Lecture

Mathematical Optimization and Polyhedral Approaches
Section 3 : Non-compact MILP and Branch&Cut

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Combinatorial Optimization Problem

To find a greatest (smallest) element within a valuated finite set.

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Given :
- a finite subset of elements $E = \{e_1, \ldots, e_n\}$
- a **solution set** $\mathcal{F}$ of subsets of E
- a weight $c = (c(e_1), \ldots, c(e_n))$

a **Combinatorial Optimization Problem** is to find a solution $F \in \mathcal{F}$ whose weight $c(F) = \sum_{e \in F} c(e)$ is maximum (or min.),

$$\textit{i.e.} \quad \max \{c(F) \mid F \in \mathcal{F}\}.$$

"Natural" MIP formulation

$$\text{Binary variable } x_e = \begin{cases} 1 & \text{if } e \text{ is chosen} \\ 0 & \text{otherwise} \end{cases} \quad \text{for every } e \in E.$$

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$$\max \sum_{e \in E} c(e) x_e$$
$$Ax \leq b$$
$$x_e \in \{0, 1\} \quad \forall e \in E.$$

We will suppose here that :
- $Ax \leq b$ is known
- the linear relaxation of this MIP can be obtained

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- $Ax \leq b$ is known
- the linear relaxation of this MIP can be obtained

Important remark : $Ax \leq b$ can be **non-compact**,
i.e. can contain an exponential number of inequalities !

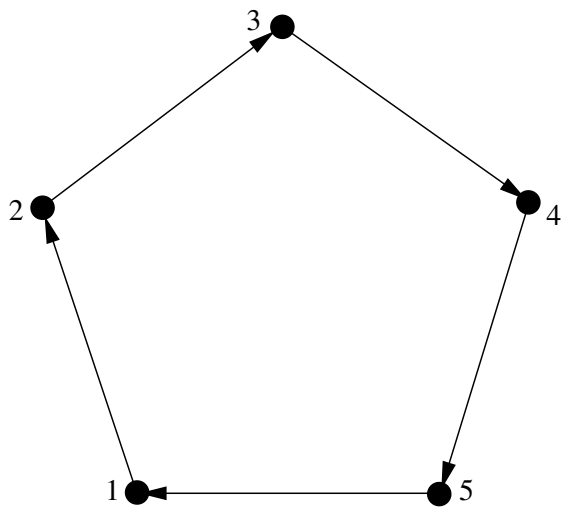
The Acyclic Induced subgraph problem

Let $G = (V, A)$ be a directed graph
with $n = |V|$ nodes and $m = |A|$ arcs.

A **circuit** of G : a sequence of arcs

$$C = (i_1 i_2, i_2 i_3, ..., i_{k-1} i_1)$$

Notation :
$V(C)$ is the set of nodes of C.



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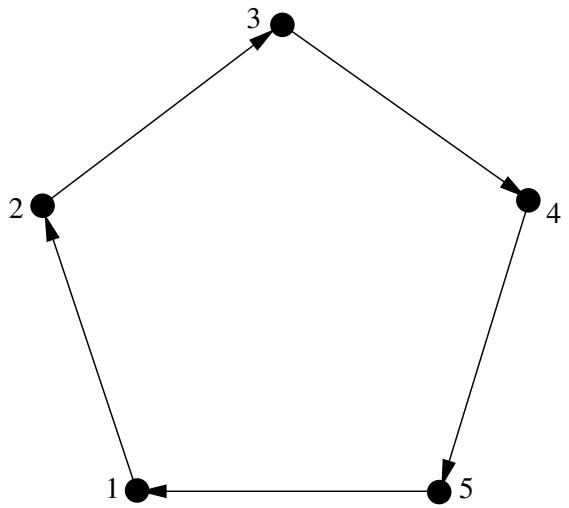


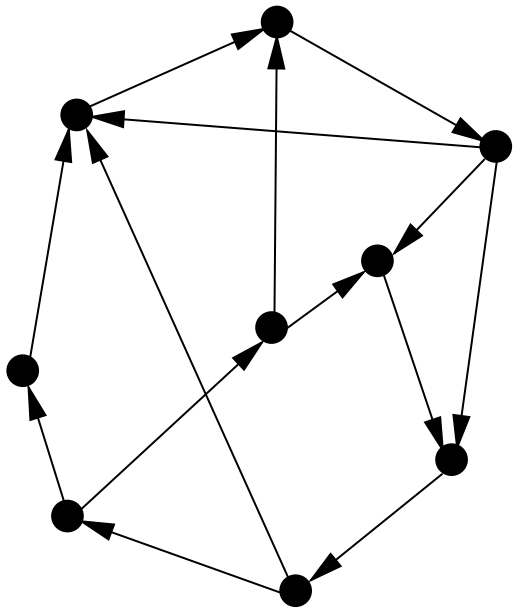
A graph is **acyclic** if it contains no circuit.

