Lecture

Mathematical Optimization and Polyhedral Approaches
Section 0 : Introduction

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1. Definition and complexity
   - Two first problems
   - Combinatorial explosion
   - Problem complexity
   - $\mathcal{NP}$-hard
   - Strongly or weakly $\mathcal{NP}$-hard
2. List of OC problems

A **Combinatorial Optimization Problem**

is to

find a greatest (smallest) element
within a valuated finite set.

Combinatorial Optimization Problem

To find a greatest (smallest) element within a valuated finite set.

Given :
- a finite subset of elements $E = \{e_1, \ldots, e_n\}$
- a **solution set** $\mathcal{F}$ of subsets of E
- a weight $c = (c(e_1), \ldots, c(e_n))$

a **Combinatorial Optimization Problem** is to find a solution $F \in \mathcal{F}$ whose weight $c(F) = \sum_{e \in F} c(e)$ is maximum (or min.),

$$\textit{i.e.} \quad \max \{c(F) \mid F \in \mathcal{F}\}.$$

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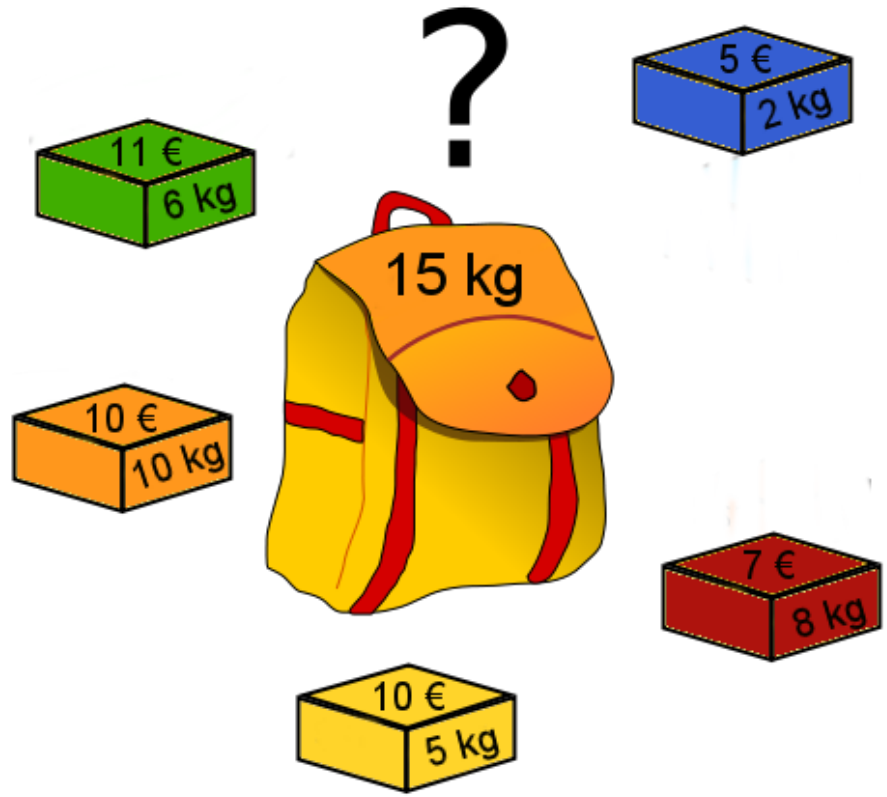
The knapsack problem

Which boxes to chose
to maximize the profit
without exceeding 15kg ?



Définition

Let us consider n **objects**.
Each object $i \in \{1, \ldots, n\}$
- with a **profit** g_i
- with a **weight** p_i
to be put inside a knapsack of **maximum total weight** P.

Then we need to find a subset $S \subset \{1, \ldots, n\}$ such that

its weight $\sum_{i \in S} p_i$ is non-greater than P *(Knapsack Constraint)*

its profit $\sum_{i \in S} g_i$ is maximum. *(Objective function)*

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How to incode a knapsack solution ?

