Lecture

Mathematical Optimization and Polyhedral Approaches

Section 2 : Polyhedral approaches

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What are the “best” valid inequalities ?

Given the variable set of a formulation,

- What are the “best” valid inequalities ?
- How to have “often” integer Branch&Bound nodes ?

...

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- How to have “often” integer Branch&Bound nodes ?

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- Can we know when a linear formulation produces integer solutions ?

Integer polytope

- Solving a (bounded) linear formulation

$$(\tilde{F}) \begin{cases} \max \ c^T x \\ \quad Ax \leq b \end{cases}$$

reduces to find an optimal extreme point
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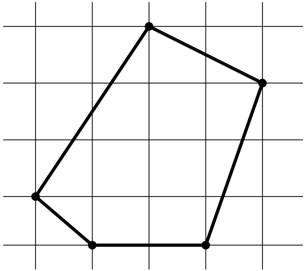
- An **integer polytope** is a polytope with integer extreme points.



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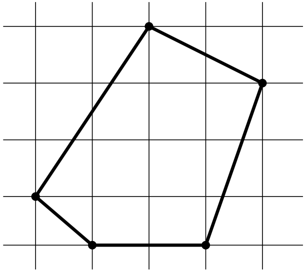

- An **integer polytope** is a polytope with integer extreme points.

- A rational polytope P is integer $\Leftrightarrow \forall c \in \mathbb{Z}^n$, $\max\{c^T x \mid x \in P\}$ is integer.

A 2-dimensional example

$$
(F)\begin{cases}
\max\ z = 2x_1 + x_2 \\
x_1 - 4x_2 \leq 0 \\
3x_1 + 4x_2 \leq 15 \\
x_1 \geq 0 \\
x_2 \geq 0 \\
x_1, x_2 \in \boldsymbol{N}
\end{cases}
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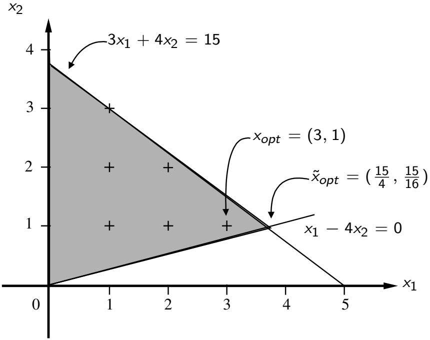

x_{opt} optimal integer solution of (F)

z^{opt} optimal integer value : 7

$\tilde{x}_{opt}$ optimal fractional solution of linear relaxation $(\tilde{F})$

z^*_{opt} optimal fractional value : $8 + \frac{7}{16}$

Note that $\tilde{x}_{opt}$ is the (only) optimal extreme point of $\tilde{F}$.

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Remark : the integer solutions of $(\tilde{F})$ are the solutions of (F).

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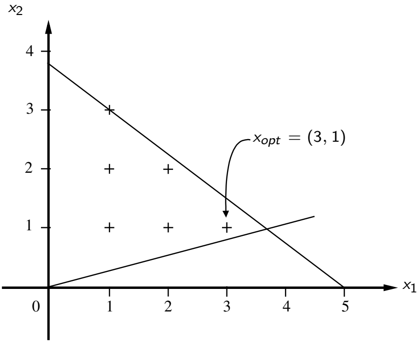

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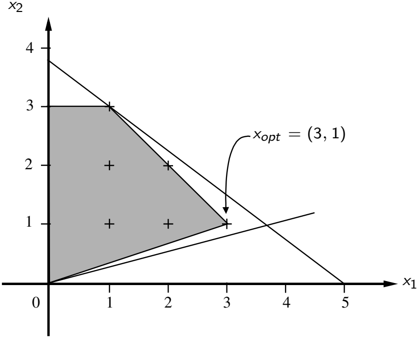

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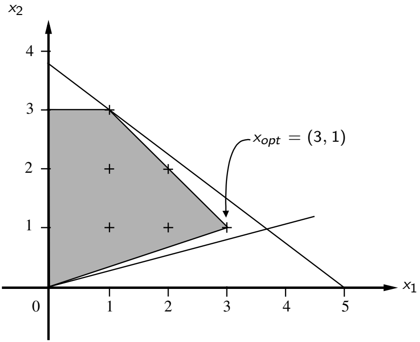

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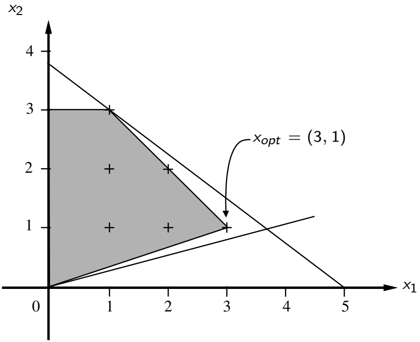

Let's take an elastic and wrap it around these integer points...

