Lecture

Mathematical Optimization and Polyhedral Approaches

Section 2 : Polynomial cases in Integer Linear Programming

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Here are some special cases where the linear relaxation of an ILP yields an integer solution... more precisely where the simplex algorithm systematically provides an integer solution.

Polyhedron and extreme points

A polyhedron is simply a geometric figure defined as the region bounded by ”planes”. In one-dimensional space, the ”planes” are points and the polyhedra are connected intervals. In two-dimensional space, the ”planes” are lines and the polyhedra are squares, rectangles,... In three-dimensional space, the ”planes” are planes and the polyhedra are cubes, dodecahedra,...

In fact, in R^n, we call a *hyperplane* H a subspace of R^n defined as the set of points satisfying a linear equation, i.e., there exist $a_1, ..., a_n, b \in R$ such that $H = \{x \in R^n \mid + a_1 x_1 + ... + a_n x_n = b\}$.

Definition

A *polyhedron* $P \subseteq R^n$ is the set of solutions of a finite system of linear inequalities, i.e.,

$$P = \{x \in R^n \mid Ax \leq b\},$$

where A is an $m \times n$ matrix (m and n positive integers) and $b \in R^m$.

We then say that the system $Ax \leq b$ **defines** or **characterizes** the polyhedron P.

In this course, we consider only rational polyhedra, i.e., those for which the coefficients of the system $Ax \leq b$ are all rational.
If A is an $m \times n$ matrix, we denote by A_i (resp. A^j) the i^{th} row (resp. j^{th} column) of A for $i = 1, ..., m$ (resp.$j = 1, ..., n$).

A **point** of a polyhedron is therefore defined by coordinates $\tilde{x} \in R^n$ such that $A\tilde{x} \leq b$.

A **polytope** is a bounded polyhedron, i.e., a polyhedron $P \subseteq R^n$ is a polytope if there exist $x^1, x^2 \in R^n$ such that $x^1 \leq x \leq x^2$, for all $x \in P$.

Definition

A point x of a polyhedron P is called an **extreme point** (or sometimes *vertex*) of P if there do not exist two solutions x^1 and x^2 of P, $x^1 \neq x^2$, such that $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$.

In other words, an extreme point of P is a point of P that is not the midpoint of a segment contained in P.

Theorem

Let $P = \{Ax \leq b\}$ be a polyhedron in R^n. Let $\tilde{x} \in R^n$ be a point. We denote by $\tilde{A}x \leq \tilde{b}$ the submatrix of constraints from $Ax \leq b$ formed by the inequalities satisfied with equality by $\tilde{x}$.
Then $\tilde{x}$ is an extreme point of P if and only if the set $\tilde{A}x \leq \tilde{b}$ has rank n.

Recall from linear algebra :
- a set of vectors is said to be *linearly independent* if none of them can be obtained as a linear combination of the others.
- the rank of a matrix is the maximum number of linearly independent rows of the matrix.

In other words, a point $\tilde{x}$ is extreme if we can produce n inequalities from the matrix A that are satisfied by $\tilde{x}$ with equality **and** that are linearly independent.

Fractional points of the knapsack problem

Consider the following knapsack problem :

$$\begin{aligned} \text{Max } & c_1 x_1 + c_2 x_2 + c_3 x_3 \\ & 2x_1 + 2x_2 + 5x_3 \leq 8 \\ & x_i \leq 1 \quad i = 1, ..., 3, \\ & x_i \geq 0 \quad i = 1, ..., 3, \\ & x_i \in N \quad i = 1, ..., 3. \end{aligned}$$

We can show that this problem admits fractional extreme points and give a case where it is optimal.

Answer : Consider the point $\tilde{x} = (1, 1, \frac{4}{5})$.
We can note that this point satisfies with equality 3 inequalities of the linear relaxation of the ILP : $x_1 = 1$, $x_2 = 1$ and the main inequality.
Moreover, these 3 inequalities are clearly linearly independent.
So this point is a fractional extreme point of the problem.
For example, with the weight $\tilde{c} = (10, 10, 1)$ this point is clearly optimal.

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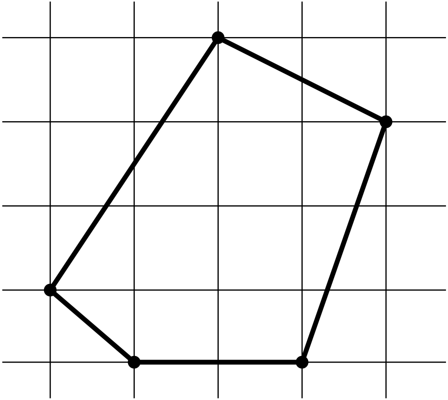
Integer polyhedra

A point in R^n is **integer** if its coordinates are integers.
A polyhedron is said to be **integer** if all its extreme points are integers.



