A_1, A_2, we denote by $A_1 \,\&\, A_2$ their disjoint union $(\{1\} \times A_1) \cup (\{2\} \times A_2)$. Hereafter we adopt the following convention.

Convention 5.2 *We consider the canonical bijection between $\mathcal{M}_f(A_1)\times\mathcal{M}_f(A_2)$ and $\mathcal{M}_f(A_1 \,\&\, A_2)$ as an equality. Therefore, we will still denote by (m_1, m_2) the corresponding element of $\mathcal{M}_f(A_1 \& A_2)$.*

Theorem 5.3 *The category* **MRel** *is a Cartesian closed category.*

Proof. The terminal object $\mathbb{1}$ is the empty set $\emptyset$, and the unique element of $\mathbf{MRel}(A, \emptyset)$ is the empty relation.

Given two sets A_1 and A_2, their categorical product in **MRel** is their disjoint union $A_1 \,\&\, A_2$ and the projections π_1, π_2 are given by:

$$\pi_i = \{([(i,a)], a) \mid a \in A_i\} \in \mathbf{MRel}(A_1 \& A_2, A_i), \text{ for } i = 1, 2.$$

It is easy to check that this is actually the categorical product of A_1 and A_2 in **MRel**; given $s \in \mathbf{MRel}(B, A_1)$ and $t \in \mathbf{MRel}(B, A_2)$, the corresponding morphism $\langle s, t\rangle \in \mathbf{MRel}(B, A_1 \& A_2)$ is given by:

$$\langle s, t\rangle = \{(m, (1, a)) \mid (m, a) \in s\} \cup \{(m, (2, b)) \mid (m, b) \in t\}\,.$$

Given two objects A and B, the exponential object $A \Rightarrow B$ is $\mathcal{M}_f(A) \times B$ and the evaluation morphism is given by:

$$\mathrm{ev}_{AB} = \{((([(m, b)], m), b) \mid m \in \mathcal{M}_f(A) \text{ and } b \in B\} \in \mathbf{MRel}((A \Rightarrow B) \& A, B)\,.$$

Again, it is easy to check that in this way we defined an exponentiation. Indeed, given any set C and any morphism $s \in \mathbf{MRel}(C \& A, B)$, there is exactly one morphism $\Lambda(s) \in \mathbf{MRel}(C, A \Rightarrow B)$ such that:

$$\mathrm{ev}_{AB} \circ (\Lambda(s) \times \mathrm{Id}_S) = s.$$

which is $\Lambda(s) = \{(p, (m, b)) \mid ((p, m), b) \in s\}$. ■

Theorem 5.4 *The category* **MRel** *is a Cartesian closed differential category.*

Proof. By Theorem 5.3 **MRel** is Cartesian closed. It is Cartesian closed left-additive since every homset $\mathbf{MRel}(A, B)$ can be endowed with the following additive structure $(\mathbf{MRel}(A, B), \cup, \emptyset)$.

Finally, given $f \in \mathbf{MRel}(A, B)$ we can define its derivative as follows:

$$D(f) = \{((([\alpha], m), \beta) \mid (m \uplus [\alpha], \beta) \in f\} \in \mathbf{MRel}(A \& A, B).$$

It is not difficult to check that $D(-)$ satisfies *(D1-7)*. We now show that also *(D-curry)* holds. Let $f \subseteq (\mathcal{M}_f(C) \times \mathcal{M}_f(A)) \times B$. On the one side we have:

$$D(\Lambda(f)) = \{((([\gamma], m_1), (m_2, \beta)) \mid ((m_1 \uplus [\gamma], m_2), \beta) \in f\}.$$

On the other side we have $D(f) = f_1 \cup f_2$, where:

$$f_1 = \{((([\gamma], []), (m_1, m_2)), \beta) \mid ((m_1 \uplus [\gamma], m_2), \beta) \in f\},$$
$$f_2 = \{((([], [\alpha]), (m_1, m_2)), \beta) \mid ((m_1, m_2 \uplus [\alpha]), \beta) \in f\}.$$

Since **MRel** is left-additive we have that

$$(f_1 \cup f_2) \circ \langle \pi_1 \times 0, \pi_2 \times \mathrm{Id} \rangle = (f_1 \circ \langle \pi_1 \times 0, \pi_2 \times \mathrm{Id} \rangle) \cup (f_2 \circ \langle \pi_1 \times 0, \pi_2 \times \mathrm{Id} \rangle)$$

Easy calculations give:

$$f_1 \circ \langle \pi_1 \times 0, \pi_2 \times \mathrm{Id} \rangle = \{((([\gamma], m_1), m_2), \beta)) \mid ((m_1 \uplus [\gamma], m_2), \beta) \in f\}$$
$$f_2 \circ \langle \pi_1 \times 0, \pi_2 \times \mathrm{Id} \rangle = \emptyset.$$

We then get $\Lambda(D(f) \circ \langle \pi_1 \times 0, \pi_2 \times \mathrm{Id} \rangle) = \Lambda(f_1 \circ \langle \pi_1 \times 0, \pi_2 \times \mathrm{Id} \rangle) = D(\Lambda(f))$. ■

The operator $\star$ can be directly defined in **MRel** as follows:

$$f \star g = \{((m_1 \uplus m_2, m), \beta) \mid (m_1, \alpha) \in g,\ ((m_2, m \uplus [\alpha]), \beta) \in f\} \in \mathbf{MRel}(C \& A, B).$$

We now provide a characterization of the linear morphisms of **MRel**.

Lemma 5.5 *A morphism $f \in \mathbf{MRel}(A, B)$ is linear iff for all $(m, \beta) \in f$ we have that m is a singleton.*

Proof. Easy calculations give $f \circ \pi_1 = \{((m, []), \beta) \mid (m, \beta) \in f\}$. This is equal to $D(f)$ if and only if m is a singleton. ■

Corollary 5.6 *In* **MRel** *every isomorphism is linear.*

Proof. Let $f \in \mathbf{MRel}(B, A)$ and $g \in \mathbf{MRel}(A, B)$ such that $f \circ g = \mathrm{Id}_A$ and $g \circ f = \mathrm{Id}_B$. Notice that f does not contain any pair $([], \alpha)$ because otherwise such a pair would also appear in $f \circ g$, and this is impossible since $f \circ g = \mathrm{Id}$. Similarly, g cannot contain any pair $([], \beta)$. Thus:

$$f \circ g = \{([\alpha], \alpha) \mid \exists \beta \in B\ ([\alpha], \beta) \in g \text{ and } ([\beta], \alpha) \in f\}.$$

Since by hypothesis $f \circ g = \{([\alpha], \alpha) \mid \alpha \in A\}$ we have that for all $\alpha \in A$ there is a $\beta \in B$ such that $([\beta], \alpha) \in f$. Suppose now, by the way of contradiction, that there is a $([\alpha_1, \ldots, \alpha_k], \beta) \in g$ such that $k > 1$. From the property above there are $\beta_1, \ldots, \beta_k \in B$ such that $([\beta_i], \alpha_i) \in f$ for $1 \leq i \leq k$, thus we would have $([\beta_1, \ldots, \beta_k], \beta) \in f \circ g = \mathrm{Id}_B$, which is impossible. By Lemma 5.5 we conclude that g is linear. Analogous considerations show that also f is linear. ■

5.1.1. An Extensional Relational Model

In this section we build a reflexive object $\mathscr{D}$ in **MRel** which is extensional by construction, and hence linear by Corollary 5.6. We first give some preliminary definitions.

Recall that $\mathbf{N}$ denotes the set of natural numbers. An $\mathbf{N}$-indexed sequence $\sigma = (m_1, m_2, \ldots)$ of multisets is *quasi-finite* if $m_i = []$ holds for all but a finite number of indices i. If A is a set, we denote by $\mathcal{M}_f(A)^{(\omega)}$ the set of all quasi-finite $\mathbf{N}$-indexed sequences of finite multisets over A. Notice that the only inhabitant of $\mathcal{M}_f(\emptyset)^{(\omega)}$ is the sequence $([], [], [], \ldots)$.

We now define a family of sets $\{D_n\}_{n \in \mathbf{N}}$ as follows:

- $D_0 = \emptyset$,
- $D_{n+1} = \mathcal{M}_f(D_n)^{(\omega)}$.

Since the operation $A \mapsto \mathcal{M}_f(A)^{(\omega)}$ is monotonic on sets, and since $D_0 \subseteq D_1$, we have $D_n \subseteq D_{n+1}$ for all $n \in \mathbf{N}$. Finally, we set $D = \cup_{n\in\mathbf{N}} D_n$.

So we have $D_0 = \emptyset$ and $D_1 = \{([], [], \dots)\}$. The elements of D_2 are quasi-finite sequences of multisets over a singleton, i.e., quasi-finite sequences of natural numbers. More generally, an element of D can be represented as a finite tree which alternates two kinds of layers:

- ordered nodes (the quasi-finite sequences), where immediate subtrees are indexed by distinct natural numbers,
- unordered nodes where subtrees are organized in a *non-empty* multiset.

In order to define an isomorphism in **MRel** between D and $(D \Rightarrow D) = \mathcal{M}_f(D) \times D$ it is enough to remark that every element $\sigma \in D$ is canonically associated with the pair $(\sigma_0, (\sigma_1, \sigma_2, \dots))$ and *vice versa*. Given $\sigma \in D$ and $m \in \mathcal{M}_f(D)$, we write $m :: \sigma$ for the element $\tau = (\tau_1, \tau_2, \dots) \in D$ such that $\tau_1 = m$ and $\tau_{i+1} = \sigma_i$. This defines a bijection between $\mathcal{M}_f(D) \times D$ and D, and hence an isomorphism in **MRel** as follows:

Proposition 5.7 *The triple $\mathscr{D} = (D, \mathcal{A}, \lambda)$ where:*

- $\lambda = \{([(m, \sigma)], m :: \sigma) \mid m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D \Rightarrow D, D)$,
- $\mathcal{A} = \{([m :: \sigma], (m, \sigma)) \mid m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D, D \Rightarrow D)$,

is an extensional categorical model of differential λ-calculus.

Proof. It is trivial that $\lambda \circ \mathcal{A} = \mathrm{Id}_D$ and $\mathcal{A} \circ \lambda = \mathrm{Id}_{D \Rightarrow D}$. We conclude by Corollary 5.6. ■

5.2. Interpreting the Differential Lambda Calculus in $\mathscr{D}$

In Section 4, we have defined the interpretation of a differential λ-term in any linear reflexive object of a Cartesian closed differential category. We provide the result of the corresponding computation, when it is performed in $\mathscr{D}$.