An instance is given by :

i	1	2	3	4	5	
g_i	5	7	10	11	10	
p_i	2	8	10	6	5	≤ 15

A solution $S \subset \{1, \ldots, n\}$ is corresponding to **an incidence vector** χ^S such that

$$\chi^S[i] = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

$S_1 = \{1, 2, 5\}$

$\chi^{S_1} =$

1	1	0	0	1

profit : 5 7 10 $= 22$

weight : 2 8 5 $= 15$

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How to test whether a vector is a solution ?

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Recognition Algorithm :

$pds \leftarrow 0$

For i from 1 to n

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If $pds \leq P$ **Then** True

Else False

$S_2 = \{1, 4, 5\}$

$\chi^{S_2} =$ | 1 | 0 | 0 | 1 | 1 |

weight : $2 \quad 6 \quad 5 \quad = 13$

Solution of profit 26

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weight : 2 8 6 $= 16$

Not a solution.

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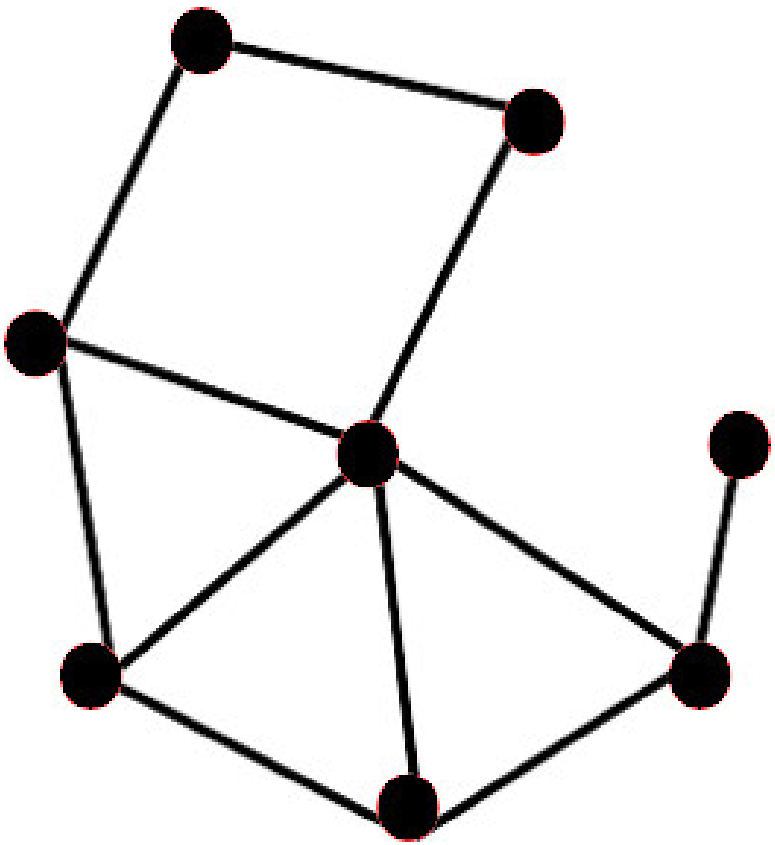
Not a solution.

Huge knapsacks...