We obtain a **new polytope !** (the convex hull of the integer points)
And this polytope is **integer** by construction.

Convex hull

Given a set S of points of $\boldsymbol{R}^n$.
the **convex hull** of S, denoted by $conv(S)$
is the smallest convex set containing S.

Theorem (of Minkowski)

A set $P \subseteq \boldsymbol{R}^n$ is a polytope
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Consequently :

- $conv(S)$ is a polytope
- there exists a finite subset of inequalities $Dx \leq \beta$ such that

$$conv(S) = \{x \in \boldsymbol{R}^n \mid Dx \leq \beta\}$$

- $\max\{c\chi \mid \chi \in conv(S)\}$ is a linear program

Combinatorial polytope

Let $\mathcal{P}$ be a combinatorial optimization problem :
- over n decisions corresponding to n integer variables.
- with a function cost c.

Let S the set of the incidence vectors of the solutions of $\mathcal{P}$.

Problem $\mathcal{P}$ is

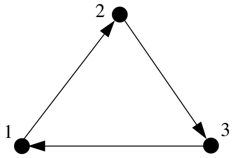
$$\max \{c\chi \mid \chi \in S\}$$

Let us consider the **linear program**

$$\max \{c\chi \mid \chi \in conv(S)\}$$

A 3-dimensional example

Let us consider the AISP on a triangle



The solutions are

$$\emptyset \quad \{1\} \quad \{2\} \quad \{3\} \quad \{1,2\} \quad \{1,3\} \quad \{2,3\}$$

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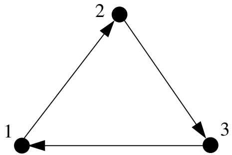

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$$\begin{array}{rl} \chi^{\emptyset} = & [\ 0 \ \ 0 \ \ 0\ ] \\ \chi^{\{1\}} = & [\ 1 \ \ 0 \ \ 0\ ] \\ \chi^{\{2\}} = & [\ 0 \ \ 1 \ \ 0\ ] \\ \chi^{\{3\}} = & [\ 0 \ \ 0 \ \ 1\ ] \\ \chi^{\{1,2\}} = & [\ 1 \ \ 1 \ \ 0\ ] \\ \chi^{\{1,3\}} = & [\ 1 \ \ 0 \ \ 1\ ] \\ \chi^{\{2,3\}} = & [\ 0 \ \ 1 \ \ 1\ ] \end{array}$$

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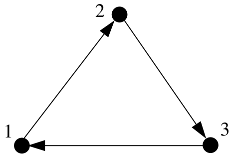

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However point

$$\chi^{\{1,2,3\}} = [\ 1 \ 1 \ 1\ ]$$

does not correspond to a solution

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What is the convex hull of these 7 points ?

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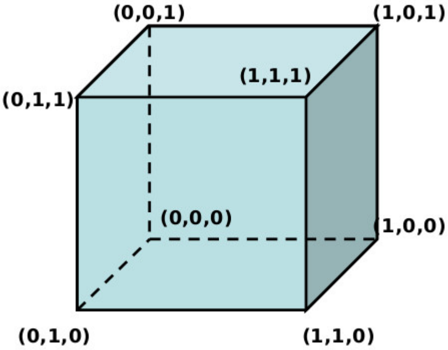

This convex hull is included into the hypercube (of dimension 3)

The hypercube is characterized by

$$\begin{aligned} x_1 &\leq 1 \\ x_2 &\leq 1 \\ x_3 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$

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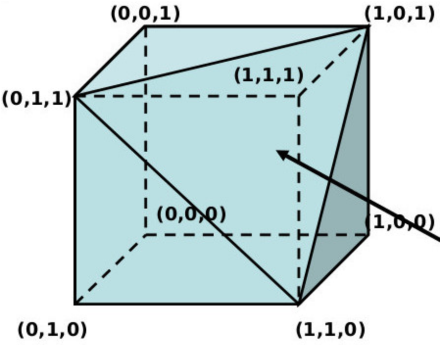

$x_1 + x_2 + x_3 \leq 2$

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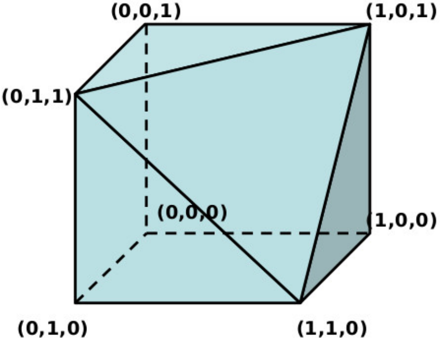