Given a differential λ-term S and a sequence $\vec{x} = x_1, \dots, x_n$ adequate for S, the interpretation $[\![S]\!]_{\vec{x}}$ is an element of $\mathbf{MRel}(D^{\vec{x}}, D)$, i.e., $[\![S]\!]_{\vec{x}} \subseteq \mathcal{M}_f(D)^n \times D$. The interpretation $[\![S]\!]_{\vec{x}}$ is defined by structural induction on S as follows:

- $[\![x_i]\!]_{\vec{x}} = \{(([], \dots, [], [\sigma], [], \dots, []), \sigma) \mid \sigma \in D\}$, where the only non-empty multiset occurs in the i-th position.
- $$\begin{array}{lll} [\![sT]\!]_{\vec{x}} = \{((m_1, \dots, m_n), \sigma) \mid \exists k \in \mathbf{N} & & \\ & \exists (m_1^j, \dots, m_n^j) \in \mathcal{M}_f(D)^n & \text{for } j = 0, \dots, k \\ & \exists \sigma_1, \dots, \sigma_k \in D & \text{such that} \\ & m_i = m_i^0 \uplus \dots \uplus m_i^k & \text{for } i = 1, \dots, n \\ & ((m_1^0, \dots, m_n^0), [\sigma_1, \dots, \sigma_k] :: \sigma) \in [\![s]\!]_{\vec{x}} & \\ & ((m_1^j, \dots, m_n^j), \sigma_j) \in [\![T]\!]_{\vec{x}} & \text{for } j = 1, \dots, k\}, \end{array}$$
- $[\![\lambda z.s]\!]_{\vec{x}} = \{((m_1, \dots, m_n), m :: \sigma) \mid ((m_1, \dots, m_n, m), \sigma) \in [\![s]\!]_{\vec{x},z}\}$, where we assume that z does not occur in $\vec{x}$,
- $[\![\mathsf{D}^1 s \cdot (t)]\!]_{\vec{x}} = \{((m_1 \uplus m'_1, \dots, m_n \uplus m'_n), m :: \beta) \mid \exists \alpha \in D\ ((m_1, \dots, m_n), \alpha) \in [\![t]\!]_{\vec{x}}$ and $((m'_1, \dots, m'_n), m \uplus [\alpha] :: \beta) \in [\![s]\!]_{\vec{x}}\}$,
- $[\![\mathsf{D}^{n+1} s \cdot (t_1, \dots, t_{n+1})]\!]_{\vec{x}} = \{((m_1 \uplus m'_1, \dots, m_n \uplus m'_n), m :: \beta) \mid \exists \alpha \in D\ ((m_1, \dots, m_n), \alpha) \in [\![t_{n+1}]\!]_{\vec{x}}$ and $((m'_1, \dots, m'_n), m \uplus [\alpha] :: \beta) \in [\![\mathsf{D}^n s \cdot (t_1, \dots, t_n)]\!]_{\vec{x}}\}$,

- $[\![0]\!]_{\vec{x}} = \emptyset$,
- $[\![s + S]\!]_{\vec{x}} = [\![s]\!]_{\vec{x}} \cup [\![S]\!]_{\vec{x}}$.

Note that if S is a *closed* differential λ-term then $[\![S]\!] \subseteq D$. Moreover, it is easy to check that $[\![\Omega]\!] = \emptyset$ (actually from [33] we know that the interpretation of all unsolvable ordinary λ-terms is empty). In the next subsection we will prove some general properties of $\mathrm{Th}(\mathscr{D})$.

5.3. An Extensional Model of Taylor Expansion

In [33] we characterized the equational theory of $\mathscr{D}$, seen as a model of the untyped λ-calculus. More precisely we proved that $\mathrm{Th}(\mathscr{D}) = \mathcal{H}^\star$, the theory equating two λ-terms M, N whenever they behave in the same way in every context. This is not surprising since Ehrhard proved in [20] that the continuous semantics [43] can be seen as the extensional collapse of the category **MRel** and that $\mathscr{D}$ corresponds to Scott's $\mathscr{D}_\infty$ under this collapse.

In this subsection we give a partial characterization of the theory of $\mathscr{D}$ seen as a model of the differential λ-calculus.

Remark 5.8 *Given an arbitrary set I and an I-indexed family of relations $\{f_i\}_{i\in I}$ from $\mathcal{M}_f(A)$ to B we have that $\cup_{i\in I} f_i \subseteq \mathcal{M}_f(A) \times B$. In particular,* **MRel** *has countable sums.*

Proposition 5.9 **MRel** *models the Taylor expansion.*

Proof. Let $f \subseteq \mathcal{M}_f(C) \times (\mathcal{M}_f(A) \times B)$ and $g \subseteq \mathcal{M}_f(C) \times A$. Easy calculations give:

$$\begin{array}{rll}
\mathrm{ev} \circ \langle f, g\rangle = & \{(m,\gamma) \mid \exists k \in \mathbf{N} & \\
& \quad \exists m_j \in \mathcal{M}_f(C) & \text{for } j = 0,\dots,k \\
& \quad \exists \alpha_1,\dots,\alpha_k \in A & \text{such that} \\
& \quad m = m_0 \uplus \dots \uplus m_k & \text{for } i = 1,\dots,n \\
& \quad (m_0, ([\alpha_1,\dots,\alpha_k],\gamma)) \in f & \\
& \quad (m_j, \alpha_j) \in g & \text{for } j = 1,\dots,k\} \\
= & \bigcup_{k\in\mathbf{N}}\{(m,\gamma) \mid \exists m_j \in \mathcal{M}_f(C) & \text{for } j = 0,\dots,k \\
& \quad \exists \alpha_1,\dots,\alpha_k \in A & \text{such that} \\
& \quad m = m_0 \uplus \dots \uplus m_k & \text{for } i = 1,\dots,n \\
& \quad (m_0, ([\alpha_1,\dots,\alpha_k],\gamma)) \in f & \\
& \quad (m_j, \alpha_j) \in g & \text{for } j = 1,\dots,k\} \\
= & \sum_{k\in\mathbf{N}}((\cdots(\Lambda^-(f) \underbrace{\star g)\cdots)\star g}_{k \text{ times}}) \circ \langle \mathrm{Id}_A, \emptyset\rangle &
\end{array}$$

■

Corollary 5.10 *Every categorical model $\mathscr{U}$ of the differential λ-calculus living in* **MRel** *satisfies $\mathcal{E} \subseteq \mathrm{Th}(\mathscr{U})$.*

Another easy corollary is that the relational semantics is *incomplete*. We recall that a semantics **C** is called *complete* if for all differential λ-theories $\mathcal{T}$ there is a model $\mathscr{U}$ living in **C** such that $\mathrm{Th}(\mathscr{U}) = \mathcal{T}$. As we know that in **MRel** only theories including $\mathcal{E}$ are representable, it follows that no non-trivial recursively enumerable[5] differential λ-theory is representable in **MRel**, and since there exists a continuum of recursively enumerable differential λ-theories we get the following result.

[5] A differential λ-theory $\mathcal{T}$ is *recursively enumerable* if the $\mathcal{T}$-equivalence class of every differential λ-term is; it is called *trivial* if it equates all differential λ-terms.

Corollary 5.11 *The relational semantics is hugely incomplete: there are $2^{\aleph_0}$ differential λ-theories that are not representable in* **MRel**.

From Corollary 5.10 we get the following (partial) characterization of $\mathrm{Th}(\mathscr{D})$.

Corollary 5.12 *The theory of $\mathscr{D}$ includes both $\lambda\beta\eta^d$ and $\mathcal{E}$.*

These preliminary results and the work in [8] lead us to the following conjecture.

Conjecture 1 *We conjecture that*

$$\mathrm{Th}(\mathscr{D}) = \{(S,T) \in \Lambda^d \times \Lambda^d \mid \textit{ for all contexts } C(\cdot),\ C(S) \textit{ is solvable iff } C(T) \textit{ is solvable } \},$$

where a context is a differential λ-term with a hole denoted by $(\cdot)$, and $C(S)$ denotes the result of substituting S (possibly with capture of variables) for the hole in C. 'Solvable' here has to be intended as may-solvable[6] *(i.e., a sum of terms converges if at least one of its components converges).*

A complete syntactical characterization of the theory of $\mathscr{D}$ is difficult to provide, and it is kept for future works.

5.3.1. A Differentially Extensional but Non-Extensional Relational Model

In this subsection we briefly present an example of a model $\mathscr{E}$ in the category **MRel** satisfying the axiom (η_∂) but not the axiom (η). This model, whose construction is similar to that of $\mathscr{D}$, was first introduced by Hyland et al. in [28] and has been studied by de Carvalho in his PhD thesis (presented as a type system called *System R*, see [17, §6.3.3]).

Let us fix a non-empty set A of "atoms" such that A does not contain any pair. Define a family of sets $\{E_n\}_{n\in\mathbf{N}}$ as follows:

- $E_0 = \emptyset$,
- $E_{n+1} = (\mathcal{M}_f(E_n) \times E_n) \cup A$.

Finally, we set $E = \cup_{n\in\mathbf{N}} E_n$ and $\mathscr{E} = (E, \mathcal{A}, \lambda)$ where $\mathcal{A}, \lambda$ are the obvious morphisms performing the retraction $(E \Rightarrow E) \lhd E$.

Remark 5.13 *It is easily verified that $\mathscr{E}$ is linear, therefore it is a model of the differential λ-calculus, and non-extensional because the atoms in $A \subseteq E$ cannot be sent injectively into $\mathcal{M}_f(E) \times E$.*

As remarked in [28], the model $\mathscr{E}$ is a relational analogue of Engeler's graph-model [25] in the same spirit as $\mathscr{D}$ is the analogue of Scott's $\mathscr{D}_\infty$. The interpretation of a differential λ-term S in $\mathscr{E}$ is defined as usual and gives, up to isomorphism, a subset $[\![S]\!]_{\vec{x}} \subseteq \mathcal{M}_f(E)^n \times E$.

Lemma 5.14 *The model $\mathscr{E}$ is differentially extensional.*

Proof. In $\mathscr{E}$ the interpretation of the linear application does not contain any atom, in the sense that $(\vec{m}, \alpha) \in [\![\mathsf{D}S \cdot T]\!]_{\vec{x}}$ entails $\alpha = (m', \beta)$. Presenting the model as a type system (m', β) would be an arrow type $m' \to \beta$. This guarantees that the η_∂-expansion does not modify the interpretation. ■

[6] May and must solvability have been studied in [38] in the context of the resource calculus.

6. The Resource Calculus

In this section we present the resource calculus [6,7] (using the formalization *à la Tranquilli* given in [39]) and we show that every model of the differential λ-calculus is also a model of the resource calculus. We then discuss the (tight) relationship existing between the differential λ-calculus and the resource calculus.

6.1. Its Syntax

The resource calculus has three syntactical categories: *resource λ-terms* (Λ^r) that are in functional position; *bags* (Λ^b) that are in argument position and represent multisets of resources, and *sums* that represent the possible results of a computation. A *resource* ($\Lambda^{(!)}$) can be linear or reusable, in the latter case it is written with a ! superscript. An *expression* (Λ^e) is either a term or a bag.

Formally, we have the following grammar:

$$\begin{array}{llcll}
\Lambda^r : & M, N, L & ::= & x \mid \lambda x.M \mid MP & \text{resource } \lambda\text{-terms} \\
\Lambda^{(!)} : & M^{(!)}, N^{(!)} & ::= & M \mid M^! & \text{resources} \\
\Lambda^b : & P, Q, R & ::= & [M_1^{(!)}, \dots, M_n^{(!)}] & \text{bags} \\
\Lambda^e : & A, B & ::= & M \mid P & \text{expressions}
\end{array}$$

Hereafter, resource λ-terms are considered up to α-conversion and permutation of resources in the bags. Intuitively, linear resources are available exactly once, while reusable resources can be used zero or many times.

Definition 6.1 *Given an expression $A \in \Lambda^e$ the set* $\mathrm{FV}(A)$ *of* free variables of A *is defined by induction on A as follows:*

- $\mathrm{FV}(x) = \{x\}$,
- $\mathrm{FV}(\lambda x.M) = \mathrm{FV}(M) - \{x\}$,
- $\mathrm{FV}(MP) = \mathrm{FV}(M) \cup \mathrm{FV}(P)$,
- $\mathrm{FV}([]) = \emptyset$,
- $\mathrm{FV}([M^{(!)}] \uplus P) = \mathrm{FV}(M) \cup \mathrm{FV}(P)$.