Recap : Linear Programming

Recall two results from linear programming :

- All optimal solutions of an LP lie on one of the hyperplanes defining the solution polyhedron.
- The set of extreme points of the solution polyhedron contains at least one optimal solution. Therefore, we can limit the search for optimal solutions to the extreme points of the solution polyhedron.

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Theorem (Equivalence Integer Polyhedron/ILP)

A rational polytope P is integer **if and only if**
for every integer vector c, the optimal value of $Max\{c^T x \mid x \in P\}$ *is integer.*

Proof : The forward direction is immediate.
Conversely, consider $v = (v_1, \ldots, v_n)^T$ an extreme point of P (there exists one because it is pointed). Assume we can prove that there exists an integer vector w such that v is the unique optimal solution of $max\{w^T x \mid x \in P\}$ (let's assume this). Take $\lambda \in \mathbb{Z}$ such that $\lambda w^T v \geq \lambda w^T u + u_1 - v_1$ for every u extreme point of P. Note that v is still the unique optimal solution of $max\{\lambda w^T x \mid x \in P\}$. Thus, by setting the weight $\bar{w} = (\lambda w_1 + 1, \lambda w_2, \ldots, \lambda w_n)^T$, we see that v is also an optimal solution of $max\{\bar{w}^T x \mid x \in P\}$ because $\lambda w^t v > x$ for all $x \in P$. But by construction $\bar{w}^T v = \lambda w^T v + v_1$ and since, by hypothesis, $\bar{w}^T v$ and $\lambda w^T v$ are integers, then v_1 is integer. We can repeat this for all components so v is integer. □

(It is possible to extend this result to unbounded polyhedra.)

Definitions :

- A polyhedron is said to be *pointed* if it contains at least one extreme point. For example, the polyhedron $\{(x_1, x_2) \in R^2 : x_1 \geq 0\}$ contains no extreme points.
- A polyhedron is said to be *rational* if it can be defined by a system where all inequalities have rational coefficients.

Here we consider only rational pointed polyhedra, which is not restrictive from a computational perspective.

A simple example

Consider the following knapsack problem where b is integer.

$$\begin{aligned} \text{Max} \quad & \sum_{i=1}^{n} c_i x_i \\ & \sum_{i=1}^{n} x_i \leq b \\ & x_i \leq 1 \quad i = 1, ..., n, \\ & x_i \geq 0 \quad i = 1, ..., n, \\ & x_i \in N \quad i = 1, ..., n. \end{aligned}$$

Let's show that this system is integer.

Answer : An extreme point of this system must satisfy with equality n linearly independent inequalities of this system. But there are few inequalities here that can be satisfied with equality !

There are two cases :

- either the main inequality is not satisfied with equality : in this case, only the trivial inequalities are tight and the point is integer.
- or the main inequality is satisfied with equality and then there are $n-1$ trivial inequalities tight. So the point consists of $n-1$ integer components 0 or 1. Let N^+ denote the components equal to 1. The last unknown component therefore has the value $b - |N^+|$: note that this quantity is necessarily positive, integer, and at most 1. So this last component is also integer.

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Unimodularity

It would be very interesting to be able to characterize matrices corresponding to integer polyhedra.

A **square** matrix A is said to be **unimodular** if A is integer and if its determinant is $+1$ or -1.

Lemma

Let A be an integer square matrix of size m, invertible. Then $A^{-1}b$ is an integer vector for every integer vector b of size m if and only if A is unimodular.

Proof : By a classic result of linear algebra, we know that $A^{-1} = \frac{A^{adj}}{det(A)}$ where A^{adj} is the adjugate matrix of A, i.e., the matrix obtained by transposing the matrix of cofactors $C_{ij} = (-1)^{i+j}M_{ij}$ and where M_{ij} is the determinant of the submatrix obtained from A by deleting row i and column j. So if A is integer, A^{adj} is also integer. So if A is unimodular, A^{-1} is integer and thus $A^{-1}b$ is an integer vector.
Conversely, if $A^{-1}b$ is an integer vector for every integer vector b of size m, then in particular $A^{-1}e_i$ is integer for e_i the i$^{\text{th}}$ unit vector for all $i = 1, \ldots, m$. So A^{-1} is integer and therefore $det(A)$ and $det(A^{-1})$ are both integers. Since $det(A).det(A^{-1}) = 1$, we have $det(A) = 1$ or -1. □

Unimodularity

The previous lemma leads to the following definition.