The convex hull of these 7 points is characterized by

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Indeed

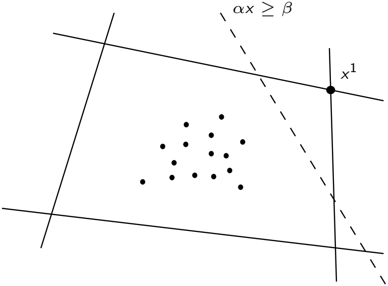
- Every extreme points of the convex hull $conv(S)$ are integer by construction.
- The optimal points of S are among the extreme points of polytope $conv(S)$.

Get around the combinatorial explosion

Optimizing (P) reduces to optimizing a linear program on $conv(S)$.

The convex hull is then the "unknown value" to found.

$$(F)\begin{cases} \max cx \\ Ax \leq b \\ \alpha x \leq \beta \\ x \text{ integer} \end{cases}$$



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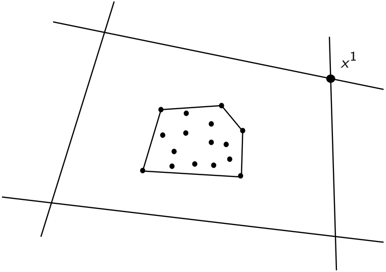

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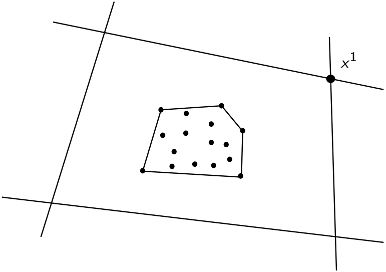

Unfortunately we cannot use this process in polynomial time...
Unless P=NP, finding the convex hull of a combinatorial polytope is NP-hard !

But even a “partial” knowledge of polytope $conv(S)$ is very useful

Example : The acyclic induced subgraph polytope

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acycl(G) : family of all node subsets inducing an acyclic subgraph of G.

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Given a solution $W \in acycl(G)$, the **incidence vector** χ^W is

$$\chi^W[i] = \begin{cases} 1 & \text{if } i \in W \\ 0 & \text{otherwise} \end{cases}$$

.

Some solutions :

- Pour $\emptyset \in acycl(G)$

$$\chi^{\emptyset} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $\{i\} \in acycl(G)\ \forall i \in V$

$$\chi^{\{i\}} = \begin{bmatrix} 0 & 0 & \underset{i}{1} & 0 & 0 \end{bmatrix}$$

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$P(G)$: **the acyclic induced subgraph polytope** of G
i.e. is the convex hull of the incidence vectors of the solutions
i.e. $P(G) = conv\{\chi^W \mid W \in acycl(G)\}$.

Dimension

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Definition

• A set of points $x^1, ..., x^k \in \boldsymbol{R}^n$ are **affinely independent** if vectors $x^2 - x^1, ..., x^k - x^1$ are linearly independent.

• A polytope P in $\boldsymbol{R}^n$ is of **dimension** d (denoted $dim(P) = d$) if P contains at least $d + 1$ affinely independent points.

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Examples :
- a plane in 3D-space is not full dimensional.
- an hypercube $[0, 1]^n$ is full dimensional in $\boldsymbol{R}^n$ but not in $\boldsymbol{R}^{n'}$ if $n' < n$

Example : The acyclic induced subgraph polytope

Lemma

The AIS polytope $P(G)$ is full-dimensional for every graph G.

Proof.

It is sufficient to produce $n+1$ affinely independent points of $P(G)$.
For instance, the incidence vectors of
- the empty set $\emptyset$
- the singletons $\{i\}$ $\forall i \in V$.

Moreover, the vectors $\chi^{\{u\}} - \chi^{\emptyset}$ are linearly independent since they form the identity matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

□

"Degree of freedom"

Let us assume that the characterization of a polytope $P \subseteq \boldsymbol{R}^n$ is given by

$$P = \left\{ x \in \boldsymbol{R}^n \mid \begin{array}{ll} A_i x \leq b_i, & i = 1, ..., m_1 \\ B_j x = d_j, & j = 1, ..., m_2 \end{array} \right\}.$$

where every inequality $A_i x \leq b_i$ is a "true" inequality,
i.e. there exists $\tilde{x} \in P$ such that $A_i \tilde{x} < b_i$.

Theorem

If $P \neq \emptyset$, then $dim(P) = n - rang(B)$.