The Acyclic Induced subgraph problem

Given a node subset $W \subset V$,

$A(W)$: arcs with both endnodes in W

$(W, A(W))$: **subgraph induced by** W



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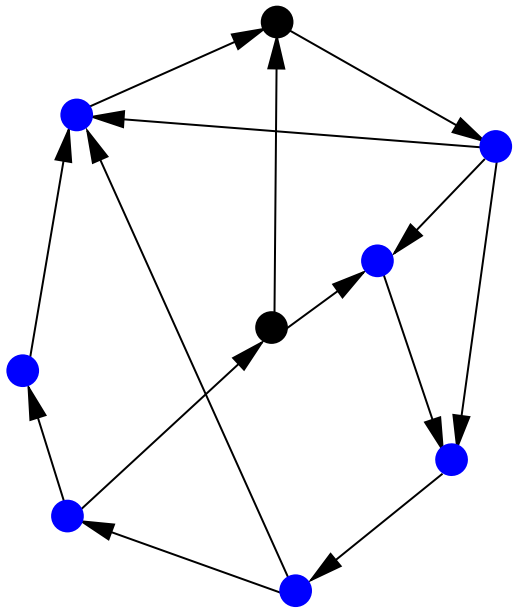

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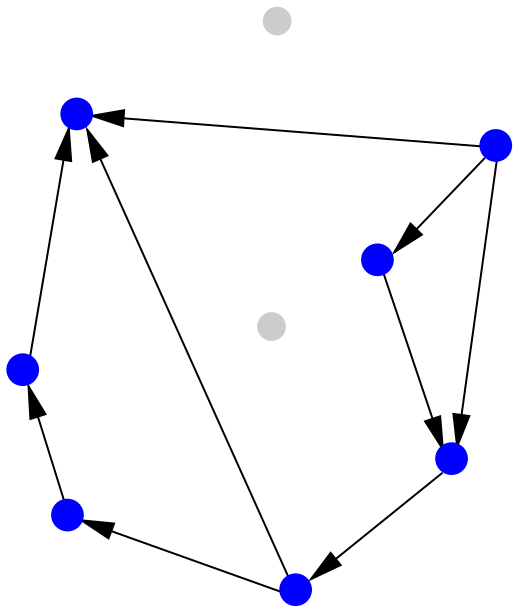

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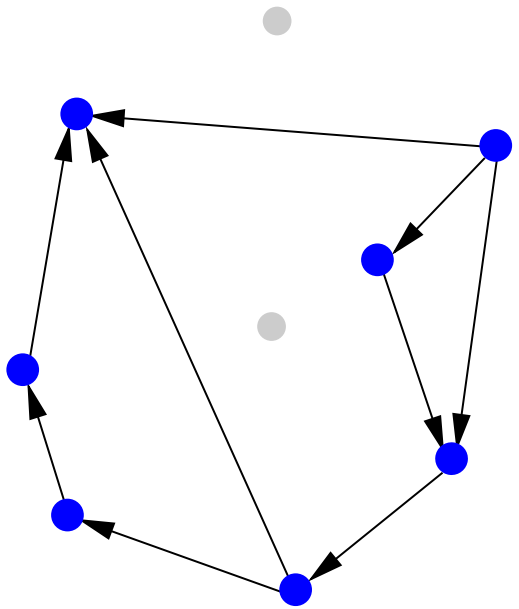

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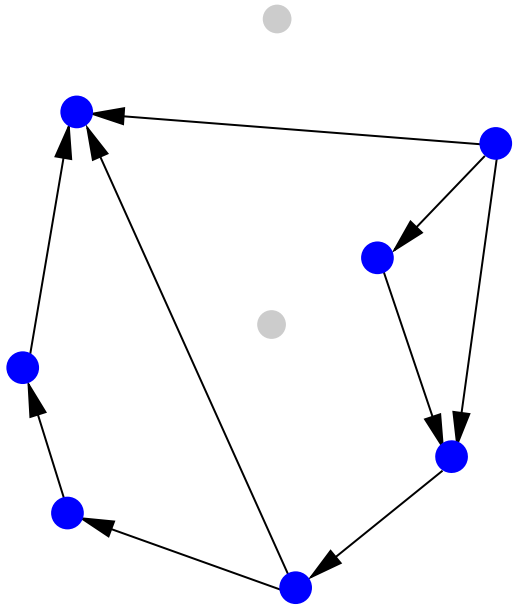

The **Acyclic Induced Subgraph Problem (AISP)** is to find
a node subset W inducing an acyclic subgraph with $|W|$ maximum
or
a node subset W' "breaking" every circuit of G with $|W'|$ min.

The Acyclic Induced subgraph problem

- The AISP is NP-hard.

Indeed :
given a non-directed graph G
construct a directed graph G' by replacing one edge by two arcs

then finding an acyclic subgraph in G' is as hard as finding a maximum stable set in G.

- The AISP is polynomial for graphs of maximum degree 3 [Baiou, Barahona]

A compact formulation

Two types of variables :

- Binary variables $x_i = \begin{cases} 1 & \text{if node } i \text{ is chosen} \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in V.$

- Continuous variables $u_i \ \forall i \in V$.

A Miller-Tucker-Zemlin (MTZ) formulation :

$$(F_{MTZ}) \begin{cases} \max \sum_{i \in V} x_i & \\ u_i - u_j + 1 \leq n(2 - x_i - x_j) & \forall ij \in A \\ 1 \leq u_i \leq n & \forall i \in V \\ u_i \in \boldsymbol{R} & \forall i \in V \\ x_i \in \{0, 1\} & \forall i \in V \end{cases}$$

A compact formulation

- The MTZ formulation is equivalent to the AISP.