Given expressions $A_1, \dots, A_k$ we set $\mathrm{FV}(A_1, \dots, A_k) = \mathrm{FV}(A_1) \cup \dots \cup \mathrm{FV}(A_k)$.

Concerning sums, $\mathbf{N}\langle\Lambda^r\rangle$ (resp. $\mathbf{N}\langle\Lambda^b\rangle$) denotes the set of finite formal sums of terms (resp. bags). As usual, we suppose that the sum is commutative and associative, and that 0 is its neutral element.

$$\mathbb{M}, \mathbb{N} \in \mathbf{N}\langle\Lambda^r\rangle \qquad \mathbb{P}, \mathbb{Q} \in \mathbf{N}\langle\Lambda^b\rangle \qquad \mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathbf{N}\langle\Lambda^e\rangle = \mathbf{N}\langle\Lambda^r\rangle \cup \mathbf{N}\langle\Lambda^b\rangle \qquad \text{sums}$$

Note that in writing $\mathbf{N}\langle\Lambda^e\rangle$ we are abusing the notation, as it does not denote the $\mathbf{N}$-module generated over $\Lambda^e = \Lambda^r \cup \Lambda^b$ but rather the union of the two $\mathbf{N}$-modules. In other words, sums must be taken only in the same sort.

The definition of $\mathrm{FV}(\cdot)$ is extended to elements of $\mathbf{N}\langle\Lambda^e\rangle$ in the obvious way.

In the grammar for resource λ-terms, bags and expressions sums do not appear, indeed in this calculus they may arise only on the "surface" (while in the differential λ-calculus sums may appear in the right argument of an application). Nevertheless, as a syntactic sugar and not as actual syntax, we extend all the constructors to sums as follows.

Notation 6.2 *We set the following abbreviations on* $\mathbf{N}\langle\Lambda^e\rangle$.

- $\lambda x.\sum_{i=1}^k M_i = \sum_{i=1}^k \lambda x.M_i$,
- $(\sum_{i=1}^k M_i)(\sum_{j=1}^n P_j) = (\sum_{i,j} M_i P_j)$,
- $[(\sum_{i=1}^k M_i)] \uplus P = \sum_{i=1}^k [M_i] \uplus P$,
- $[(\sum_{i=1}^k M_i)^!] \uplus P = [M_1^!, \dots, M_k^!] \uplus P$.

These equalities make sense since all constructors, but the $(\cdot)^!$, are linear. Notice the difference between these rules and the analogous ones for the differential λ-calculus introduced in Notation 2.4. In the differential λ-calculus the application operator is only linear in its left component while here it is bilinear.

The 0-ary version of the above equalities give us $\lambda x.0 = 0$, $\mathbb{M}0 = 0$, $0\mathbb{P} = 0$, $[0] \uplus P = 0$, $[0^!] \uplus P = P$ and $0 \uplus P = 0$. Therefore 0 annihilates everything except when it lies under a $(\cdot)^!$.

Definition 6.3 *Let A be an expression and N be a resource λ-term.*

- $A\{N/x\}$ *is the usual substitution of N for x in A. It is extended to sums as in* $\mathbb{A}\{\mathbb{N}/x\}$ *by linearity*[7] *in* $\mathbb{A}$, *and using Notation 6.2 for* $\mathbb{N}$.
- $A\langle N/x\rangle$ *is the* linear substitution *defined inductively as follows:*

$$\begin{array}{ll} y\langle N/x\rangle = \begin{cases} N & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} & \begin{array}{l}(\lambda y.M)\langle N/x\rangle = \lambda y.M\langle N/x\rangle \\ (MP)\langle N/x\rangle = M\langle N/x\rangle P + M(P\langle N/x\rangle)\end{array} \\ [M]\langle N/x\rangle = [M\langle N/x\rangle] & []\langle N/x\rangle = 0 \\ [M^!]\langle N/x\rangle = [M\langle N/x\rangle, M^!] & (P \uplus R)\langle N/x\rangle = P\langle N/x\rangle \uplus R + P \uplus R\langle N/x\rangle \end{array}$$

It is extended to $\mathbb{A}\langle\mathbb{N}/x\rangle$ *by bilinearity*[8] *in both* $\mathbb{A}$ *and* $\mathbb{N}$.

The operation $M\langle N/x\rangle$ on resource λ-terms is roughly equivalent to the operation $\frac{\partial S}{\partial x}\cdot T$ on differential λ-terms (cf. Lemma 6.11 below). Notice that in defining $[M^!]\langle N/x\rangle$ we morally extract a linear copy of M from the infinitely many represented by $M^!$, that receives the substitution, and we keep the other ones unchanged.

Example 6.4

1. $x\langle M/x\rangle = M$ *and* $y\langle M/x\rangle = 0$,
2. $(x[x])\langle M+N/x\rangle = (M+N)[x] + x[M+N] = M[x] + N[x] + x[M] + x[N]$,
3. $(x[x^!])\langle M+N/x\rangle = (M+N)[x^!] + x[(M+N), x^!] = M[x^!] + N[x^!] + x[M, x^!] + x[N, x^!]$,
4. $(x[x^!])\{M+N/x\} = (M+N)[(M+N)^!] = M[M^!, N^!] + N[M^!, N^!]$.

As a matter of notation, we will write $\vec{L}$ for $L_1, \dots, L_k$ and $\vec{N}^!$ for $N_1^!, \dots, N_n^!$. We will also abbreviate $M\langle L_1/x\rangle \cdots \langle L_k/x\rangle$ in $M\langle\vec{L}/x\rangle$. Moreover, given a sequence $\vec{L}$ and an index $1 \le i \le k$ we will write $\vec{L}_{-i}$ for $L_1, \dots, L_{i-1}, L_{i+1}, \dots, L_k$.

Remark 6.5 *Every applicative resource λ-term MP can be written in a unique way as* $M[\vec{L}, \vec{N}^!]$.

[7] A unary operator $F(\cdot)$ is extended by linearity by setting $F(\Sigma_i A_i) = \Sigma_i F(A_i)$.
[8] A binary operator $F(\cdot,\cdot)$ is extended by bilinearity by setting $F(\Sigma_i A_i, \Sigma_j B_j) = \Sigma_{i,j} F(A_i, B_j)$.

6.2. Resource Lambda Theories

We now define the equational theories of the resource calculus, namely the *resource λ-theories*. To begin with, we present the main axiom associated with this calculus (for $\vec{L} = L_1, \dots, L_k$ and $\vec{N} = N_1, \dots, N_n$):

$$(\beta^r) \quad (\lambda x.M)[\vec{L}, \vec{N}^!] = M\langle \vec{L}/x\rangle\{\Sigma_{i=1}^n N_i/x\}$$

Notice that, when $n = 0$, this rule becomes $(\lambda x.M)[\vec{L}] = M\langle\vec{L}/x\rangle\{0/x\}$. Once oriented from left to right, the (β^r)-conversion expresses the way of calculating a function $\lambda x.M$ applied to a bag containing linear resources $\vec{L}$ and reusable resources $\vec{N}$.

Remark 6.6 *The left-to-right oriented version of (β^r) corresponds to the* giant-step *reduction, in the terminology of [39]. In the same paper the authors also consider a* baby-step *reduction rule. They prove that both reductions are confluent and that every giant-step can be emulated by several baby-steps. For our purposes we can consider the rule (β^r) without loss of generality, because both reductions generate the same equational theory.*

In the resource calculus the axiom equating all resource λ-terms having the same extensional behaviour has the shape:

$$(\eta^r) \quad \lambda x.M[x^!] = M, \text{ where } x \notin \mathrm{FV}(M).$$

In this context the axiom (η_∂) of the differential λ-calculus has no analogue since the application of a resource λ-term to a bag morally corresponds to a sequence of linear applications always followed by a classic application (see Definition 6.10, below). Therefore the linear application where (η_∂) should act is hidden.

The resource calculus can be seen as a proper extension of the classic λ-calculus.

Remark 6.7 *The classic λ-calculus can be easily injected within the resource calculus. Indeed, given an ordinary λ-term M, it is sufficient to translate every subterm of M of shape PQ into $P[Q^!]$. In this restricted system, the rules (β^r) and (η^r) are completely equivalent to the classic (β) and (η)-conversions, respectively.*

We now define the equational theories associated with this calculus, namely the *resource λ-theories.*

A *λ^r-relation* $\mathcal{R}$ is any set of equations between sums of resource λ-terms (resp. bags). Thus $\mathcal{R}$ can be thought of as a binary relation on $\mathbf{N}\langle\Lambda^e\rangle$.

A λ^r-relation $\mathcal{R}$ is called:

- an *equivalence* if it is closed under the following rules (for all $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathbf{N}\langle\Lambda^e\rangle$):

$$\frac{}{\mathbb{A} = \mathbb{A}}\ \text{reflexivity} \qquad \frac{\mathbb{B} = \mathbb{A}}{\mathbb{A} = \mathbb{B}}\ \text{symmetry} \qquad \frac{\mathbb{A} = \mathbb{B} \quad \mathbb{B} = \mathbb{C}}{\mathbb{A} = \mathbb{C}}\ \text{transitivity}$$

- *compatible* if it is closed under the following structural rules (for all $\mathbb{M}, \mathbb{N}, \mathbb{M}_i, \mathbb{N}_i \in \mathbf{N}\langle\Lambda^r\rangle$ and $\mathbb{P}, \mathbb{Q} \in \mathbf{N}\langle\Lambda^b\rangle$):

$$\frac{\mathbb{M} = \mathbb{N}}{\lambda x.\mathbb{M} = \lambda x.\mathbb{N}}\ \text{lambda} \qquad \frac{\mathbb{M} = \mathbb{N} \quad \mathbb{Q} = \mathbb{P}}{\mathbb{M}\mathbb{P} = \mathbb{N}\mathbb{Q}}\ \text{app}$$

$$\frac{\mathbb{M} = \mathbb{N} \quad \mathbb{P} = \mathbb{Q}}{[\mathbb{M}^{(!)}] \uplus \mathbb{P} = [\mathbb{N}^{(!)}] \uplus \mathbb{Q}}\ \text{bag} \qquad \frac{\mathbb{M}_i = \mathbb{N}_i \quad \text{for all } 1 \le i \le n}{\sum_{i=1}^n \mathbb{M}_i = \sum_{i=1}^n \mathbb{N}_i}\ \text{sum}$$

As a matter of notation, we will write $\mathcal{R} \vdash \mathbb{M} = \mathbb{N}$ or $\mathbb{M} =_{\mathcal{R}} \mathbb{N}$ for $\mathbb{M} = \mathbb{N} \in \mathcal{R}$.

Definition 6.8 *A* resource λ-theory *is any compatible* λ^r*-relation* $\mathcal{R}$ *which is an equivalence relation and includes* (β^r)*.* $\mathcal{R}$ *is called* extensional *if it also contains* (η^r)*. We say that* $\mathcal{R}$ satisfies sum idempotency *whenever* $\mathcal{R} \vdash M + M = M$*.*

We denote by $\lambda\beta^r$ (resp. $\lambda\beta\eta^r$) the minimum resource λ-theory (resp. the minimum extensional resource λ-theory).