An $m \times n$ matrix A with $n \geq m$ of rank m is said to be **unimodular** if A is integer and if the matrix associated with each of its bases has determinant $+1$ or -1.
(A basis of A is a set of m linearly independent column vectors and the matrix associated with a basis is thus an invertible $m \times m$ square submatrix).

Theorem

[Veinott and Dantzig]
Let A be an integer $m \times n$ matrix of full row rank. The polyhedron defined by $Ax = b, x \geq 0$ is integer for every integer vector b if and only if A is unimodular.

Total unimodularity

A matrix is called **totally unimodular** (TU) if all its square submatrices have determinant 0, 1, or -1.
Thus, for a unimodular matrix, the coefficients are therefore only 0, 1, and -1.

We can note that in fact an $m \times n$ matrix A is TU if and only if the matrix $[AI]$ of size $m \times (m+n)$ (which is obtained by appending an identity matrix to A) is unimodular.

A consequence of the previous theorem gives then the following important theorem.

Theorem

[Hoffman-Kruskal]
Let A be an $m \times n$ TU matrix. Then the polyhedron defined by $Ax \leq b, x \geq 0$ is integer for every integer vector b.

Note that this theorem is not a characterization of integer polyhedra.
Moreover, it is not easy to detect whether a matrix is TU. An essential (and complex) result by Seymour (1980) proves in fact that these matrices can be constructed according to a particular scheme.

Special cases of TU matrices

While this result is quite complex, there is a special case of TU matrices that is very easy to recognize.

Theorem (Poincaré (1900))

Let A be a matrix whose coefficients are 0, 1, or -1 and such that each column contains at most one coefficient 1 and at most one coefficient -1.
Then A is TU.

And so, if the theorem applies, any solution of an LP using A as the coefficient matrix of the inequalities corresponds to an integer polyhedron.

Special cases of TU matrices

This more general theorem generalizes that of Poincaré

Theorem

Let A be a matrix whose coefficients are 0, 1, or -1 and such that

- *each column contains at most two non-zero coefficients.*
- *the rows of A can be partitioned into two sets I_1 and I_2 such that*
  - *if a column has two coefficients of opposite signs then their rows are in the same set I_1 or I_2*
  - *if a column has two coefficients of the same sign then their rows are one in I_1 and the other in I_2.*

Then A is TU.

This second theorem implies the first when $I_2 = \emptyset$.

Minimum cost flow problem (without capacity)

Consider a **network** $G = (V, A)$ which is a directed graph with a vertex s, called source, without predecessors from which every vertex of G can be reached and a vertex t, called sink, which is reachable from every vertex of G. An *s-t-flow* is a positive vector $x \in R^m$ if it satisfies the "flow conservation" constraint

$$\sum_{a \in \delta^+(u)} x(a) - \sum_{a \in \delta^-(u)} x(a) = 0 \quad \forall u \in v \setminus \{s, t\}.$$

We associate a cost $w(a) \in R_+$ to each arc $a \in A$.

The **minimum cost flow problem** or **min-cost flow** consists in finding a flow such that the value $\sum_{a \in A} w(a)x(a)$ is minimized.

Minimum cost flow problem (without capacity)

Consider the following LP :

$$\text{Min} \quad \sum_{a \in A} w(a)x(a)$$

$$\sum_{a \in \delta^+(u)} x(a) - \sum_{a \in \delta^-(u)} x(a) = 0 \quad \forall u \in V \setminus \{s, t\}$$

Poincaré's result proves that the constraint matrix of this LP, called the **flow matrix**, is TU.

Indeed, the variable $x(a)$ appears twice in its column : once for each of its endpoints, either with a coefficient -1 or with a coefficient +1.

So this LP corresponds to an integer polyhedron : it will always have an integer solution.

But this example, where the flow is unbounded, has an optimal solution of zero... it's not a good example !

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Minimum cost flow problem (the real one)

Let a network G be equipped with a weight w on the arcs
... and a minimum **integer** flow capacity $b(a) \in N$ associated with each arc $a \in A$.

This is formulated as

$$\begin{aligned} \text{Min} \quad & \sum_{a \in A} w(a)x(a) \\ & \sum_{a \in \delta^+(u)} x(a) - \sum_{a \in \delta^-(u)} x(a) = 0 \quad \forall u \in V \setminus \{s,t\} \\ & x(a) \geq b(a) \quad \forall a \in A. \end{aligned}$$

We can prove that this LP also corresponds to an integer polyhedron.