Indeed :
Given a solution (x, u) of (F_{MTZ}), let $W = \{i \in V \mid x_i = 1\}$.
If W induces a circuit C : $u_i + 1 \leq u_j \quad \forall ij \in C$, a contradiction.

- (F_{MTZ}) is compact
- n binary variables and n continuous variables
- m inequalities

Let's go with Cplex !

A non-compact formulation

Same binary variables $x_i = \begin{cases} 1 & \text{if node } i \text{ is chosen} \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in V.$

$$(F_C) \begin{cases} \max \sum_{i \in V} x_i & \\ \sum_{i \in V(C)} x_i \leq |C| - 1 & \forall C \text{ circuit of } G \\ x_i \in \{0,1\} & \forall i \in V \end{cases}$$

These inequalities are called the **circuit inequalities**.

Formulation (F_C) is clearly equivalent to the AISP.

A non-compact formulation

Circuit inequalities are in **exponential number** with respect to the number of nodes.

Formulation (F_C) cannot be directly used as compact formulation in Cplex :

- Is this formulation really better than a compact one ?
- How to use such a non-compact formulation ?

For an integer linear formulation

$$(F) \quad \begin{cases} \max c^T x \\ \quad Ax \leq b \\ \quad x \in \mathbb{Z} \end{cases}$$

with n variables
with a exponential number of inequalities with respect to n.

For an integer linear formulation

$$(F) \quad \begin{cases} \max c^T x \\ Ax \leq b \\ x \in \mathbb{Z} \end{cases} \qquad (\tilde{F}) \quad \begin{cases} \max c^T x \\ Ax \leq b \\ \cancel{x \in \mathbb{Z}} \end{cases}$$

with n variables
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How can we solve the linear relaxation $(\tilde{F})$ of (F) ?

Polyhedron

A *hyperplane* of $\boldsymbol{R}^n$ is the set of points $\tilde{x} \in \boldsymbol{R}^n$ satisfying a linear equality $ax = \alpha$.

A *halfspace* of $\boldsymbol{R}^n$ is the set of points $\tilde{x} \in \boldsymbol{R}^n$ satisfying a linear equality $ax \leq \alpha$.

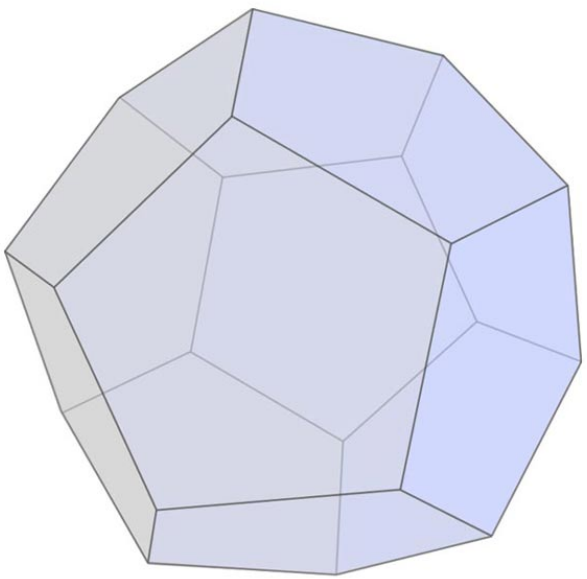
A **polyhedron** P is the intersection of a finite number of halfspaces
i.e.

$$P = \{\tilde{x} \in \boldsymbol{R}^n \mid A\tilde{x} \leq b\}$$

with $Ax \leq b$ system of linear inequalities.

Such a system $Ax \leq b$ **characterizes** a polyhedron.

A **polytope** is a bounded polyhedron.



Extreme point

• An **extreme point** (or vertex) of a polytope P is a point $x \in P$ s.t. there is no solutions $x^1 \neq x^2$ in P with $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$.

• Solving a (bounded) linear formulation

$$(\tilde{F}) \begin{cases} \max \ c^T x \\ \quad Ax \leq b \end{cases}$$

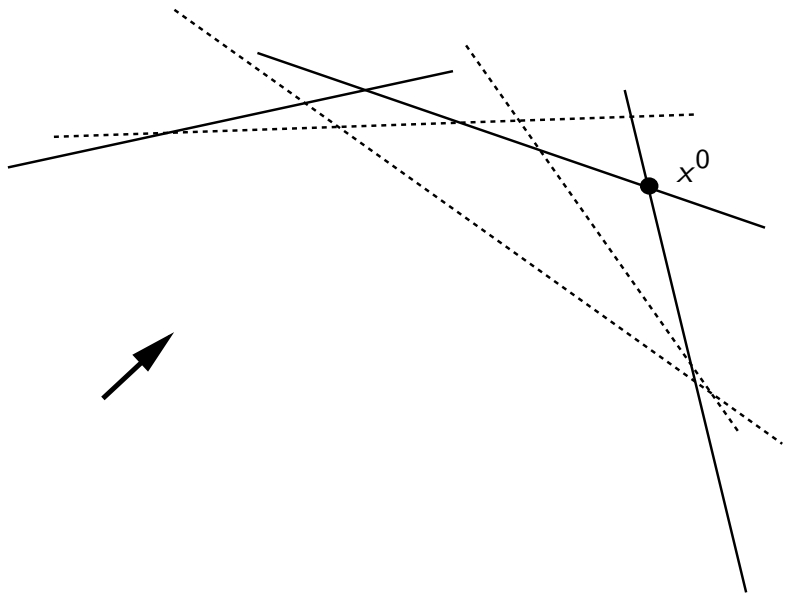
reduces to find an optimal extreme point of polytope