Example 6.9

1. $\lambda\beta^r \vdash (\lambda x.x[x])[\mathbf{I}] = 0$, $\lambda\beta^r \vdash (\lambda x.x[x])[\mathbf{I}, \mathbf{I}] = \mathbf{I}$ *and* $\lambda\beta^r \vdash (\lambda x.x[x])[\mathbf{I}, \mathbf{I}, \mathbf{I}] = 0$,
2. $\lambda\beta^r \vdash (\lambda x.x[x])[M, N] = M[N] + N[M]$,
3. $\lambda\beta^r \vdash (\lambda x.x[x, x])[(\lambda y.y[y^!])^!] = (\lambda x.x[x^!])[\lambda y.y[y^!], \lambda z.z[z^!]] = 2(\lambda y.y[y^!])[(\lambda z.z[z^!])^!]$,
4. $\lambda\beta\eta^r \vdash (\lambda xz.y[y][z^!])[] = \lambda z.y[y][z^!] = y[y]$.

6.3. From the Resource to the Differential Lambda Calculus...

In this subsection we show that every linear reflexive object living in a Cartesian closed differential category is also a sound model of the untyped resource calculus. This result is achieved by first translating the resource calculus in the differential λ-calculus, and then applying the machinery of Section 4.

Definition 6.10 *The resource calculus can be easily translated into the differential λ-calculus as follows:*

- $x^d = x$,
- $(\lambda x.M)^d = \lambda x.M^d$,
- $(M[L_1, \ldots, L_k, N_1^!, \ldots, N_n^!])^d = (\mathsf{D}^k M^d \cdot (L_1^d, \ldots, L_k^d))(\Sigma_{i=1}^n N_i^d)$.

The translation is then extended to elements in $\mathbf{N}\langle\Lambda^r\rangle$ *by setting* $(\Sigma_{i=1}^n M_i)^d = \Sigma_{i=1}^n M_i^d$*.*

The next lemma shows that this translation behaves well with respect to the differential and the usual substitution.

Lemma 6.11 *Let* $M, N \in \Lambda^r$ *and* x *be a variable. Then:*

(i) $(M\langle N/x\rangle)^d = \frac{\partial M^d}{\partial x} \cdot N^d$,

(ii) $(M\{N/x\})^d = M^d\{N^d/x\}$.

Proof. (i) By structural induction on M. The only difficult case is $M \equiv M'[\vec{L}, \vec{N}^!]$. By definition of $(-)^d$ and of linear substitution we have:

$$((M'[\vec{L}, \vec{N}^!])\langle N/x\rangle)^d = (M'\langle N/x\rangle[\vec{L}, \vec{N}^!])^d + (M'([\vec{L}, \vec{N}^!]\langle N/x\rangle))^d =$$
$$\underbrace{(M'\langle N/x\rangle[\vec{L}, \vec{N}^!])^d}_{(1)} + \underbrace{(\Sigma_{j=1}^k M'[L_j\langle N/x\rangle, \vec{L}_{-j}, \vec{N}^!])^d}_{(2)} + \underbrace{(\Sigma_{i=1}^n M'[N_i\langle N/x\rangle, \vec{L}, \vec{N}^!])^d}_{(3)}.$$

Let us consider the three summands separately.

(1) By definition of $(-)^d$ we have that $(M'\langle N/x\rangle[\vec{L},\vec{N}^!])^d = (\mathsf{D}^k(M'\langle N/x\rangle)^d\cdot(\vec{L}^d))(\Sigma_{i=1}^n N_i^d)$. By applying the induction hypothesis, this is equal to $(\mathsf{D}^k(\frac{\partial (M')^d}{\partial x}\cdot N^d)\cdot(\vec{L}^d))(\Sigma_{i=1}^n N_i^d)$.

(2) By definition of the translation map $(-)^d$ we have that $(\Sigma_{j=1}^k M'[L_j\langle N/x\rangle,\vec{L}_{-j},\vec{N}^!])^d = \Sigma_{j=1}^k(\mathsf{D}^{k-1}(\mathsf{D}(M')^d\cdot(L_j\langle N/x\rangle)^d)\cdot(\vec{L}^d_{-j}))(\Sigma_{i=1}^n N_i^d)$. By applying the induction hypothesis, this is equal to $\Sigma_{j=1}^k(\mathsf{D}^{k-1}(\mathsf{D}(M')^d\cdot(\frac{\partial L_j^d}{\partial x}\cdot N^d))\cdot(\vec{L}^d_{-j}))(\Sigma_{i=1}^n N_i^d)$.

(3) By definition of $(-)^d$ we have $(\Sigma_{j=1}^n M'[N_j\langle N/x\rangle,\vec{L},\vec{N}^!])^d = \Sigma_{j=1}^n(M'[N_j\langle N/x\rangle,\vec{L},\vec{N}^!])^d = \Sigma_{j=1}^n(\mathsf{D}^k(\mathsf{D}(M')^d\cdot(N_j\langle N/x\rangle)^d)\cdot(\vec{L}^d))(\Sigma_{i=1}^n N_i^d)$. By applying the induction hypothesis, this is equal to $\Sigma_{j=1}^n(\mathsf{D}^k(\mathsf{D}(M')^d\cdot(\frac{\partial N_j^d}{\partial x}\cdot N^d))\cdot(\vec{L}^d))(\Sigma_{i=1}^n N_i^d)$. By permutative equality this is equal to $\Sigma_{j=1}^n(\mathsf{D}(\mathsf{D}^k(M')^d\cdot(\vec{L}^d))\cdot(\frac{\partial N_j^d}{\partial x}\cdot N^d))(\Sigma_{i=1}^n N_i^d)$.

To conclude the proof it is sufficient to verify that $\frac{\partial}{\partial x}((\mathsf{D}^k(M')^d\cdot(\vec{L}^d))(\Sigma_{i=1}^n N_i^d))\cdot N^d$ is equal to the sum of (1), (2) and (3).

(ii) By straightforward induction on M. ■

The translation $(\cdot)^d$ is 'faithful' in the sense expressed by the next proposition.

Proposition 6.12 *For all $M\in\Lambda^r$ we have that $\lambda\beta^r\vdash M=N$ implies $\lambda\beta^d\vdash M^d=N^d$.*

Proof. It is easy to check that the proposition holds for the contextual rules.

Suppose then that $\lambda\beta^r\vdash M=N$ because $M\equiv(\lambda x.M')[\vec{L},\vec{N}^!]$ and $N\equiv M'\langle\vec{L}/x\rangle\{\Sigma_{i=1}^n N_i/x\}$. By definition of the map $(-)^d$ we have $((\lambda x.M')[\vec{L},\vec{N}^!])^d = (\mathsf{D}^k(\lambda x.(M')^d)\cdot(\vec{L}^d))(\Sigma_{i=1}^n N_i^d) =_{\lambda\beta^d} (\lambda x.\frac{\partial^k (M')^d}{\partial x,\dots,x}\cdot(\vec{L}^d))(\Sigma_{i=1}^n N_i^d) =_{\lambda\beta^d} (\frac{\partial^k (M')^d}{\partial x,\dots,x}\cdot(\vec{L}^d))\{\Sigma_{i=1}^n N_i^d/x\}$ which is equal to N^d by Lemma 6.11. ■

Remark 6.13 *The two results above generalize straightforwardly to sums of resource λ-terms (*i.e.*, to elements $\mathbb{M}\in\mathbf{N}\langle\Lambda^r\rangle$).*

6.3.1. Interpreting the Resource Calculus by Translation

Given a linear reflexive object $\mathscr{U}$ living in a Cartesian closed differential category $\mathbf{C}$ it is possible to interpret resource λ-terms trough their translation $(-)^d$. Indeed, it is sufficient to set

$$[\![M]\!]_{\vec{x}} = [\![M^d]\!]_{\vec{x}} : U^n\to U.$$

From this fact, Proposition 6.12 and Remark 6.13 it follows that $\mathscr{U}$ is a sound model of the untyped resource calculus.

Remark 6.14 *If $\mathscr{U}$ is an extensional model of the differential λ-calculus, then it is also an extensional model of the resource calculus. Indeed $[\![(\lambda x.M[x^!])^d]\!]_{\vec{x}} = [\![\lambda x.M^d x]\!]_{\vec{x}} = [\![M^d]\!]_{\vec{x}}$.*

For the resource calculus we are able to prove a completeness result stronger than the one for the differential λ-calculus. More precisely we can get rid of the hypothesis that the theory is differentially extensional. Indeed for every resource λ-theory $\mathcal{R}$ the differential λ-theory $\mathcal{T}$ generated[9] by

$$\{S=T\mid \exists\mathbb{M},\mathbb{N}\in\mathbf{N}\langle\Lambda^r\rangle\ S=M^d,T=N^d,\mathcal{R}\vdash M=N\}$$

is such that $\mathcal{R}\vdash\mathbb{M}=\mathbb{N}$ if and only if $\mathcal{T}\vdash\mathbb{M}^d=\mathbb{N}^d$. When $\mathcal{R}$ satisfies sum idempotency also $\mathcal{T}$ does, thus we can apply the construction described in Subsection 4.4, and get a Cartesian

[9]The differential λ-theory generated by a set E of equations is the smallest differential λ-theory including E.

closed differential category $\mathbf{C}_{\mathcal{T}}$ where $\mathbf{I}$ is a linear reflexive object. Then one can prove the following lemma, which is similar to Proposition 4.19 except that the axiom η_∂ does not play a role anymore. This is due to the fact that in the translation of the resource calculus the linear application is always followed by a regular application, therefore the η_∂-expansion disappears by (β^r)-conversion.

Lemma 6.15 *For every* $\mathbb{M} \in \mathbf{N}\langle\Lambda^r\rangle$ *we have (for some* $z \notin \mathrm{FV}(\mathbb{M})$*):*

$$[\![\mathbb{M}]\!]_{\vec{x}} = \lambda z.\mathbb{M}^d\{\pi^{\vec{x}}_{x_1}z/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}z/x_n\} : \mathbf{I}^{\vec{x}} \to \mathbf{I}.$$

Proof. The only interesting case is $\mathbb{M} = M[\vec{L}, \vec{N}^!]$. In the following we denote by σ_r the sequence of substitutions $\{\pi^{\vec{x}}_{x_1}r/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}r/x_n\}$. We have:

$$\begin{array}{lll}
[\![M[\vec{L},\vec{N}^!]]\!]_{\vec{x}} & = [\![(\mathsf{D}^k M^d\cdot(\vec{L}^d))(\Sigma_i N_i^d)]\!]_{\vec{x}} & \text{by definition of } (\cdot)^d \\
& = \mathrm{ev}\circ\langle \mathbf{1}\circ[\![(\mathsf{D}^k M^d\cdot(\vec{L}^d)]\!]_{\vec{x}}), [\![\Sigma_i N_i^d]\!]_{\vec{x}}\rangle & \text{by definition of } [\![\cdot]\!]_{\vec{x}} \\
& = \mathrm{ev}\circ\langle \lambda ry.(\mathsf{D}^k([\![M^d]\!]_{\vec{x}}r)\cdot([\![\vec{L}]\!]_{\vec{x}}r))y, \Sigma_i[\![N_i^d]\!]_{\vec{x}}\rangle & \text{by calculations} \\
& = \mathrm{ev}\circ\langle \lambda ry.(\mathsf{D}^k(M^d\sigma_r)\cdot(\vec{L}^d\sigma_r))y, \Sigma_i\lambda r.N_i^d\sigma_r\rangle & \text{by induction hypothesis} \\
& = \lambda z.((\lambda ry.(\mathsf{D}^k(M^d\sigma_r)\cdot(\vec{L}^d\sigma_r))y)z)(\Sigma_i(\lambda r.N_i^d\sigma_r)z) & \text{by calculations} \\
& = \lambda z.(\lambda y.(\mathsf{D}^k(M^d\sigma_z)\cdot(\vec{L}^d\sigma_z))y)(\Sigma_i N_i^d\sigma_z) & \text{by } (\beta^r)\text{-conversion} \\
& = \lambda z.(\mathsf{D}^k(M^d\sigma_z)\cdot(\vec{L}^d\sigma_z))(\Sigma_i N_i^d\sigma_z) & \text{by } (\beta^r)\text{-conversion} \\
& = \lambda z.(M[\vec{L},\vec{N}^!])^d\{\pi^{\vec{x}}_{x_1}z/x_1\}\cdots\{\pi^{\vec{x}}_{x_n}z/x_n\} & \text{by definition of } (\cdot)^d
\end{array}$$

■

As a corollary we get the equational completeness for the resource calculus.