Proof. Let $\tilde{x}$ be an extreme point of the polyhedron. If a component $x(a)$ satisfies the capacity inequality with equality, this component is integer. Let A^f be the set of arcs of A with flow $\tilde{x}(a) > b(a)$. Consider the system derived from the flow conservation inequalities : we transform the original inequalities by fixing, to their bounds $b(a)$, the components $\tilde{x}(a)$ with $a \notin A^f$. This system consists only of the flow matrix limited to arcs A^f so the solution from this matrix will have integer components.

Bipartite matching (assignment) problem

Let a complete bipartite graph $G = (V_1 \cup V_2, E)$ be associated with a weight $c \in N^m$ associated with the edges of E.

A **matching** of G is a set of edges pairwise non-incident.

The **bipartite matching problem** consists in determining a matching that maximizes the sum of the weights of its edges.

There exist several efficient polynomial algorithms to solve this classical problem (including the famous Hungarian algorithm).

Bipartite matching (assignment) problem

Consider the following LP that formulates the problem :

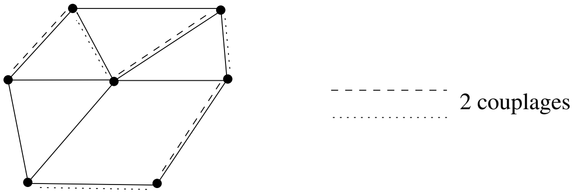
$$\begin{aligned}
\text{Max} \sum_{e \in E} c(e)x(e) \\
\sum_{e \in \delta(u)} x(e) \leq 1 \quad \forall u \in V_1 \\
\sum_{e \in \delta(u)} x(e) \leq 1 \quad \forall u \in V_2 \\
x(e) \geq 0 \quad \forall e \in E.
\end{aligned}$$

The matrix of this LP is TU by the theorem using $I_1 = V_1$ and $I_2 = V_2$.

Matching (assignment) problem

Let's generalize the bipartite matching problem.

In an undirected graph $G = (V, E)$, a **matching** is a set of edges pairwise non-adjacent (i.e., with no common vertex). The graph is equipped with a weight $w(e)$ associated with each edge $e \in E$.



The **maximum matching problem** consists in finding a matching of maximum cardinality (or maximum weight).

Matching (assignment) problem

A possible formulation that generalizes the previous one is :

$$\begin{aligned} \text{Max} \sum_{e \in E} c(e)x(e) & \\ \sum_{e \in \delta(u)} x(e) \leq 1, & \quad \forall v \in V, \\ 0 \leq x(e) \leq 1, & \quad \forall e \in E, \\ x(e) \text{ integer}, & \quad \forall e \in E. \end{aligned}$$

Indeed, in a matching, there is at most one edge incident to each vertex.

• If G is bipartite, it can be written according to the previous formulation : the LP has a TU matrix and the corresponding polyhedron is therefore integer.

• If G is not bipartite, what happens ?

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Matching (assignment) problem

We cannot find sets I_1 and I_2 satisfying the theorem.
Indeed, since the graph is not bipartite, we cannot "bipartition" the set of rows as required by the theorem.

In fact, the associated LP does not correspond to an integer LP (and thus is not TU). We can produce a counterexample by producing a graph and a **fractional extreme point** (i.e., non-integer) :

Take $G = C$ limited to a cycle C (odd) of 5 vertices.
The extreme point $\tilde{x}$ assigning to each edge the value $\frac{1}{2}$ satisfies each of the 5 inequalities (1) with equality.
Moreover, these 5 inequalities (1) are linearly independent :
it is indeed a fractional extreme point of the polyhedron of the formulation.

And yet the matching problem is polynomial even for non-bipartite G !

There is no contradiction !

This formulation is not "integer" but there exists one, given by Jack Edmonds

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Recap : Duality

For a linear program $(\tilde{P})$, then called the *primal*, the *dual* is the linear program $(\tilde{D})$ as follows

$$(\tilde{P}) \begin{cases} \text{Max} & z = c^T x \\ & Ax \leq b \\ & x \geq 0 \end{cases} \qquad (\tilde{D}) \begin{cases} \text{Min} & w = b^T y \\ & A^T y \geq c \\ & y \geq 0 \end{cases}$$

In "algebraic" form :

$$(\tilde{P}) \begin{cases} \text{Max} & z = \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \ \ \forall i = 1, \ldots, m \\ & x_i \geq 0 \ \ \forall j = 1, \ldots, n. \end{cases} \qquad (\tilde{D}) \begin{cases} \text{Min} & w = \sum_{i=1}^{m} b_i y_i \\ & \sum_{i=1}^{m} a_{ij} y_i \geq c_j \ \ \forall j = 1, \ldots \\ & y_j \geq 0 \ \ \forall i = 1, \ldots, m. \end{cases}$$

The **dual variables** of $(\tilde{D})$ correspond to the inequalities of $(\tilde{P})$.
The matrix of $(\tilde{D})$ is the transpose A^T.
The costs of the objective function and the right-hand side terms of the inequalities swap roles.