$$P = \{x \in \boldsymbol{R}^n \mid Ax \leq b\}$$

Initialisation

Let $A_0 x \leq b_0$ be a subset of inequalities of $Ax \leq b$ and $(\tilde{F}^0)$ the linear program restricted to $A_0 x \leq b_0$.
Let x^0 the solution of $(\tilde{F}^0)$.

$$(\tilde{F}^0) \begin{cases} \max c^T x \\ A_0 x \leq b_0 \end{cases}$$



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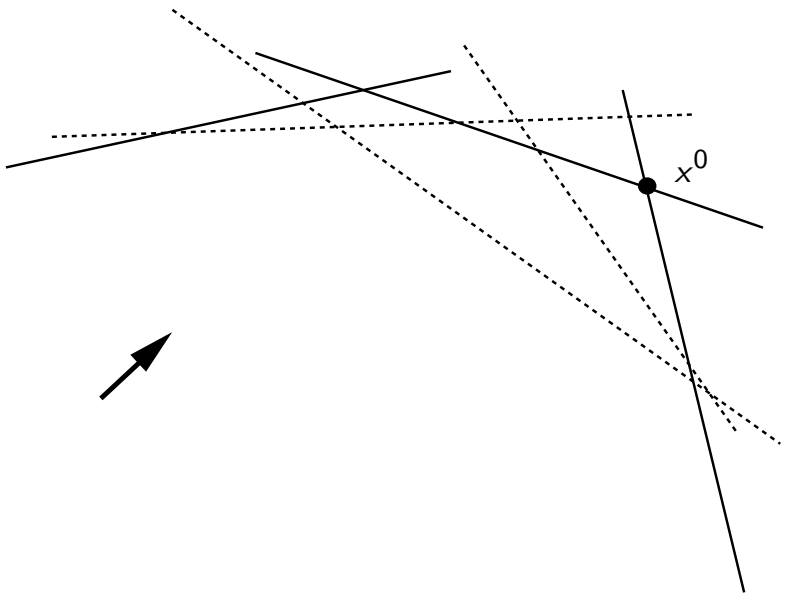


Solution x^0 is an extreme point of the polytope characterized by $A_0x \leq b_0$.

Note that :
if x^0 satisfies every inequality of $Ax \leq b$
x^0 is be an extreme point of the polytope characterized by $Ax \leq b$
and x^0 will an optimal solution !

Separation problem

Definition (Separation problem)

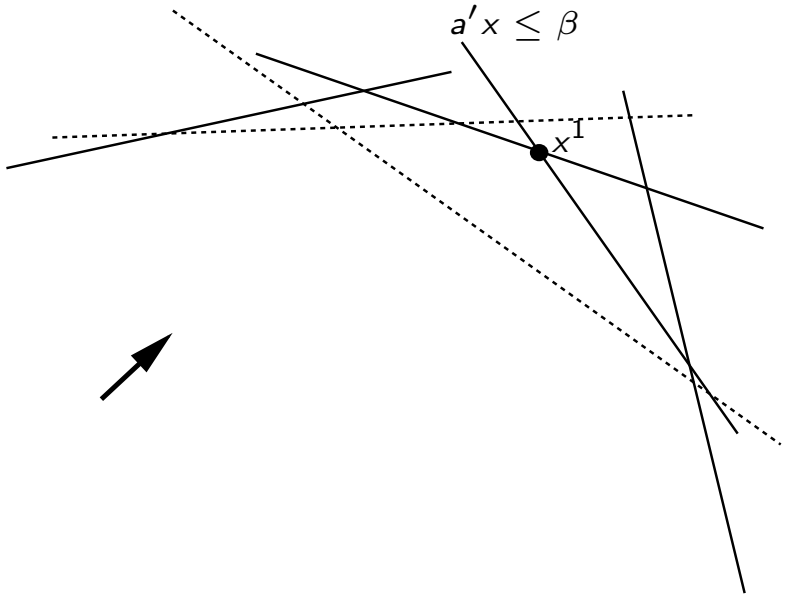
Given a point $\tilde{x} \in \boldsymbol{R}^n$,
the **separation problem** associated to $Ax \leq b$ and $\tilde{x}$ is
- to determine whether $\tilde{x}$ satisfies every inequality of $Ax \leq b$
- or to produce an inequality $ax \leq \alpha$ of $Ax \leq b$ violated by $\tilde{x}$.

An inequality $ax \leq \alpha$ of $Ax \leq b$ is **violated** by x^0 if $ax^0 > \alpha$.