Corollary 6.16 (Equational Completeness) *Every resource λ-theory* $\mathcal{R}$ *satisfying sum idempotency is the theory of a linear reflexive object in a differential Cartesian closed category.*

6.4. And Back...

In this subsection we define a translation from the differential to the resource calculus. This translation is more tricky because in the differential λ-calculus the result of the linear application $\mathsf{D}(\lambda x.s)\cdot t$ mantains the lambda abstraction (since it waits for other arguments that may substitute the remaining occurrences of x in s), while the naïvely corresponding resource λ-term $(\lambda x.M)[N]$ does erase it (since all other free occurrences of x in M are substituted by 0).

Definition 6.17 *The differential λ-calculus can be translated into the resource calculus as follows:*

$$\begin{array}{l}
x^r = x, \\
(\lambda x.s)^r = \lambda x.s^r, \\
(sT)^r = s^r[(T^r)^!], \\
(\mathsf{D}^k s\cdot(t_1,\ldots,t_k))^r = \lambda y.s^r[t_1^r,\ldots,t_k^r,y^!], \ \textit{where } y \textit{ is a fresh variable}, \\
(s+S)^r = s^r + S^r.
\end{array}$$

Notice that while the shape of the term $\lambda y.s^r[t_1^r,\ldots,t_k^r,y^!]$ looks similar to an (η^r)-expansion of $s^r[t_1^r,\ldots,t_k^r]$, it is not[10]! Indeed, in the (η^r)-axiom, $y^!$ is supposed to be in a singleton bag.

Lemma 6.18 *Let* $S, T \in \Lambda^d$ *and* x *be a variable. Then:*

(i) $(\frac{\partial S}{\partial x}\cdot T)^r = S^r\langle T^r/x\rangle$,

[10] However, a connection with (η_∂)-conversion can be found in Proposition 6.20(iii).

(ii) $(S\{T/x\})^r = S^r\{T^r/x\}$.

Proof. (i) By structural induction on S. If S is a variable, a lambda abstraction or a sum, the lemma follows straight from the induction hypothesis.

- case $S \equiv \mathsf{D}^k s \cdot (t_1, \dots, t_k)$. We have:

$$
\begin{aligned}
&(\tfrac{\partial}{\partial x}(\mathsf{D}^k s \cdot (t_1, \dots, t_k)) \cdot T)^r = \\
&\quad = \Sigma_{i=1}^k ((\mathsf{D}^k s \cdot (t_1, \dots, \tfrac{\partial t_i}{\partial x} \cdot T, \dots, t_k)))^r \\
&\qquad + ((\mathsf{D}^k (\tfrac{\partial s}{\partial x} \cdot T) \cdot (t_1, \dots, t_k)))^r && \text{by def. of } \tfrac{\partial (\cdot)}{\partial x} \cdot T \\
&\quad = \Sigma_{i=1}^k \lambda y. s^r [t_1^r, \dots, (\tfrac{\partial t_i}{\partial x} \cdot T)^r, \dots, t_k^r, y^!] \\
&\qquad + \lambda y. (\tfrac{\partial s}{\partial x} \cdot T)^r [t_1^r, \dots, t_k^r, y^!] && \text{by def. of } (\cdot)^r \\
&\quad = \Sigma_{i=1}^k \lambda y. s^r [t_1^r, \dots, t_i^r \langle T^r/x \rangle, \dots, t_k^r, y^!] \\
&\qquad + \lambda y. (s^r \langle T^r/x \rangle)[t_1^r, \dots, t_k^r, y^!] && \text{by induction hypothesis} \\
&\quad = (\lambda y. s^r [t_1^r, \dots, t_k^r, y^!]) \langle T^r/x \rangle && \text{by def. of } \langle T^r/x \rangle \\
&\quad = (\mathsf{D}^k s \cdot (t_1, \dots, t_k))^r \langle T^r/x \rangle && \text{by def. of } (\cdot)^r
\end{aligned}
$$

- case $S \equiv sU$. By definition, we have $(\frac{\partial (sU)}{\partial x} \cdot T)^r = ((\frac{\partial s}{\partial x} \cdot T)U + (\mathsf{D} s \cdot (\frac{\partial U}{\partial x} \cdot T))U)^r = ((\frac{\partial s}{\partial x} \cdot T)U)^r + ((\mathsf{D} s \cdot (\frac{\partial U}{\partial x} \cdot T))U)^r = (\frac{\partial s}{\partial x} \cdot T)^r[(U^r)^!] + (\lambda y. s^r[(\frac{\partial U}{\partial x} \cdot T)^r, y^!])[(U^r)^!]$. By induction hypothesis this is equal to $(s^r\langle T^r/x \rangle)[(U^r)^!] + (\lambda y. s^r[U^r\langle T^r/x \rangle, y^!])[(U^r)^!]$. By β-conversion this is equal to $(s^r\langle T^r/x \rangle)[(U^r)^!] + s^r[U^r\langle T^r/x \rangle, (U^r)^!]$. By definition of linear substitution this is $(s^r[(U^r)^!])\langle T^r/x \rangle = (sU)^r\langle T^r/x \rangle$.

(ii) By straightforward induction on S. ■

The next proposition shows that also the translation $(\cdot)^r$ is faithful.

Proposition 6.19 *For all $S, T \in \Lambda^d$ we have that $\lambda\beta^d \vdash S = T$ implies $\lambda\beta^r \vdash S^r = T^r$.*

Proof. It is easy to check that the proposition holds for the contextual rules.

Suppose that $\lambda\beta^d \vdash S = T$ holds because $S \equiv \mathsf{D}^k(\lambda x.s) \cdot (u_1, \dots, u_k)$ and $T \equiv \lambda x. \frac{\partial^k s}{\partial x, \dots, x} \cdot (u_1, \dots, u_k)$. Then we have

$$
\begin{array}{rcll}
S^r & = & \lambda y. (\lambda x. s^r)[u_1^r, \dots, u_k^r, y^!] & \text{by def. of } (\cdot)^r \\
 & =_{\lambda\beta^r} & \lambda y. s^r \langle u_1^r/x \rangle \cdots \langle u_k^r/x \rangle \{y/x\} & \text{by } \beta^r\text{-conversion} \\
 & \equiv & \lambda x. s^r \langle u_1^r/x \rangle \cdots \langle u_k^r/x \rangle & \text{by } \alpha\text{-conversion} \\
 & = & \lambda x. \big(\frac{\partial^k s}{\partial x, \dots, x} \cdot (u_1, \dots, u_k) \big)^r & \text{by Lemma 6.18(i)} \\
 & = & T^r & \text{by def. of } (\cdot)^r
\end{array}
$$

■

The two translations $(\cdot)^d$ and $(\cdot)^r$ are not exactly one the inverse of the other one. The next proposition presents the properties that they do satisfy, which are summarized in Figure 1 in terms of retractions and isomorphisms between the two calculi.

Proposition 6.20 *The translations $(\cdot)^d$ and $(\cdot)^r$ enjoy the following properties:*

(i) $(s^r)^d \equiv s$*, for all usual λ-terms s,*

(ii) $(S^r)^d \not\equiv S$ *and* $(\mathbb{M}^d)^r \not\equiv \mathbb{M}$*, for some $S \in \Lambda^d$ and $\mathbb{M} \in \mathbf{N}\langle \Lambda^r \rangle$,*

(iii) $\lambda\beta\eta_\partial^d \vdash (S^r)^d = S$*, for all $S \in \Lambda^d$,*

The differential λ-calculus		The resource calculus
with $=_{\lambda\beta\eta^d_\partial}$	$\lhd$	with $=_{\lambda\beta^r}$
with $=_{\lambda\beta^d}$	$\rhd$	with $=_{\lambda\beta^r}$
with $=_{\lambda\beta\eta^d}$	$\cong$	with $=_{\lambda\beta\eta^r}$

Figure 1: Relationships between the differential and resource calculus.

(iv) $\lambda\beta^r \vdash (\mathbb{M}^d)^r = \mathbb{M}$, *for all* $\mathbb{M} \in \mathbf{N}\langle\Lambda^r\rangle$.

Proof. (i) By straightforward induction on the structure of s.

(ii) For instance $((\mathsf{D}x \cdot x)^r)^d = (\lambda y.x[x, y^!])^d = \lambda y.(\mathsf{D}x \cdot x)y \not\equiv \mathsf{D}x \cdot x$. On the other hand we have $((x[L])^d)^r = ((\mathsf{D}x \cdot y)0)^r = (\lambda z.x[y, z^!])0 \not\equiv x[L]$.

(iii) By induction on the structure of S.

- case $S \equiv \mathsf{D}^k s \cdot (t_1, \ldots, t_k)$. By definition of $(\cdot)^r$ we have that $((\mathsf{D}^k s \cdot (t_1, \ldots, t_k))^r)^d$ is equal to $(\lambda y.s^r[t_1^r, \cdots, t_k^r, y^!])^d = \lambda y.(\mathsf{D}^k (s^r)^d \cdot ((t_1^r)^d, \cdots, (t_k^r)^d))y$. By induction hypothesis we have $(s^r)^d =_{\lambda\beta\eta^d_\partial} s$ and $(t_i^r)^d =_{\lambda\beta\eta^d_\partial} t_i$ for all $1 \le i \le k$. Therefore we get $\lambda y.(\mathsf{D}^k (s^r)^d \cdot ((t_1^r)^d, \cdots, (t_k^r)^d))y =_{\lambda\beta\eta^d_\partial} \lambda y.(\mathsf{D}^k s \cdot (t_1, \ldots, t_k))y =_{\lambda\beta\eta^d_\partial} \mathsf{D}^k s \cdot (t_1, \ldots, t_k)$.

- case $S \equiv sT$. We have $((sT)^r)^d = (s^r[(T^r)^!])^d = (s^r)^d(T^r)^d$. By induction hypothesis, we know that $(s^r)^d =_{\lambda\beta\eta^d_\partial} s$ and $(T^r)^d =_{\lambda\beta\eta^d_\partial} T$, thus we conclude $(s^r)^d(T^r)^d =_{\lambda\beta\eta^d_\partial} sT$.

- All other cases are trivial.