Recap : Duality

- **Weak duality theorem :**
If $(\tilde{P})$ and $(\tilde{D})$ each admit a solution $\tilde{x}$ and $\tilde{y}$,
then $c^T\tilde{x} \leq b^T\tilde{y}$.

- **Duality theorem :**
If $(\tilde{P})$ admits an optimal (finite) solution,
then $(\tilde{D})$ also does and moreover they "coincide", *i.e.*

$$(\tilde{P}) \quad max\{c^T x \mid Ax \leq b\} = min\{b^T y \mid y^T A = c, y \geq 0\} \quad (\tilde{D}).$$

Min-Max bound

Consider an ILP (P) and its linear relaxation $(\tilde{P})$ which is an LP.
Consider an LP $(\tilde{D})$ dual of $(\tilde{P})$ and its ILP version (D).
An integer solution x of (P) is a solution of $(\tilde{P})$.
An integer solution y of (D) is a solution of $(\tilde{D})$.
Let x^* be an optimal solution of $(\tilde{P})$ then (by weak duality)

$$c^T x \leq c^T x^* \leq b^T y^* \leq b^T y$$

In the VERY special case where we know both :
- an algorithm giving an integer solution x (approximate) for (P)
- and an algorithm giving an integer solution y (approximate) for (D)
then we have a **Min-Max bound** $[x, y]$ for the integer solution of (P) (and of (D)).

Totally dual integrality

A natural question is to know when such a bound is an equality !

Consider an ILP (P) with linear relaxation $(\tilde{P}) = \{c^T x \mid Ax \leq b\}$. And the dual of $(\tilde{P})$ is $(\tilde{D}) = \{b^T y \mid A^T y \geq c\}$.

The system $Ax \leq b$ is **totally dual integral** (TDI)
if, for every integer vector c such that there exists an optimal solution of $(\tilde{P})$, then this solution can be obtained by an integer vector y in $(\tilde{D})$.

Theorem

Let $Ax \leq b$ be a TDI system with $(\tilde{P}) = \{cx \mid Ax \leq b\}$ rational and b integer. Then $(\tilde{P})$ is an integer polytope, i.e. $(P) = (\tilde{P})$.

Proof : By the definition of $(\tilde{P})$ being TDI, for every optimal solution x of the problem, there exists an integer vector y solution of the dual that achieves this optimum, i.e., such that $c^T x = y^T b$. But since b is integer, if y is integer, $c^T x$ is therefore integer. So by the theorem [Equivalence Integer Polyhedron/ILP], x is integer. Therefore $(\tilde{P})$ is integer. □

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Consider an ILP (P) with linear relaxation $(\tilde{P}) = \{c^T x \mid Ax \leq b\}$. And the dual of $(\tilde{P})$ is $(\tilde{D}) = \{b^T y \mid A^T y \geq c\}$.

The system $Ax \leq b$ is **totally dual integral** (TDI)
if, for every integer vector c such that there exists an optimal solution of $(\tilde{P})$, then this solution can be obtained by an integer vector y in $(\tilde{D})$.

Theorem

Let $Ax \leq b$ be a TDI system with $(\tilde{P}) = \{cx \mid Ax \leq b\}$ rational and b integer. Then $(\tilde{P})$ is an integer polytope, i.e. $(P) = (\tilde{P})$.

Proof : By the definition of $(\tilde{P})$ being TDI, for every optimal solution x of the problem, there exists an integer vector y solution of the dual that achieves this optimum, i.e., such that $c^T x = y^T b$. But since b is integer, if y is integer, $c^T x$ is therefore integer. So by the theorem [Equivalence Integer Polyhedron/ILP], x is integer. Therefore $(\tilde{P})$ is integer. □

One could also be interested in characterizing TDI systems. But this does not really make sense relative to solving a combinatorial optimization problem : indeed, for every rational system $Ax \leq b$, there exists a positive integer t such that $\frac{1}{t}Ax \leq \frac{1}{t}b$ is TDI. The existence of such a system therefore says nothing about the structure of the associated polyhedron P. In fact, we have the following result.