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$dim(P)$ gives the "degree of freedom" of a problem :
$n - dim(P)$ variables can be obtained by fixing the $dim(P)$ others

Redundant inequality

Definition

Let P a polytope characterized by a system $Ax \leq b$.

An inequality $ax \leq alpha$ of $Ax \leq b$ is **redundant**
if the system "$Ax \leq b$ minus $ax \leq \alpha$" still characterizes P.

A non-redundant inequality is then **essential**.

What are the essential inequalities ?

Facet of a polytope

Let $ax \leq \alpha$ is a valid inequality for the problem corresponding to a polytope P.

Definition

- The **face** of $ax \leq \alpha$ is the set of points of P satisfying $ax \leq \alpha$ to equality,

$$i.e.\ F = \{x \in P \mid ax = \alpha\}$$

- A face F is a **facet** of P
  if $\emptyset \neq F \neq P$ and $dim(F) = dim(P) - 1$.

Theorem

- *If $P \neq \emptyset$, then a non-facet inequality of P is redundant.*
- *Every facet of P corresponds to one inequality of a characterization of P.*

Trivial facet of the AIS polytope $P(G)$

Given a node $i_0 \in V$, the trivial inequality

$$x_{i_0} \geq 0$$

defines a facet of $P(G)$.

The corresponding face is $F_{i_0} = \{\chi^W \in \boldsymbol{R}^n \mid W \in acycl(G) \text{ and } \chi^W[i_0] = 0\}$.

• $\chi^\emptyset \in F_{i_0}$ then $F_{i_0} \neq \emptyset$

• $\chi^{\{i_0\}} \notin F_{i_0}$ then $F_{i_0} \neq P(G)$

• The vectors $\chi^\emptyset$ and $\chi^{\{i\}}$, $i \neq i_0$, are n affinely independent points of F_{i_0}
then $dim(F_{i_0}) = n - 1$
Hence $x_{i_0} \geq 0$ is a facet of $P(G)$. □

Clique inequality of the AIS polytope

Given a clique K of G, the clique inequality is

$$\sum_{i \in K} x_i \leq 1$$

If there exists K' a clique of G such that $K \subset K'$
Then

$$\begin{array}{rcll} \sum_{i \in K'} x_i & \leq & 1 & \\ -x_i & \leq & 0 & \forall i \in K' \setminus K \\ \hline \sum_{i \in K} x_i & \leq & 1 & \end{array}$$

Thus the clique inequality associated to K is redundant and not-facet defining.

Clique inequality of the AIS polytope

Lemma

A clique inequality on K defines a facet
if and only if K is inclusion-wise maximal.

The consequence of this theorem is that maximal cliques are “better inequalities” to add to strenghen a cutting plane algorithm.

Indeed facet defining inequalities are called **“the deepest cuts”** !

In practice, the heuristic method we present for clique inequalities always produce maximal cliques.

Characterization

For some polynomial combinatorial problem $\mathcal{P}$ and its associated combinatorial polytope P.

Some methods to show that a system $Ax \leq b$ characterizes P

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- to show that every facet of P corresponds to at least one inequality among $Ax \leq b$
- ... and many others (polyhedral decomposition, extended formulation+projection, critical extreme point study,...)

Bipartite matching problem

Let $G = (V_1 \cup V_2, E)$ be a bipartite (undirected) graph
Let $c \in \boldsymbol{R}^m$ a cost associated to the edges of$E$.

A **matching** of G is a set of pairwise disjoint edges.
The **matching problem** on bipartite graph G is to find a matching of maximal cost.

Theorem

The following linear program is integer and is equiv. to the bipartite matching problem.

$$\max \sum_{e \in E} c(e)x(e)$$
$$\sum_{e \in \delta(u)} x(e) \leq 1 \quad \forall u \in V_1$$
$$\sum_{e \in \delta(u)} x(e) \leq 1 \quad \forall u \in V_2$$
$$x(e) \geq 0 \quad \forall e \in E.$$

(The matrix is totally unimodular)

The matching problem

Let $G = (V, E)$ be an undirected graph.

The previous formulation on G is not integer !

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The **matching polytope** is the convex hull of the incidence vectors of the matchings *i.e.*

$$P_M(G) = conv\{\chi^M \in \mathbb{R}^n \mid M \text{ matching of } G\}.$$

Theorem (Jack Edmonds (1965))

The matching polytope is characterized by

$$\sum_{e \in \delta(u)} x(e) \leq 1 \qquad \forall v \in V$$

$$\sum_{e \in E(S)} x(e) \leq \frac{|S| - 1}{2} \qquad \forall S \subseteq V \text{ with } |S| \text{ odd}$$

$$x(e) \geq 0 \qquad \forall e \in E.$$