(iv) By induction on the structure of $\mathbb{M}$. The only interesting case is $\mathbb{M} \equiv M[\vec{L}, \vec{N}^!]$. We have $((M[\vec{L}, \vec{N}^!])^d)^r = ((\mathsf{D}^k M^d \cdot (\vec{L}^d))(\Sigma_{i=1}^n N_i^d))^r = (\lambda y.(M^d)^r[(\vec{L}^d)^r, y^!])[((\vec{N}^d)^r)^!]$. By induction hypothesis we know that $(M^d)^r =_{\lambda\beta^r} M$, $(L_j^d)^r =_{\lambda\beta^r} L_j$ and $(N_i^d)^r =_{\lambda\beta^r} N_i$, thus $(\lambda y.(M^d)^r[(\vec{L}^d)^r, y^!])[((\vec{N}^d)^r)^!] =_{\lambda\beta^r} (\lambda y.M[\vec{L}, y^!])[(\vec{N})^!]$. Since $y \notin \mathrm{FV}(M, \vec{L})$ we have that $(\lambda y.M[\vec{L}, y^!])[(\vec{N})^!] =_{\lambda\beta^r} M[\vec{L}, \vec{N}^!]$. ■

7. Discussion, Further Works and Related Works

In this paper we proposed a general categorical definition of model of the untyped differential λ-calculus, namely the notion of linear reflexive object living in a Cartesian closed differential category. We have proved that this notion of model is: (i) sound, i.e. the equational theory induced by a model is actually a differential λ-theory; (ii) inhabited, indeed we gave concrete examples of such a definition namely the models $\mathscr{D}$ and $\mathscr{E}$ living in **MRel** and all the syntactic models built trough the revised Scott-Koymans' construction; (iii) equationally complete, provided that we restrict to differentially extensional differential λ-theories satisfying sum idempotency.

Finally, we have shown that the equational theories of the differential λ-calculus and of the resource calculus are tightly connected. Formally, we have provided faithful translations between the two calculi, thus showing that they share the same notion of model. In particular, this shows that linear reflexive objects in Cartesian closed differential categories are also sound models of the untyped resource calculus. For the resource calculus we were able to prove an even stronger equational completeness theorem, in the sense that it holds for all resource λ-theories satisfying sum idempotency.

7.1. Related Works

This paper is morally a continuation of the work on (Cartesian) differential categories done in [4,5] and can be considered as a long version of [12]. Note however that all the calculi under consideration in those papers were simply typed. Moreover, our aim here was to find a suitable notion of semantics for Ehrhard and Regnier's differential λ-calculus (thus supposing the calculus as given), while in [4,5] the goal was to provide a categorical axiomatization of a differential operator and then find a calculus (namely, the term logic) that suits the categories under consideration. In particular, the differential calculus presented in [5] by Blute, Cockett and Seely was slightly different from Ehrhard and Regnier's differential λ-calculus in some key points.

On the one side, the calculus defined in [5] has no λ-abstraction, then it is not an extension of λ-calculus, on the other side it has explicit substitutions and constructors for the pairing, the projections and every n-ary function. Also the treatment of differentiation is different — in the Leibniz-style approach of [5] the notation for differentiation becomes

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash s : A \quad \Gamma \vdash u : A}{\Gamma \vdash \frac{\mathsf{d}t}{\mathsf{d}x}(s) \cdot u : B}\ (\partial)$$

where the variable x is bound in t. Hence differential terms are built using the binder $\frac{\mathsf{d}(\cdot)}{\mathsf{d}x}$. Intuitively $\frac{\mathsf{d}t}{\mathsf{d}x}(s)$ denotes the derivative of t at s[11] and determines a linear transformation, so that it could be typed as $\frac{\mathsf{d}t}{\mathsf{d}x}(s) : A \multimap B$, while u is the point where the derivative is calculated. The lack of λ-abstraction in this system is not a true difference because λ-terms could be added without problems. Besides the usual equations from λ-calculus, one should just add

$$\lambda x.(s + t) = \lambda x.s + \lambda x.t \qquad \frac{\mathsf{d}(\lambda y.s)}{\mathsf{d}x}(t) \cdot u = \lambda y.\frac{\mathsf{d}s}{\mathsf{d}x}(t) \cdot u$$

and the resulting system is conjectured to have linear reflexive objects living in Cartesian closed differential categories as sound and complete models. To better understand it, we sketch the translation from the differential λ-calculus into this system:

$$(\mathsf{D}\, s \cdot t)^\circ = \lambda x_0.\Big(\frac{\mathsf{d}(s^\circ x)}{\mathsf{d}x}(x_0) \cdot t^\circ\Big), \qquad \big(\tfrac{\partial s}{\partial x} \cdot t\big)^\circ = \frac{\mathsf{d}s^\circ}{\mathsf{d}x}(x) \cdot t^\circ.$$

where x_0 is some fresh variable. This calculus is certainly more standard from a mathematical point of view, while we think the differential and resource calculi are more standard from a computer scientist point of view. We believe that to obtain a completeness result in the simply typed setting the choice of the language in [5] would be more promising; on the other hand in the untyped case the language would suffer the same problems we encountered in Theorem 4.20, namely the completeness only for differentially extensional theories satisfying sum idempotency. Finally, once stripped the calculus of types and constructors (since in this paper we are interested in the pure untyped setting) it becomes quite similar to the differential λ-calculus. For all these reasons we decided not to analyze the calculus of [5] further.

7.2. Other Examples of Cartesian Closed Differential Categories

In Section 5 we have presented **MRel** (and cited **MFin** in Remark 5.1) as an instance of the definition of Cartesian closed differential category. We briefly discuss here other examples of such categories that have been recently defined in the literature.

[11] I.e., the Jacobian matrix.

Actually *game semantics* is an inexhaustible source of differential categories, indeed resource usage is represented rather explicitly in games and strategies. In collaboration with Laird and McCusker we have shown in [31] that the games model $\mathbf{G}^{\otimes}$ of Idealized Algol with non-determinism introduced in [27] contains a (definable) differential operator giving it a structure of Cartesian closed differential category. The category $\mathbf{G}^{\otimes}$ is cpo-enriched, has arenas as objects and suitable non-deterministic strategies as morphisms. Intuitively, this category is additive since non-deterministic strategies are closed under union and the linearity of a strategy on a certain component is captured by the fact that the strategy plays *exactly once* in that component.

Moreover in [31] we provided a general categorical construction for building differential categories. Its key step takes a symmetric monoidal category with countable biproducts, embeds it in its *Karoubi envelope* and then constructs the *cofree cocommutative comonoid* on this category (following the recipe in [37]) and a differential operator on the Kleisli category of the corresponding comonad. Since biproducts may be added to any category by free constructions, this gives a way of embedding any symmetric monoidal (closed) category in a Cartesian (closed) differential category. This construction allows to recover both the category **MRel**, starting from the terminal symmetric monoidal closed category (one object, one morphism), and $\mathbf{G}^{\otimes}$ starting from a symmetric monoidal category of *exhausting games*.

The category $\mathbf{G}^{\otimes}$, just like **MRel**, models the Taylor expansion. Natural examples of differential Cartesian closed categories that *do not* model the Taylor expansion have been recently defined in [13] by introducing new exponential operations on **Rel**. The intuition behind this construction is rather simple: the authors replace the set of natural numbers (that are used for counting multiplicities of elements in multisets) with more general semi-rings containing elements ω such that $\omega + 1 = \omega$ (i.e., elements that are morally infinite). In these models with infinite multiplicities all differential constructions are available, but the Taylor formula does not hold. Indeed, in these categories it is possible to find a morphism $f \neq 0$ such that, for all $n \in \mathbf{N}$, the n-th derivative of f evaluated on 0 is equal to 0: the Taylor expansion of such an f is the 0 map, and hence the morphism is different from its Taylor expansion. In particular, the authors exhibit models where the interpretation of Ω is different from 0.

7.3. Algebraic Approach

Another interesting line of research would be to provide an algebraic definition of model of the differential λ-calculus. In other words we would like to introduce a class of algebras modeling the differential λ-calculus in the same way combinatory algebras model the regular one. This would open the way to generalize the powerful techniques developed in [32,35,40] for analyzing combinatory algebras. For instance, in collaboration with Salibra, we proved that combinatory algebras satisfy good algebraic properties, like a Stone representation theorem stating that every combinatory algebra is decomposable in a weak Boolean product of indecomposable algebras [35]. This allowed, among other things, to give a uniform proof of incompleteness for the main semantics of λ-calculus (i.e. the continuous, stable and strongly stable semantics).

A first attempt to provide algebraic models of the resource calculus has been recently done by Carraro, Ehrhard and Salibra in [14]. In that paper the authors introduce the notion of "resource λ-models" and show that they are suitable to model the *finite* resource calculus (i.e., the promotion-free fragment). At the moment, a generalization allowing to model the full fragment of resource calculus (or, equivalently, the differential λ-calculus) does not seem easy, and is kept for future work.

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A. Technical Appendix

This technical appendix is devoted to provide the full proofs of the two main lemmas in Subsection 3.3. These proofs are not particularly difficult, but quite long and require some preliminary notations.

Notation A.1 *We will adopt the following notations:*

- *Given a sequence of indices* $\vec{i} = i_1, \dots, i_k$ *with* $i_j \in \{1, 2\}$ *we write* $\pi_{\vec{i}}$ *for* $\pi_{i_1} \circ \cdots \circ \pi_{i_k}$. *Thus* $\pi_{1,2} = \pi_1 \circ \pi_2$.
- *For brevity, when writing a Cartesian product of objects as subscript of* 0 *or* Id, *we will replace the operator* $\times$ *by simple juxtaposition. For instance, the morphism* $\mathrm{Id}_{(A\times B)\times(C\times D)}$ *will be written* $\mathrm{Id}_{(AB)(CD)}$.

Hereafter "(proj)" will refer to the rules $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$ that hold in every Cartesian category. We recall that $\mathrm{sw}_{ABC} = \langle\langle \pi_{1,1}, \pi_2\rangle, \pi_{2,1}\rangle : (A \times B) \times C \to (A \times C) \times B$.

Lemma A.2 *(Lemma 3.17) Let* $f : (C \times A) \times D \to B$, $g : C \to A$, $h : C \to B'$.

(i) $\pi_2 \star g = g \circ \pi_1$,

(ii) $(h \circ \pi_1) \star g = 0$,

(iii) $\Lambda(f) \star g = \Lambda(((f \circ \mathrm{sw}) \star (g \circ \pi_1)) \circ \mathrm{sw})$.

Proof. (i)

$$\begin{array}{rll}
\pi_2 \star g = & D(\pi_2) \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle & \text{by def. of } \star \\
= & \pi_2 \circ \pi_1 \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle & \text{by D3} \\
= & \pi_2 \circ \langle 0_C, g \circ \pi_1\rangle & \text{by (proj)} \\
= & g \circ \pi_1 & \text{by (proj)}
\end{array}$$

(ii)

$$\begin{array}{rll}
(h \circ \pi_1) \star g = & D(h \circ \pi_1) \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle & \text{by def. of } \star \\
= & D(h) \circ \langle D(\pi_1), \pi_{1,2}\rangle\langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle & \text{by D5} \\
= & D(h) \circ \langle \pi_1 \circ \pi_1, \pi_{1,2}\rangle \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{C\times A}\rangle & \text{by D3} \\
= & D(h) \circ \langle 0_C, \pi_1\rangle & \text{by (proj)} \\
= & 0 & \text{by D2}
\end{array}$$

(iii) We first prove the following claim.