Theorem

[Giles and Pulleyblank]
Let P be a rational polyhedron. Then there exists a TDI system $Ax \leq b$ with A integer such that $P = \{Ax \leq b\}$. Moreover, if P is integer, then b can be chosen integer.

This result tells us that there always exists a TDI system for every integer polyhedron P and that consequently, there exists a system that always has integer solutions.

- Determining a TDI system allows proving the integrality of a polyhedron (and its LP).

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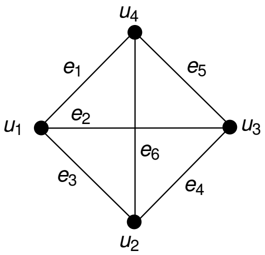
A small non-TDI example

Consider the following (very simple) problem.
We call a *multi-set*, a set of elements where each element can be represented multiple times.
Given a complete graph on 4 vertices $K_4 = (V, E)$ whose edges are equipped with a weight $w(e)$, $e \in E$, determine a multi-set of edges of maximal weight of K_4 such that, for each vertex of the graph, there are at most 2 edges incident to this vertex (we can call this problem the maximal "2-matching").



A small non-TDI example

This problem is formulated by the following ILP (P) :

$$\text{Max} \sum_{e \in E} w(e)x(e)$$

$$\sum_{e \in \delta(u)} x(e) \leq 2 \quad \forall u \in V,$$

$$x(e) \geq 0 \quad \forall e \in E,$$

$$x(e) \in N \quad \forall e \in E.$$

Note that $x(e)$ is indeed taken in N.

Look at the optimal solution for $w(e) = 1$ for $e \in E$:
the optimal solution of this problem is clearly 4.
Indeed,
- either an edge is taken 2 times : without loss of generality consider e_1, then only e_4 can be added at most 2 times.
- or each edge is taken at most 1 time : an optimal solution is then clearly to take 4 edges forming a square.

A small non-TDI example

Let's show that the system formed by inequalities (1-1) is not TDI.

Answer : For this, consider the dual of this system based on variables y associated with inequalities (1), i.e., with the vertices of the graph :

$$\text{Min} \sum_{v \in V} 2y(v)$$
$$y(u) + y(v) \geq w_e \quad \forall e = uv \in E,$$
$$y(v) \geq 0 \quad \forall v \in V.$$

We can note that, for the weight $w(e) = 1, e \in E$, it is impossible that there exists an integer dual solution of value 4. Therefore there does not exist an integer solution of the dual corresponding to an optimal solution of the primal of value 4 : the system is therefore not TDI.

Indeed, any integer solution of this dual must have at least 3 of the variables y equal to at least 1 (otherwise one of the inequalities would not be satisfied for one of the edges) : so any integer solution of this dual is at least of value 6.

(A less obvious question would be to search how to change the formulation of the problem to make it TDI.)

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Maximum flow problem

Consider a **network** $G = (V, A)$ which is a directed graph with a vertex s, called source, without predecessors from which every vertex of G can be reached and a vertex t, called sink, which is reachable from every vertex of G. An *s-t-flow* is a positive vector $x \in R^m$ if it satisfies the "flow conservation" constraint

$$\sum_{a \in \delta^+(u)} x(a) - \sum_{a \in \delta^-(u)} x(a) = 0 \quad \forall u \in v \setminus \{s, t\}.$$

We associate a maximum capacity $b(a) \in N$ to each arc $a \in A$. A flow is said to be *feasible* if $x(a) \leq b(a)$ for every arc $a \in A$.

Let $v = \sum_{a \in \delta^+(s)} x(a)$ be the value of the flow entering s (which is exactly also the value of the flow leaving p : $v = \sum_{a \in \delta^-(s)} x(a)$).

The **maximum capacity flow problem** or **maximum flow problem** consists in finding a flow such that its value is maximized.

We know from the Ford-Fulkerson theorem that if c is integer, then there exists an integer optimal flow and we know how to determine it in strongly polynomial time.

Max-Flow/Min-Cut duality

Recalling the statement of the maximum flow problem, we can define a second problem called the **minimum cut problem** or **min-cut**.

A ***s*-*t*-cut** (or rather here directed s-t-cut) is a set of arcs C leaving a set of vertices W such that $s \in W$ and $t \notin W$, *i.e.* $C = \delta^+(W)$. (similarly $C = \delta^-(\bar{W})$ where $\bar{W} = V \setminus W$.)

The **capacity of a cut** C is the sum of the capacities of the arcs in the cut : $b(C) = \sum_{a \in C} b(a)$.