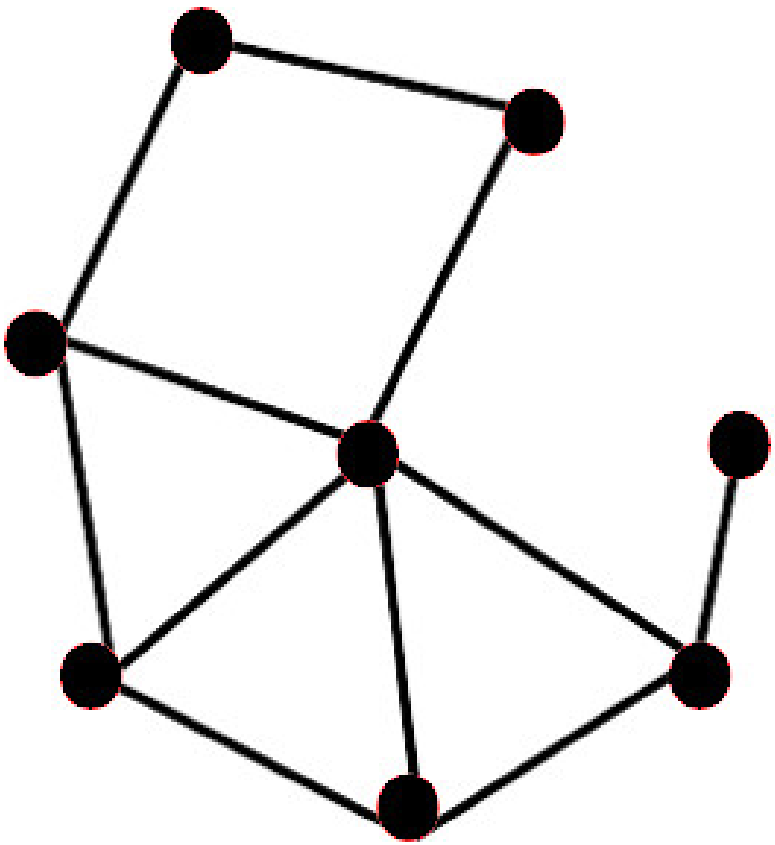
Graph



A **graph** $G = (V, E)$ is a pair where
- V is the node set
- $E \subseteq V \times V$ is the edge set.

Two nodes are **adjacent** if they are linked by an edge.

Graph



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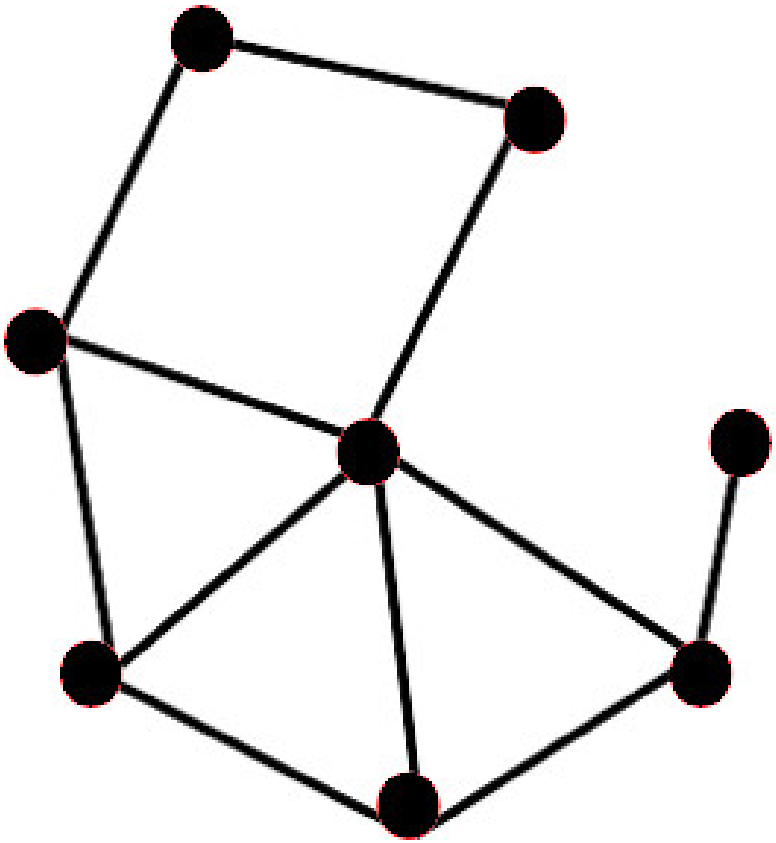
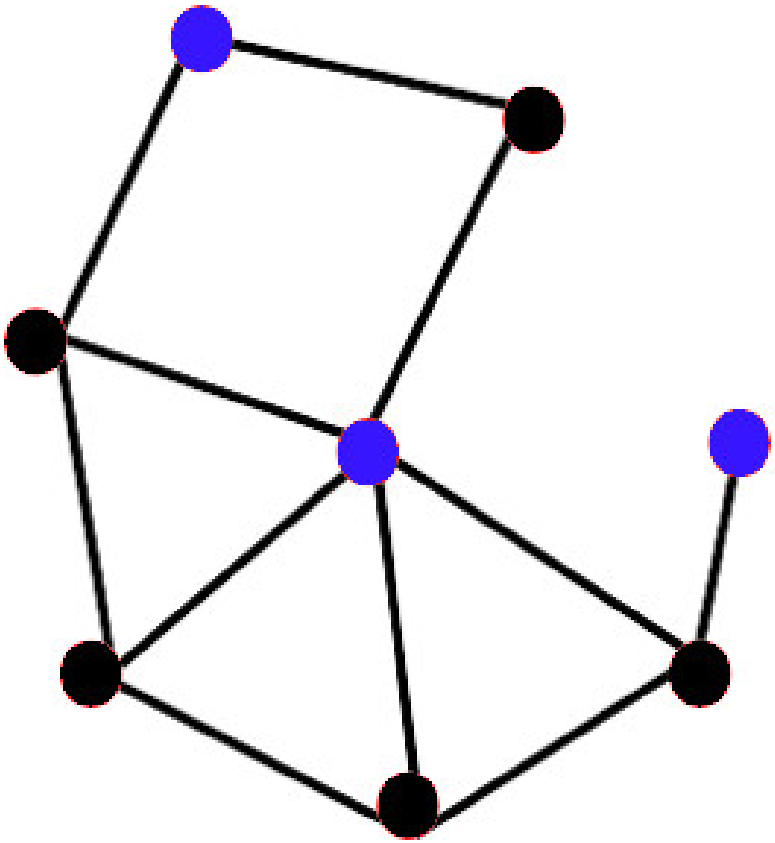