Claim A.3 *Let* $g : C \to A$, *then the following diagram commutes:*

$$\begin{array}{ccccc}
(C\times A)\times D & \xrightarrow{\langle \pi_1, \mathrm{Id}_{C\times A}\rangle \times \mathrm{Id}_D} & (C\times(C\times A))\times D & \xrightarrow{(\langle 0_C, g\rangle \times \mathrm{Id}_{C\times A})\times \mathrm{Id}_D} & ((C\times A)\times(C\times A))\times D \\
\Big\downarrow{\scriptstyle \langle \pi_1 \times \mathrm{Id}_D, \mathrm{sw}\rangle} & & & & \Big\downarrow{\scriptstyle \langle \pi_1 \times 0_D, \pi_2 \times \mathrm{Id}_D\rangle} \\
(C\times D)\times((C\times D)\times A) & \xrightarrow{\langle 0_{C\times D}, g\circ\pi_1\rangle \times \mathrm{Id}_{(C\times D)\times A}} & ((C\times D)\times A)\times((C\times D)\times A) & \xrightarrow{\langle D(\mathrm{sw}), \mathrm{sw}\circ\pi_2\rangle} & ((C\times A)\times D)\times((C\times A))\times D)
\end{array}$$

Sub-proof.

$$\begin{array}{l}
\langle \pi_1 \times 0_D, \pi_2 \times \mathrm{Id}_D\rangle \circ ((\langle 0_C, g\rangle \times \mathrm{Id}_{CA}) \times \mathrm{Id}_D) \circ (\langle \pi_1, \mathrm{Id}_{CA}\rangle \times \mathrm{Id}_D) = \\
\langle\langle\langle 0_C, g \circ \pi_{1,1}\rangle, 0_D\rangle, \langle \pi_{2,1}, \pi_2\rangle\rangle \circ \langle\langle \pi_1, \langle \pi_1, \pi_2\rangle\rangle \circ \pi_1, \pi_2\rangle = \\
\langle\langle\langle 0_C, g \circ \pi_{1,1}\rangle, 0_D\rangle, \langle \pi_{2,1}, \pi_2\rangle\rangle \circ \langle\langle \pi_{1,1}, \langle \pi_{1,1}, \pi_{2,1}\rangle\rangle, \pi_2\rangle = \\
\langle\langle\langle 0_C, g \circ \pi_{1,1}\rangle, 0_D\rangle, \langle\langle \pi_{1,1}, \pi_{2,1}\rangle, \pi_2\rangle\rangle = \\
\langle\langle\langle 0_C, g \circ \pi_{1,1}\rangle, 0_D\rangle, \langle\langle \pi_{1,1,2}, \pi_{2,2}\rangle, \pi_{2,1,2}\rangle\rangle \circ \langle \pi_1 \times \mathrm{Id}_D, \mathrm{sw}\rangle = \\
\langle\langle\langle \pi_{1,1,1}, \pi_{2,1}\rangle, \pi_{2,1,1}\rangle, \langle\langle \pi_{1,1,2}, \pi_{2,2}\rangle, \pi_{2,1,2}\rangle\rangle \circ \langle\langle 0_{CD}, g \circ \pi_{1,1}\rangle, \pi_2\rangle \circ \langle \pi_1 \times \mathrm{Id}_D, \mathrm{sw}\rangle = \\
\langle D(\mathrm{sw}), \mathrm{sw} \circ \pi_2\rangle \circ (\langle 0_{CD}, g \circ \pi_1\rangle \times \mathrm{Id}_{(CD)A}) \circ \langle \pi_1 \times \mathrm{Id}_D, \mathrm{sw}\rangle
\end{array}$$

We can now conclude the proof as follows:

$$\begin{array}{rll}
\Lambda(f) \star g = & D(\Lambda(f)) \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle & \text{by def. of } \star \\
= & \Lambda(D(f) \circ \langle \pi_1 \times 0_D, \pi_2 \times \mathrm{Id}_D\rangle) \circ \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle & \text{by (D-curry)} \\
= & \Lambda(D(f) \circ \langle \pi_1 \times 0_D, \pi_2 \times \mathrm{Id}_D\rangle \circ ((\langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle) \times \mathrm{Id}_D)) & \text{by (Curry)} \\
= & \Lambda(D(f) \circ \langle D(\mathrm{sw}), \mathrm{sw} \circ \pi_2\rangle \circ (\langle 0_{CD}, g \circ \pi_1\rangle \times \mathrm{Id}_{(CD)A}) \circ \langle \pi_1 \times \mathrm{Id}_D, \mathrm{sw}\rangle) & \text{by Claim A.3} \\
= & \Lambda(D(f \circ \mathrm{sw}) \circ (\langle 0_{CD}, g \circ \pi_1\rangle \times \mathrm{Id}_{(CD)A}) \circ \langle \pi_1, \mathrm{Id}\rangle \circ \mathrm{sw}) & \text{by D5} \\
= & \Lambda(((f \circ \mathrm{sw}) \star (g \circ \pi_1)) \circ \mathrm{sw}) & \text{by def. of } \star
\end{array}$$

■

Lemma A.4 *(Lemma 3.18) Let* $f : C \times A \to (D \Rightarrow B)$, $g : C \to A$, $h : C \times A \to D$

(i) $(\mathrm{ev} \circ \langle f, h\rangle) \star g = \mathrm{ev} \circ \langle f \star g + \Lambda(\Lambda^-(f) \star (h \star g)), h\rangle$

(ii) $\Lambda(\Lambda^-(f) \star h) \star g = \Lambda(\Lambda^-(f \star g) \star h) + \Lambda(\Lambda^-(f) \star (h \star g))$

(iii) $\Lambda(\Lambda^-(f) \star h) \circ \langle \mathrm{Id}_C, g\rangle = \Lambda(\Lambda^-(f \circ \langle \mathrm{Id}_C, g\rangle) \star (h \circ \langle \mathrm{Id}_C, g\rangle))$

Proof.

(i) Let us set $\varphi \equiv \langle\langle 0_C, g \circ \pi_1\rangle, \mathrm{Id}_{CA}\rangle$. Then we have:

$$\begin{array}{ll}
(\mathrm{ev} \circ \langle f, h\rangle) \star g = & \text{by def. of } \star \\
D(\mathrm{ev} \circ \langle f, h\rangle) \circ \varphi = & \text{by (D-eval)} \\
(\mathrm{ev} \circ \langle D(f), h \circ \pi_2\rangle + D(\Lambda^-(f)) \circ \langle\langle 0_{CA}, D(h)\rangle, \langle \pi_2, h \circ \pi_2\rangle\rangle) \circ \varphi = & \text{by Def. 3.2} \\
\mathrm{ev} \circ \langle D(f), h \circ \pi_2\rangle \circ \varphi + D(\Lambda^-(f)) \circ \langle\langle 0_{CA}, D(h) \circ \varphi\rangle, \langle \mathrm{Id}_{CA}, h\rangle\rangle = & \text{by def. of } \star \\
\mathrm{ev} \circ \langle D(f) \circ \varphi, h\rangle + D(\Lambda^-(f)) \circ \langle\langle 0_{CA}, (h \star g) \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle \circ \langle \mathrm{Id}_{CA}, h\rangle = & \text{by def. of } \star \\
\mathrm{ev} \circ \langle f \star g, h\rangle + (\Lambda^-(f) \star (h \star g)) \circ \langle \mathrm{Id}, h\rangle = & \text{by (beta-cat)} \\
\mathrm{ev} \circ \langle f \star g, h\rangle + \mathrm{ev} \circ \langle \Lambda(\Lambda^-(f) \star (h \star g)), h\rangle = & \text{by Lemma 3.8} \\
\mathrm{ev} \circ \langle f \star g + \Lambda(\Lambda^-(f) \star (h \star g)), h\rangle &
\end{array}$$

(ii) We first simplify the equation $\Lambda(\Lambda^-(f)\star h)\star g = \Lambda(\Lambda^-(f\star g)\star h)+\Lambda(\Lambda^-(f)\star(h\star g))$ to get rid of the Cartesian closed structure. The right side can be rewritten as $\Lambda((\Lambda^-(f \star g) \star h) + \Lambda^-(f) \star (h \star g))$. By taking a morphism $f' : (C \times A) \times D \to B$ such that $f = \Lambda(f')$ and by applying Lemma 3.17(*iii*) we discover that it is equivalent to show that:

$$((f' \star h) \circ \mathrm{sw}) \star (g \circ \pi_1) \circ \mathrm{sw} = (((f' \circ \mathrm{sw}) \star (g \circ \pi_1)) \circ \mathrm{sw}) \star h + f' \star (h \star g).$$

By definition of $\star$ we have:

$$((f' \star h) \circ \mathrm{sw}) \star (g \circ \pi_1) \circ \mathrm{sw} = D(D(f') \circ \langle\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle, \mathrm{sw}\rangle) \circ \langle\langle 0_{CD}, g \circ \pi_{1,1}\rangle, \mathrm{sw}\rangle$$

Let us call now $\varphi \equiv \langle\langle 0_{CD}, g \circ \pi_{1,1}\rangle, \mathrm{sw}\rangle$ and write $D^2(f)$ for $D(D(f))$. Then we have:

$$\begin{array}{ll}
D^2(f') \circ \langle\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle, \mathrm{sw}\rangle) \circ \varphi = & \text{by D5} \\
D^2(f') \circ \langle D(\langle\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle, \mathrm{sw}\rangle), \langle\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle, \mathrm{sw}\rangle \circ \pi_2\rangle \circ \varphi = & \text{by (pair)} \\
D^2(f') \circ \langle D(\langle\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle, \mathrm{sw}\rangle) \circ \varphi, \langle\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle, \mathrm{sw}\rangle \circ \pi_2 \circ \varphi\rangle = & \text{by D4} \\
D^2(f') \circ \langle\langle D(\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle) \circ \varphi, D(\mathrm{sw}) \circ \varphi\rangle, \langle\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle, \mathrm{sw}\rangle \circ \mathrm{sw}\rangle = & \text{by Rem. 3.16} \\
D^2(f') \circ \langle\langle D(\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle) \circ \varphi, D(\mathrm{sw}) \circ \varphi\rangle, \langle\langle 0_{CA}, h \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle &
\end{array}$$

Since $\langle D(\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle) \circ \varphi, D(\mathrm{sw}) \circ \varphi\rangle = \langle 0, D(\mathrm{sw}) \circ \varphi\rangle + \langle D(\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle) \circ \varphi, 0\rangle$ we can apply D2 and rewrite the expression above as a sum of two morphisms:

$$\begin{array}{lc}
(1) & D^2(f') \circ \langle\langle 0_{(CA)D}, D(\mathrm{sw}) \circ \varphi\rangle, \langle\langle 0_{CA}, h \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle\ + \\
(2) & D^2(f') \circ \langle\langle D(\langle 0_{CA}, h \circ \langle \pi_{1,1}, \pi_2\rangle\rangle) \circ \varphi, 0_{(CA)D}\rangle, \langle\langle 0_{CA}, h \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle
\end{array}$$

We now show that $(1) = (((f' \circ \mathrm{sw}) \star (g \circ \pi_1)) \circ \mathrm{sw}) \star h$. Indeed, we have:

$$\begin{array}{ll}
D^2(f') \circ \langle\langle 0_{(CA)D}, D(\mathrm{sw}) \circ \varphi\rangle, \langle\langle 0_{CA}, h \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle = & \text{by Rem. 3.16} \\
D^2(f') \circ \langle\langle 0_{(CA)D}, \mathrm{sw} \circ \pi_1 \circ \varphi\rangle, \langle\langle 0_{CA}, h \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle = & \text{by (proj)} \\
D^2(f') \circ \langle\langle 0_{(CA)D}, \mathrm{sw} \circ \langle 0_{CD}, g \circ \pi_{1,1}\rangle\rangle, \langle\langle 0_{CA}, h \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle = & \text{by Rem. 3.16} \\
D^2(f') \circ \langle\langle 0_{(CA)D}, \langle\langle 0_C, g \circ \pi_{1,1}\rangle, 0_D\rangle\rangle, \langle\langle 0_{CA}, h \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle = & \text{by D7} \\
D^2(f') \circ \langle\langle\langle\langle 0_C, 0_A\rangle, 0_D\rangle, \langle 0_{CA}, h \circ \pi_1\rangle\rangle, \langle\langle\langle 0_C, g \circ \pi_{1,1}\rangle, 0_D\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle = & \text{by D2} \\
D^2(f') \circ \langle\langle\langle\langle 0_C, D(g) \circ \langle 0_C, \pi_{1,1}\rangle\rangle, 0_D\rangle, \langle 0_{CA}, h \circ \pi_1\rangle\rangle, \langle\langle\langle 0_C, g \circ \pi_{1,1}\rangle, 0_D\rangle, \mathrm{Id}_{(CA)D}\rangle\rangle. &
\end{array}$$