Iteration

If the separation problem for x^0 produces an inequality $a'x \leq \beta$ violated by x^0

$$(\tilde{F}_1) \begin{cases} \max c^T x \\ A_0 x \leq b_0 \\ a'x \leq \beta \end{cases}$$



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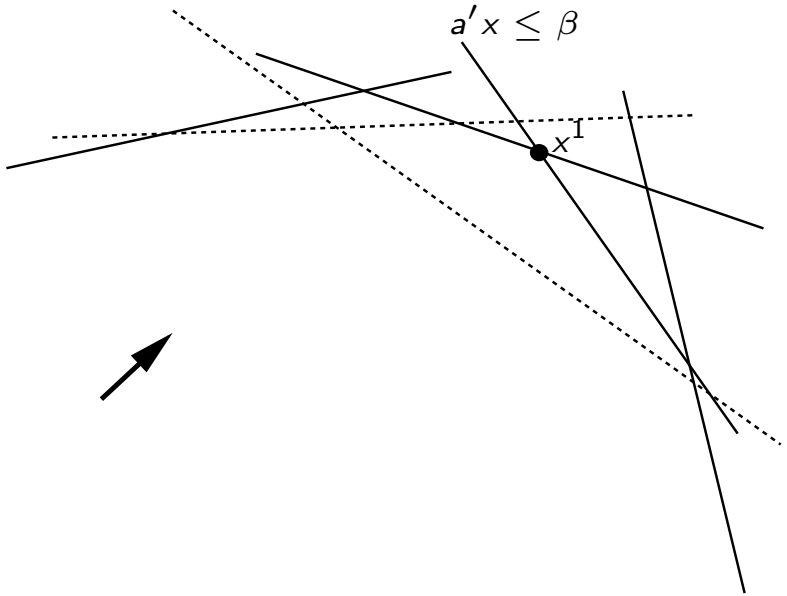

And so on !

Cutting plane based algorithm

Definition (Cutting-plane based "method")

While there exists an inequality $ax \leq \alpha$ of $Ax \leq b$ violated by x^i
$(\tilde{F}_{i+1}) \leftarrow (\tilde{F}_i) + ax \leq \alpha$
Solve the linear program $\tilde{F}_{i+1}$
$x_{i+1} \leftarrow$ solution of $\tilde{F}_{i+1}$
$i \leftarrow i + 1$

• An algorithm which solves the separation problem is called a **separation algorithm** (the whole method is to reiterate the separation algorithm).

• An inequality which is violated a solution x^i is called a **cut** because the inequality **separates** an useless part of R^n from the polytope characterized by $Ax \leq b$.

Validity of the method

- *Ending*

In the worst case, the algorithm enumerates every inequality of $Ax \leq b$.

- *Validity*

At the end of the loop, the final solution x^* is an extreme point of the whole system $Ax \leq b$ and maximizes $c^T x$:
then x^* is an optimal solution of $(\tilde{F})$.

Complexity

Theorem (Grötschel, Lovász, Schrijver, 1981)

A cutting plane based method of rational system $Ax \leq b$ is polynomial
if and only if
the separation algorithm associated to $Ax \leq b$ is polynomial.

This fundamental results is :

Optimize $\Leftrightarrow$ Separate

Separation algorithm for the circuit inequalities

Theorem

The circuit inequalities

$$\sum_{i \in V(C)} x_i \leq |C| - 1 \quad \forall \text{ circuit } C$$

can be separated in polynomial time.

Note that, by setting $x' = 1 - x$,
a circuit inequality can be rewritten

$$\sum_{i \in V(C)} x'_i \geq 1$$

Separation algorithm for the circuit inequalities

Given a point $\tilde{x} \in [0,1]^n$
The separation problem for the circuit inequalities is to determine whether or not there exists a circuit inequality violated by $\tilde{x}$ (and in the last case, to produce one violated inequality).

Set $x' = 1 - \tilde{x}$.
Find a circuit $\tilde{C}$ of minimal weight with respect to x'.
(This can be done in polynomial time since $x' \geq 0$)

- If $\sum_{i \in V(\tilde{C})} x'_i < 1$: $\sum_{i \in V(\tilde{C})} x_i \leq |\tilde{C}| - 1$ is violated by $\tilde{x}$.
- If $\sum_{i \in V(\tilde{C})} x'_i \geq 1$: there is no circuit ineq. violated by $\tilde{x}$. □

Branch-and-Cut algorithm

The cutting plane based method gives in polynomial time the relaxation value of an integer formulation.

At the end, the solution is not integer (unless if $P = NP$).