According to the Ford-Fulkerson theorem concerning maximum flows, we know that for any maximum flow, we can algorithmically determine an associated minimum capacity cut such that the value of the flow equals the value of the cut.
We speak of the Max-Flow/Min-Cut duality.

Maximum flow problem

Consider the following LP formulation.

$$
\begin{aligned}
\text{Max} \quad & v \\
& \sum_{a \in \delta^+(u)} x(a) - \sum_{a \in \delta^+(u)} x(a) = 0 \quad \forall u \in V \setminus \{s, t\} \\
& \sum_{a \in \delta^+(s)} x(a) - v = 0, \\
& \sum_{a \in \delta^-(t)} x(a) + v = 0, \\
& x(a) \leq b(a) \quad \forall a \in A, \\
& v \geq 0, \\
& x(a) \geq 0 \quad \forall a \in A.
\end{aligned}
$$

Let's show that the LP of max-flow is associated with a dual LP forming a TDI system. Hint : the solutions of the min-cut problem are solutions of the dual.

Answer : Write the dual as follows : denote $\pi(u)$ the dual variables associated with the first 3 inequalities for $u \in V$ and denote $\gamma(u, v)$ the dual variables associated with the arcs $a = (u, v)$ of the capacity constraints.

$$\text{Min} \sum_{a=(u,v)\in A} b(a)\gamma(u, v)$$

$$\pi(u) - \pi(v) + \gamma(u, v) \geq 0 \quad \forall a = (u, v) \in A,$$

$$-\pi(s) + \pi(t) \geq 1,$$

$$\pi(u) \lesseqgtr 0 \quad \forall u \in V,$$

$$\gamma(u, v) \geq 0 \quad \forall a = (u, v) \in A.$$

Given a maximum flow, i.e., a solution of the primal, we know how to determine by the Ford-Fulkerson algorithm, a minimum capacity *s*-*t*-cut. We can show that this cut corresponds to an integer solution $(\tilde{\gamma}, \tilde{\pi})$ of the dual and which has the same value as the maximum flow solution : which proves that the system is TDI (well almost... see next page).

This solution is given by $\tilde{\gamma}(u, v) = 1$ if $u \in W$ and 0 otherwise and $\tilde{\pi}(u) = 1$ if $u \notin W$ and 0 otherwise.
Indeed, we can prove that this solution $(\tilde{\gamma}, \tilde{\pi})$ is a dual solution by considering the 4 possible cases of an edge's situation in the graph relative to the cut (either $u \in W, v \notin W$; $u \in W, v \in W$, $u \notin W, v \in W$ and $u \notin W, v \notin W$).

The previous page proposes an integer solution of the dual for the max-flow formulation...

Attention, to be able to say that it is TDI, we need to perform this proof for every integer cost function c on the variables v and x !
The formulation is therefore that of the flow with the objective function :

$$\text{Max} \quad c_0 v + \sum_{a \in A} c_a x(a)$$

The corresponding dual then sees its right-hand side terms change :

$$\pi(u) - \pi(v) + \gamma(u, v) \geq c(a) \quad \forall a = (u, v) \in A$$
$$-\pi(s) + \pi(t) \geq c_0$$

The solution $(\tilde{\gamma}, \tilde{\pi})$ integer solution of the dual that coincides with the value of the primal is then less obvious to determine... (but it exists).

Complete characterization

We call a **complete characterization of a problem** the description of a set of inequalities and variables of an integer linear program whose optimal solutions are the optimal solutions of the problem.

Unless P equals NP, it is not possible to give a complete characterization of an NP-hard problem.

For polynomial problems, we can as we have just seen here :
- find a TU formulation
- find a TDI system
- or more ”specific” proofs for problems : for example proving the absence of fractional extreme points, or searching for all inequalities necessary for this description...

Characterizing ”by absence of fractional points”

Example of the bipartite matching problem

Let a complete bipartite graph $G = (V_1 \cup V_2, E)$ be associated with a weight $c \in N^m$ associated with the edges of E.
A **matching** of G is a set of edges pairwise non-incident. The **bipartite matching problem** consists in determining a matching that maximizes the sum of the weights of its edges.

Consider the following LP

$$\text{Max} \sum_{e \in E} c(e)x(e)$$
$$\sum_{e \in \delta(u)} x(e) \leq 1 \quad \forall u \in V_1$$
$$\sum_{e \in \delta(u)} x(e) \leq 1 \quad \forall u \in V_2$$
$$x(e) \geq 0 \quad \forall e \in E.$$

We have already shown that this LP is TU.
This means that the inequalities of this LP are the (complete) characterization

Characterizing ”by absence of fractional points”

Example of the bipartite matching problem

Let’s show this characterization again but using a proof of the type ”by absence of fractional points”.