A **stable set**
(or independent set)
is a pairwise non-adjacent node subset.

The **stable set problem** is to find a stable set with a maximum number of nodes.

Stable Set problem



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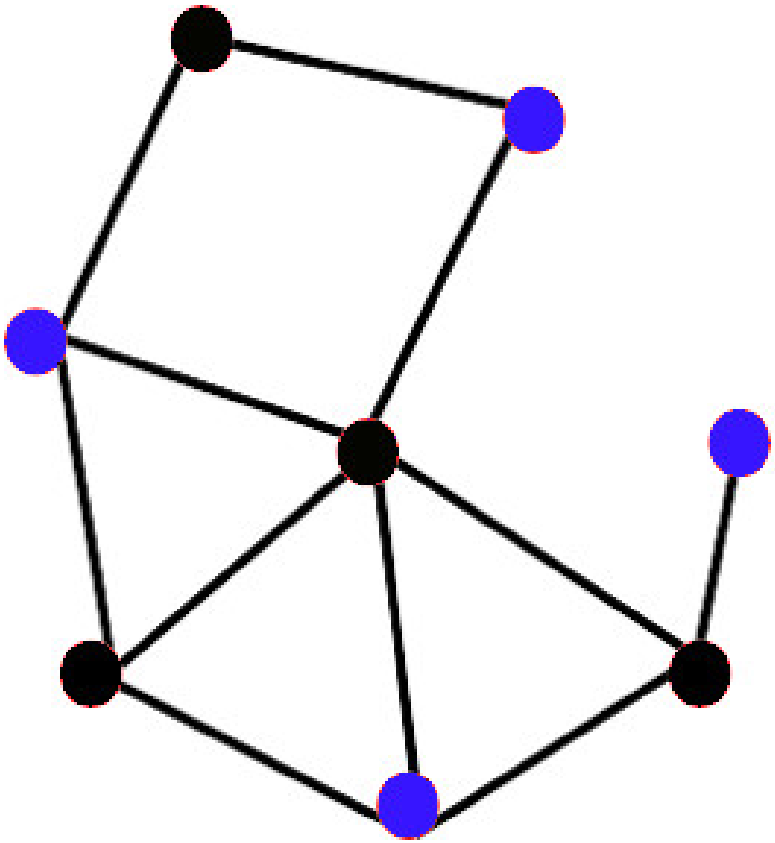