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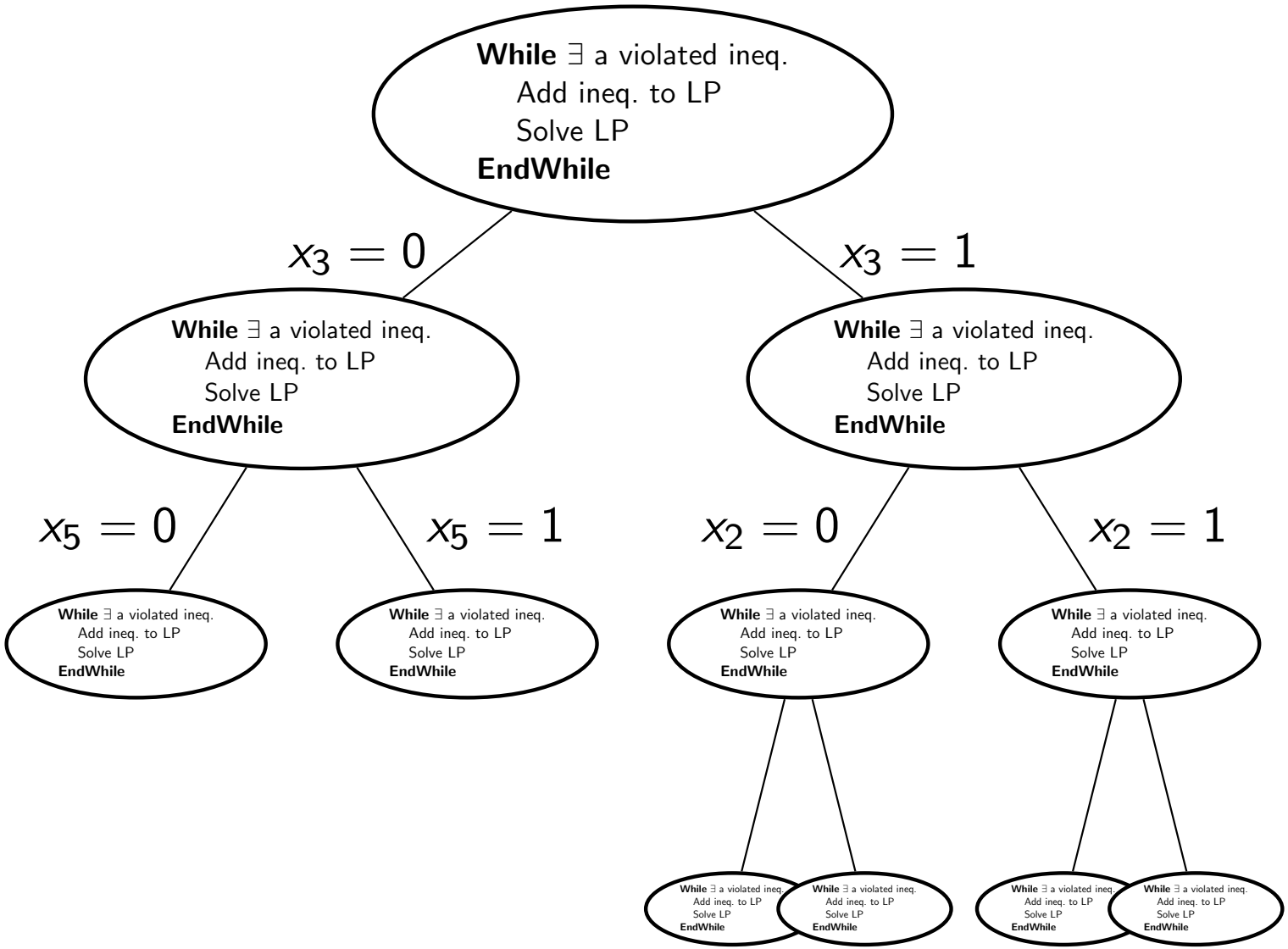
The only known general framewok to solve integer problem is... Branch&Bound !

A Branch&Bound that uses a cutting plane based algorithm in every node of the Branch&Bound tree
is a **Branch&Cut algorithm**.

Branch-and-cut algorithm



Formulation comparison

There is no generic method to compare the relaxation values of two formulations.

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Theorem

The relaxation value of the MTZ formulation (F_{MTZ})
$\leq$ *the relaxation value of the circuit formulation* (F_C).

Sketch of the proof.
First note that by summing MTZ inequalities over a circuit C, we obtain

$$\sum_{i \in V(C)} x_i \leq \left(1 - \frac{1}{2n}\right) |C|$$

Let $\tilde{x}$ be a solution of the relaxation $(\tilde{F}_C)$, then $\tilde{x}$ satisfies

$$\sum_{i \in V(C)} \tilde{x}_i \leq |C| - 1 \leq \left(1 - \frac{1}{2n}\right) |C|$$

By setting appropriate values u_i, we get a solution $(\tilde{x}, u)$ of (F_{MTZ}). □

Reinforcement

Given an integer formulation (F) $\begin{cases} \max c^T x \\ Ax \leq b \\ x \in \mathbb{Z} \end{cases}$

Definition

An inequality $ax \leq \alpha$ is **valid** for an integer formulation (F) if every integer solution of (F) satisfies $ax \leq \alpha$.

Adding a valid inequality to formulation (F) does not change the solution space of (F).

But valid inequalities can improve the relaxation value !

Obtaining valid inequalities

- Summing inequalities : Chvátal-Gomory rounding method :
- Multiplying inequalities : lift-and-project Lovász-Shrivjer meth.
- Finding particular sub-structures (stable set, knapsack,...)
- Lifting coefficients of inequalities (to obtain stronger ones)
- Disjunctive cuts, local branching inequalities,...
- ... and many others techniques ...

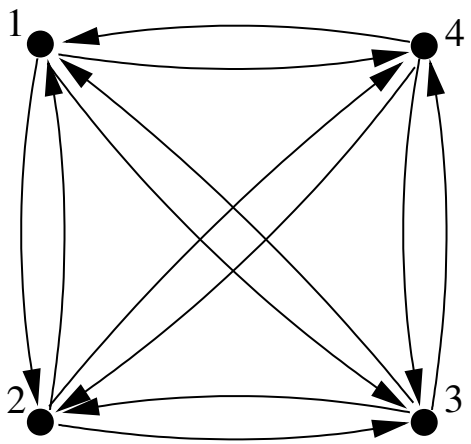
Clique inequalities

In a directed graph, a clique is a subset K of nodes inducing a complete subgraph.