Let P be the polytope associated with the previous formulation, i.e., the polytope defined by the inequalities of this formulation.
Let $x*$ be an extreme point of P
Set $E_f = \{e \in E \mid 0 < x^*(e) < 1\}$ and suppose E_f is non-empty (we want to obtain a contradiction).

Case 1 : E_f contains a cycle.
We will show that x^* cannot be an extreme point, i.e., we will construct two points y and z in P such that $x^* = \frac{1}{2}(y + z)$.
Consider a cycle C in E_f.
Actually, a graph is bipartite if and only if all its cycles are of even length.

So since G is bipartite, C is of even length.

Characterizing ”by absence of fractional points”

Example of the bipartite matching problem

Denote then $e_1, e_2, \ldots, e_{2k}$ the edges of C. Then set

$$y(e) = \begin{cases} x^* + \epsilon & \text{if } e \in \{e_1, e_3, e_5, \ldots, e_{2k-1}\} \\ x^* - \epsilon & \text{if } e \in \{e_2, e_4, e_6, \ldots, e_{2k}\} \\ x^* & \text{otherwise} \end{cases}$$

$$z(e) = \begin{cases} x^* - \epsilon & \text{if } e \in \{e_1, e_3, e_5, \ldots, e_{2k-1}\} \\ x^* + \epsilon & \text{if } e \in \{e_2, e_4, e_6, \ldots, e_{2k}\} \\ x^* & \text{otherwise} \end{cases}$$

By taking $\epsilon = min\{x^*(e), 1 - x^*(e) \mid e \in E_f\}$, we see that y and z are positive. So y and z satisfy the trivial inequalities. Moreover, $\epsilon > 0$ and y and z are distinct and distinct from x.

Characterizing "by absence of fractional points"

Furthermore, we show that y and z satisfy the other inequalities :
For $u \in V_1$, prove that y satisfies $\sum_{e \in \delta(u)} x(e) \leq 1$.
This is trivially true if u is not in the cycle C.
If u is in C, there are two edges e_i and e_{i+1} of C incident to u : we can see that then we have $\sum_{e \in \delta(u)} y(e) = \sum_{e \in \delta(u)} y(e) + \epsilon - \epsilon = \sum_{e \in \delta(u)} x^*(e) \leq 1$.
By symmetry, we see that y and z satisfy all inequalities of the formulation.

Finally, by construction, $x^* = \frac{1}{2}(y + z)$ so x is not an extreme point, a contradiction.

Characterizing ”by absence of fractional points”

Case 2 : E_f contains no cycle.

In this case, consider the longest path (in number of edges) $\mu = (e_1, e_2, \ldots, e_k)$ in E_f.
Now set $\epsilon = min\{x^*(e), 1 - x^*(e) \mid e \in E_f\}$. We have $\epsilon > 0$ and then set the points y and z defined as in Case 1 which are distinct and distinct from x.
Show that y and z indeed satisfy the inequalities of the formulation. In the same way as in Case 1, the trivial inequalities are satisfied as well as the degree inequalities for vertices outside μ and for interior vertices of μ.
We also prove that the degree inequalities are satisfied by y and z for the endpoints of μ. Indeed, let u be the first vertex of the path μ, incident to e_1. z clearly satisfies the degree inequality for u.

Now note that u cannot be incident to another edge in E_f because otherwise μ would not be a longest path. Moreover, since $x^*(e_1) > 0$, u therefore cannot be incident to an edge f such that $x^*(f) = 1$ so u is only incident to edges f such that $x^*(f) = 0$. Therefore y also satisfies the degree inequality for u (symmetric case for the other endpoint).

Characterizing ”by absence of fractional points”

Similarly to Case1, we can conclude that this formulation contains no fractional points : it is indeed a characterization of the matching problem in bipartite graphs. □

Conclusions with a definition

Consider a Combinatorial Optimization Problem $(\mathcal{P})$.
The expression *Integer formulation* for $(\mathcal{P})$ is "dangerous" :
- either it designates an Integer Linear Program (ILP) modeling $(\mathcal{P})$: which is a priori NP-hard to solve.
- or it designates a Linear Program all of whose extreme points are integer, which is potentially polynomial.

Here we will prefer the expression **"formulation without fractional extreme points"** to avoid this ambiguity.

And most of the time, the linear relaxations of an ILP possess fractional extreme points !