Let us set $\psi \equiv \langle\langle 0_{CA}, h \circ \pi_1\rangle, \mathrm{Id}_{(CA)D}\rangle$. Then we have:

$D(D(f))\circ\langle\langle\langle\langle 0_C, D(g)\circ\langle 0_C,\pi_{1,1}\rangle\rangle, 0_D\rangle,\langle 0_{CA}, h\circ\pi_1\rangle\rangle,\langle\langle\langle 0_C, g\circ\pi_{1,1}\rangle, 0_D\rangle,\mathrm{Id}_{(CA)D}\rangle\rangle =$	by (proj)
$D(D(f))\circ\langle\langle\langle\langle 0_C, D(g)\circ\langle\pi_{1,1,1},\pi_{1,1,2}\rangle\rangle, 0_D\rangle,\pi_1\rangle,\langle\langle\langle 0_C, g\circ\pi_{1,1,2}\rangle, 0_D\rangle,\pi_2\rangle\rangle\circ\psi =$	by D3
$D(D(f))\circ\langle\langle\langle\langle 0_C, D(g)\circ\langle D(\pi_{1,1}),\pi_{1,1,2}\rangle\rangle, 0_D\rangle,\pi_1\rangle,\langle\langle\langle 0_C, g\circ\pi_{1,1,2}\rangle, 0_D\rangle,\pi_2\rangle\rangle\circ\psi =$	by D5
$D(D(f))\circ\langle\langle\langle\langle 0_C, D(g\circ\pi_{1,1})\rangle, 0_D\rangle,\pi_1\rangle,\langle\langle\langle 0_C, g\circ\pi_{1,1,2}\rangle, 0_D\rangle,\pi_2\rangle\rangle\circ\psi =$	by D1
$D(D(f))\circ\langle\langle\langle\langle D(0_C), D(g\circ\pi_{1,1})\rangle\rangle, D(0_D)\rangle, D(\mathrm{Id}_{(CA)D})\rangle,\langle\langle\langle 0_C, g\circ\pi_{1,1,2}\rangle, 0_D\rangle,\pi_2\rangle\rangle\circ\psi =$	by D4
$D(D(f))\circ\langle D(\langle\langle\langle 0_C, g\circ\pi_{1,1}\rangle, 0_D\rangle,\mathrm{Id}_{(CA)D}\rangle),\langle\langle\langle 0_C, g\circ\pi_{1,1}\rangle, 0_D\rangle,\mathrm{Id}_{(CA)D}\rangle\circ\pi_2\rangle\circ\psi =$	by D5
$D(D(f)\circ\langle\langle\langle 0_C, g\circ\pi_{1,1}\rangle, 0_D\rangle,\mathrm{Id}_{(CA)D}\rangle)\circ\psi =$	by Rem. 3.16
$D(D(f)\circ\langle\mathrm{sw}\circ\langle 0_{CD}, g\circ\pi_{1,1}\rangle,\mathrm{sw}\circ\mathrm{sw}\rangle)\circ\psi =$	by (proj)
$D(D(f)\circ\langle\mathrm{sw}\circ\pi_1,\mathrm{sw}\circ\pi_2\rangle\circ\langle\langle 0_{CD}, g\circ\pi_{1,1}\rangle,\mathrm{sw}\rangle)\circ\psi =$	by Rem. 3.16
$D(D(f)\circ\langle D(\mathrm{sw}),\mathrm{sw}\circ\pi_2\rangle\circ\langle\langle 0_{CD}, g\circ\pi_{1,1}\rangle,\mathrm{sw}\rangle)\circ\psi =$	by D5
$D(D(f\circ\mathrm{sw})\circ\langle\langle 0_{CD}, g\circ\pi_{1,1}\rangle,\mathrm{Id}_{(CD)A}\rangle\circ\mathrm{sw})\circ\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle =$	by def. of $\star$
$(((f\circ\mathrm{sw})\star(g\circ\pi_1))\circ\mathrm{sw})\star h$	

We will now show that $(2) = f\star(h\star g)$, and this will conclude the proof.

$D^2(f)\circ\langle\langle D(\langle 0_{CA}, h\circ\langle\pi_{1,1},\pi_2\rangle\rangle)\circ\varphi, 0_{(CA)D}\rangle,\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle\rangle =$	by D1+4
$D^2(f)\circ\langle\langle\langle 0_{CA}, D(h\circ\langle\pi_{1,1},\pi_2\rangle)\rangle\circ\varphi, 0_{(CA)D}\rangle,\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle\rangle =$	by D5
$D^2(f)\circ\langle\langle\langle 0_{CA}, D(h)\circ\langle D(\langle\pi_{1,1},\pi_2\rangle),\langle\pi_{1,1,2},\pi_{2,2}\rangle\rangle\rangle\circ\varphi, 0_{(CA)D}\rangle,\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle\rangle =$	by D4+D3
$D^2(f)\circ\langle\langle\langle 0_{CA}, D(h)\circ\langle\langle D(\pi_{1,1}), D(\pi_2)\rangle,\langle\pi_{1,1,2},\pi_{2,2}\rangle\rangle\rangle\circ\varphi, 0_{(CA)D}\rangle,\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle\rangle =$	by D5+D3
$D^2(f)\circ\langle\langle\langle 0_{CA}, D(h)\circ\langle\langle\pi_{1,1,1},\pi_{2,1}\rangle,\langle\pi_{1,1,2},\pi_{2,2}\rangle\rangle\rangle\circ\varphi, 0_{(CA)D}\rangle,\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle\rangle =$	by (proj)
$D^2(f)\circ\langle\langle\langle 0_{CA}, D(h)\circ\langle\langle 0_C, g\circ\pi_{1,1}\rangle,\pi_1\rangle\rangle, 0_{(CA)D}\rangle,\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle\rangle =$	by D6
$D(f)\circ\langle\langle 0_{CA}, D(h)\circ\langle\langle 0_C, g\circ\pi_{1,1}\rangle,\pi_1\rangle\rangle,\mathrm{Id}_{(CA)D}\rangle =$	by (proj)
$D(f)\circ\langle\langle 0_{CA}, D(h)\circ\langle\langle 0_C, g\circ\pi_1\rangle,\mathrm{Id}_{CA}\rangle\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle =$	by def. of $\star$
$f\star(h\star g)$	

(*iii*) By (Curry) we have $\Lambda(\Lambda^-(f)\star h)\circ\langle\mathrm{Id}_C, g\rangle = \Lambda((\Lambda^-(f)\star h)\circ(\langle\mathrm{Id}_C, g\rangle\times\mathrm{Id}_D))$, thus if we show that $(\Lambda^-(f)\star h)\circ(\langle\mathrm{Id}_C, g\rangle\times\mathrm{Id}_D) = \Lambda^-(f\circ\langle\mathrm{Id}_C, g\rangle)\star(h\circ\langle\mathrm{Id}_C, g\rangle)$ we have finished.

We proceed then as follows:

$(\Lambda^-(f)\star h)\circ(\langle\mathrm{Id}_C, g\rangle\times\mathrm{Id}_D) =$	by def. of $\star$
$D(\Lambda^-(f))\circ\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle\circ(\langle\mathrm{Id}_C, g\rangle\times\mathrm{Id}_D) =$	by def. of Λ^-
$D(\mathrm{ev}\circ\langle f\circ\pi_1,\pi_2\rangle)\circ\langle\langle 0_{CA}, h\circ\pi_1\rangle,\mathrm{Id}_{(CA)D}\rangle\circ(\langle\mathrm{Id}_C, g\rangle\times\mathrm{Id}_D) =$	by D5+D4
$D(\mathrm{ev})\circ\langle\langle D(f\circ\pi_1), D(\pi_2)\rangle,\langle f\circ\pi_{1,2},\pi_{2,2}\rangle\rangle\circ\langle\langle 0_{CA}, h\rangle\circ\langle\pi_1, g\circ\pi_1\rangle,\langle\mathrm{Id}_C, g\rangle\times\mathrm{Id}_D\rangle =$	by D5+D3
$D(\mathrm{ev})\circ\langle\langle D(f)\circ\langle\pi_{1,1},\pi_{1,2}\rangle,\pi_{2,1}\rangle,\langle f\circ\pi_{1,2},\pi_{2,2}\rangle\rangle\circ\langle\langle 0_{CA}, h\circ\langle\pi_1, g\circ\pi_1\rangle\rangle,\langle\mathrm{Id}_C, g\rangle\times\mathrm{Id}_D\rangle =$	by (proj)
$D(\mathrm{ev})\circ\langle\langle D(f)\circ\langle 0_{CA},\langle\pi_1, g\circ\pi_1\rangle\rangle, h\circ\langle\pi_1, g\circ\pi_1\rangle\rangle,\langle f\circ\langle\pi_1, g\circ\pi_1\rangle,\pi_2\rangle\rangle =$	by D2
$D(\mathrm{ev})\circ\langle\langle D(f)\circ\langle\langle 0_C, D(g)\circ\langle 0_C,\mathrm{Id}_C\rangle\rangle,\langle\mathrm{Id}_C, g\rangle\rangle, h\circ\langle\pi_1, g\circ\pi_1\rangle\rangle,\langle f\circ\langle\mathrm{Id}_C, g\rangle,\mathrm{Id}_D\rangle\rangle =$	
by setting $\varphi = \langle\langle 0_C, h\circ\langle\pi_1, g\circ\pi_1\rangle\rangle,\mathrm{Id}_{CD}\rangle$	
$D(\mathrm{ev})\circ\langle\langle D(f)\circ\langle\langle\pi_{1,1}, D(g)\circ\langle\pi_{1,1},\pi_{1,2}\rangle\rangle,\langle\pi_{1,2}, g\circ\pi_{1,2}\rangle\rangle,\pi_{2,1}\rangle,\langle f\circ\langle\pi_{1,2}, g\circ\pi_{1,2}\rangle,\pi_{2,2}\rangle\rangle\circ\varphi =$	by D5
$D(\mathrm{ev})\circ\langle\langle D(f\circ\langle\pi_1, g\circ\pi_1\rangle), D(\pi_2)\rangle,\langle f\circ\langle\pi_{1,2}, g\circ\pi_{1,2}\rangle,\pi_{2,2}\rangle\rangle\circ\varphi =$	by D4
$D(\mathrm{ev})\circ\langle D(\langle f\circ\langle\pi_1, g\circ\pi_1\rangle,\pi_2\rangle),\langle f\circ\langle\pi_{1,2}, g\circ\pi_{1,2}\rangle,\pi_{2,2}\rangle\rangle\circ\varphi =$	by D5
$D(\mathrm{ev}\circ\langle f\circ\langle\pi_1, g\circ\pi_1\rangle,\pi_2\rangle)\circ\varphi =$	by def. of Λ^-
$D(\Lambda^-(f\circ\langle\mathrm{Id}_C, g\rangle))\circ\varphi =$	by def. of $\star$
$\Lambda^-(f\circ\langle\mathrm{Id}_C, g\rangle)\star(h\circ\langle\mathrm{Id}_C, g\rangle)$	

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