The **clique inequalities**

$$\sum_{i \in K} x_i \leq 1 \text{ pour every clique } K \text{ of } G$$

are valid for the formulation.



Indeed, at most one node can be taken among a clique.

Clique inequalities

Lemma

The separation problem for the clique inequalities is NP-complete.

Indeed, proving that there is no violated clique inequality is equivalent to find a maximal weighted clique in G, which is a famous NP-hard problem □

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- However, a **heuristic separation algorithm** can be used !

For instance, a simple but efficient greedy heuristic :

Given a linear relaxation value x^* :

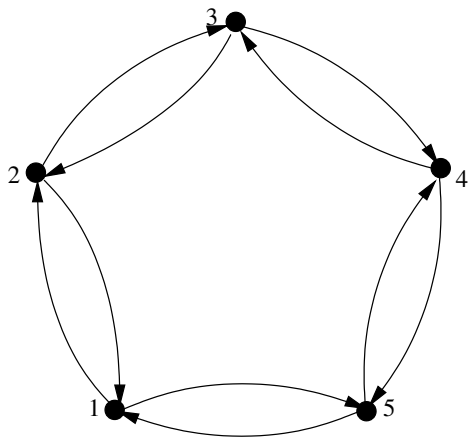
- sort the nodes with respect to decreasing values x_i^*
- $K \leftarrow \emptyset$
- iteratively try to add a node in K such that K stays a clique

Chvàtal sum techniques

For a "directed cycle" D like this one.

For each consecutive nodes i and $i+1$ there is a circuit inequality

$$x_i + x_{i+1} \leq 1$$



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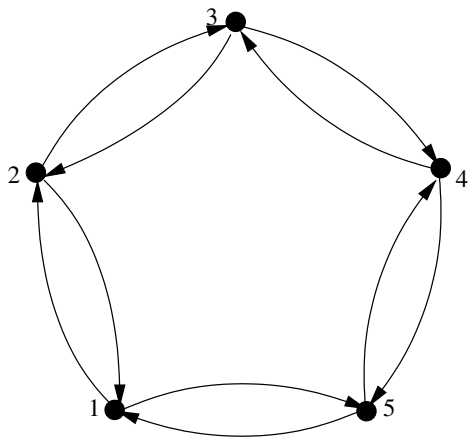

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By summing these inequalities :

$$\begin{array}{lllllll} x_1 & + & x_2 & & & \leq & 1 \\ & & x_2 & + & x_3 & \leq & 1 \\ & & & & \ldots & & \\ x_1 & & & + & x_{|D|} & \leq & 1 \end{array}$$

$$\sum_{i \in V(C)} x_i \leq \frac{|D|}{2}$$

odd cycle inequalities

The left part is integer, let's round it down

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odd cycle inequalities

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$$\sum_{i \in V(C)} x_i \leq \left\lfloor \frac{|D|}{2} \right\rfloor$$

- If $|D|$ is even, nothing new is obtained.
- If $|D|$ is odd, we obtain a new valid inequality

$$\sum_{i \in V(C)} x_i \leq \frac{|D| - 1}{2}$$

Such odd cycle inequalities can be found in the stable set polytope.

They can be separated in polynomial time.

What is inside a MIP solver ?

A MIP solver like CPLEX, GUROBI, XPRESS, SCIP,... are "Automatic Branch-and-Cut process.

- Strong preprocessing phase
- Automatic use of generic valid inequalities through efficient cutting plane based methods
- Automatic lifting operations to reinforce known inequalities and produce nex ones
- Automatic logical inference to break the symmetry of the branching tree
- Generic rounding heuristics

And it is more and more easy to add your own valid inequalities to these frameworks !

The travelling salesman problem (TSP)

n cities
c_{ij} the transportation cost between i and j
Find a Hamiltonian "tour" that visits each cities exactly once

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$$\min \sum_{i,j} c_{ij} x_{ij}$$

$$\sum_{j \in V} x_{ij} = 2 \qquad \forall i \in V,$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in V \times V.$$

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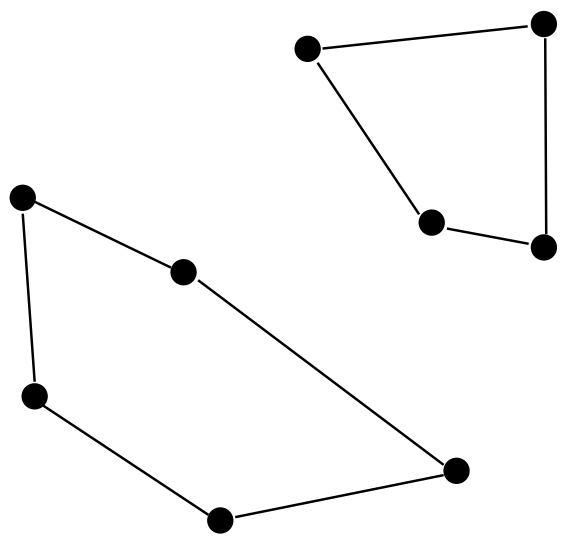

Unfortunately, the integer solutions contain "subtours".

Eliminating subtours

Menger's theorem :
a graph is connected if and only if every cut contains at least one edge.

Then "cut" inequalities

$$\sum_{e \in \delta(W)} x(e) \geq 1 \quad \forall W \subsetneq V \text{ and } W \neq \emptyset$$

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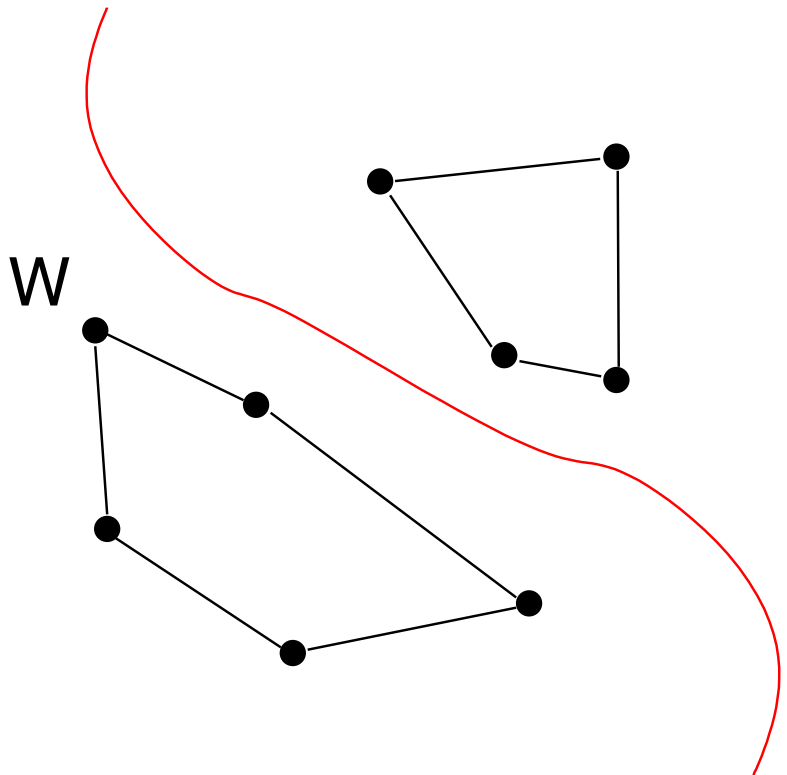

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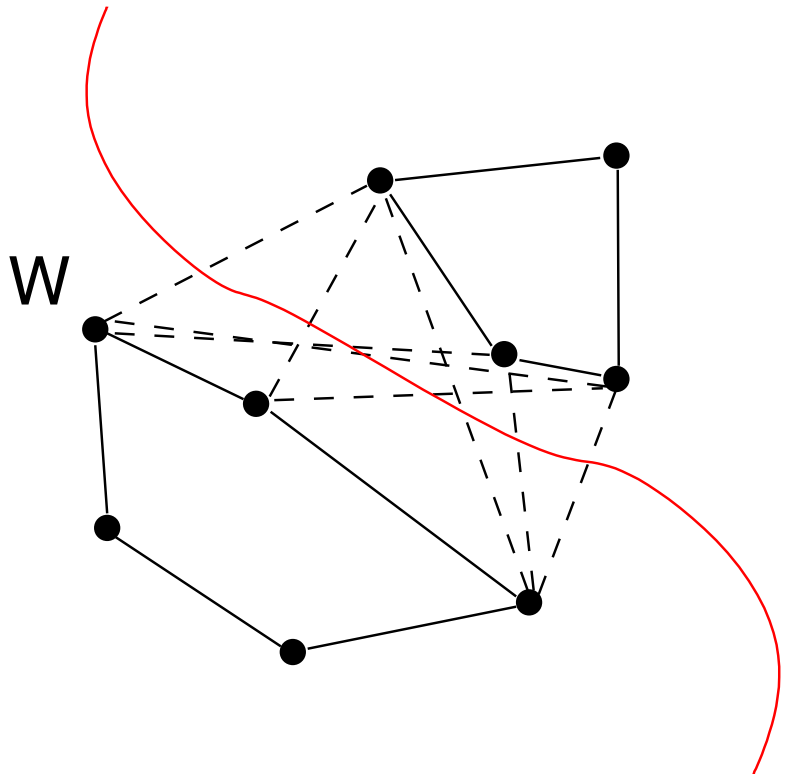
Then "cut" inequalities

$$\sum_{e \in \delta(W)} x(e) \geq 1 \quad \forall W \subsetneq V \text{ and } W \neq \emptyset$$



The TSP formulation

$$
\begin{aligned}
\min \sum_{e \in E} c(e)x(e) & \\
\sum_{e \in \delta(v)} x(e) = 2 & \quad \forall u \in V, \\
\sum_{e \in \delta(W)} x(e) \geq 2 & \quad \forall W \subsetneq V \text{ and } W \neq \emptyset, \\
x(e) \in \{0,1\} & \quad \forall e \in E.
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With (a lot of) additional facet defining inequalities,
this formulation succeed to solve instances with more than 200 000 cities.