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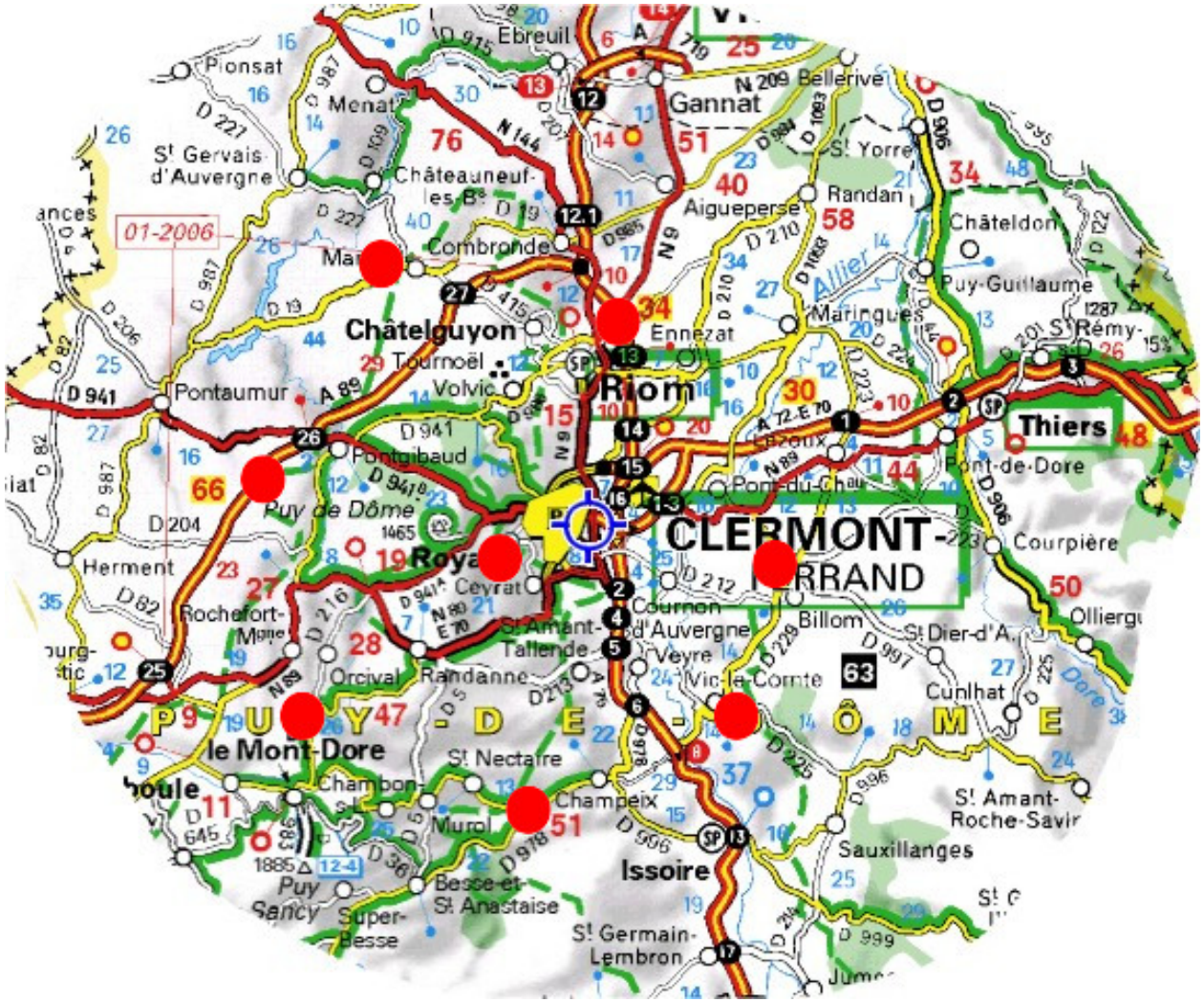
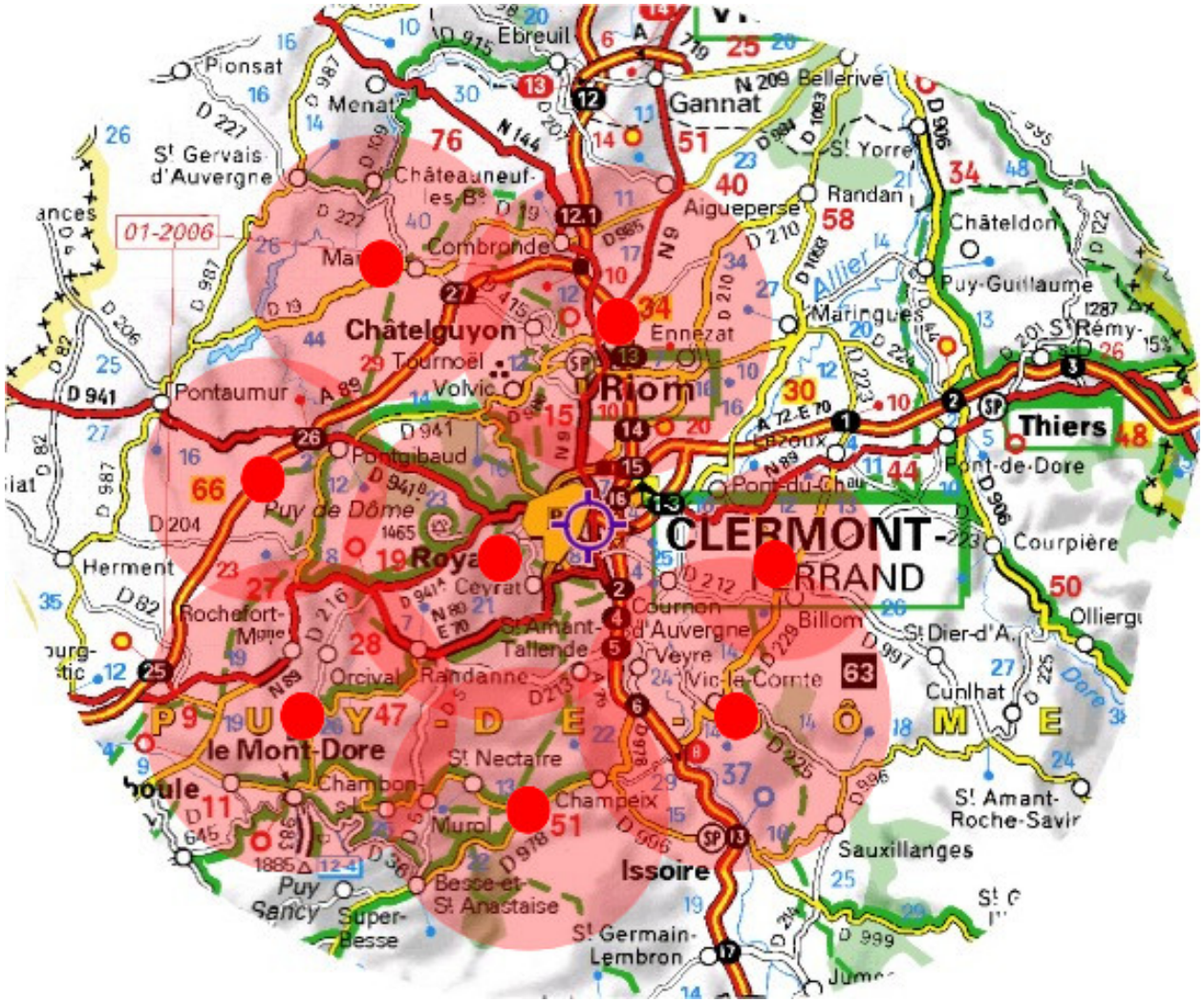
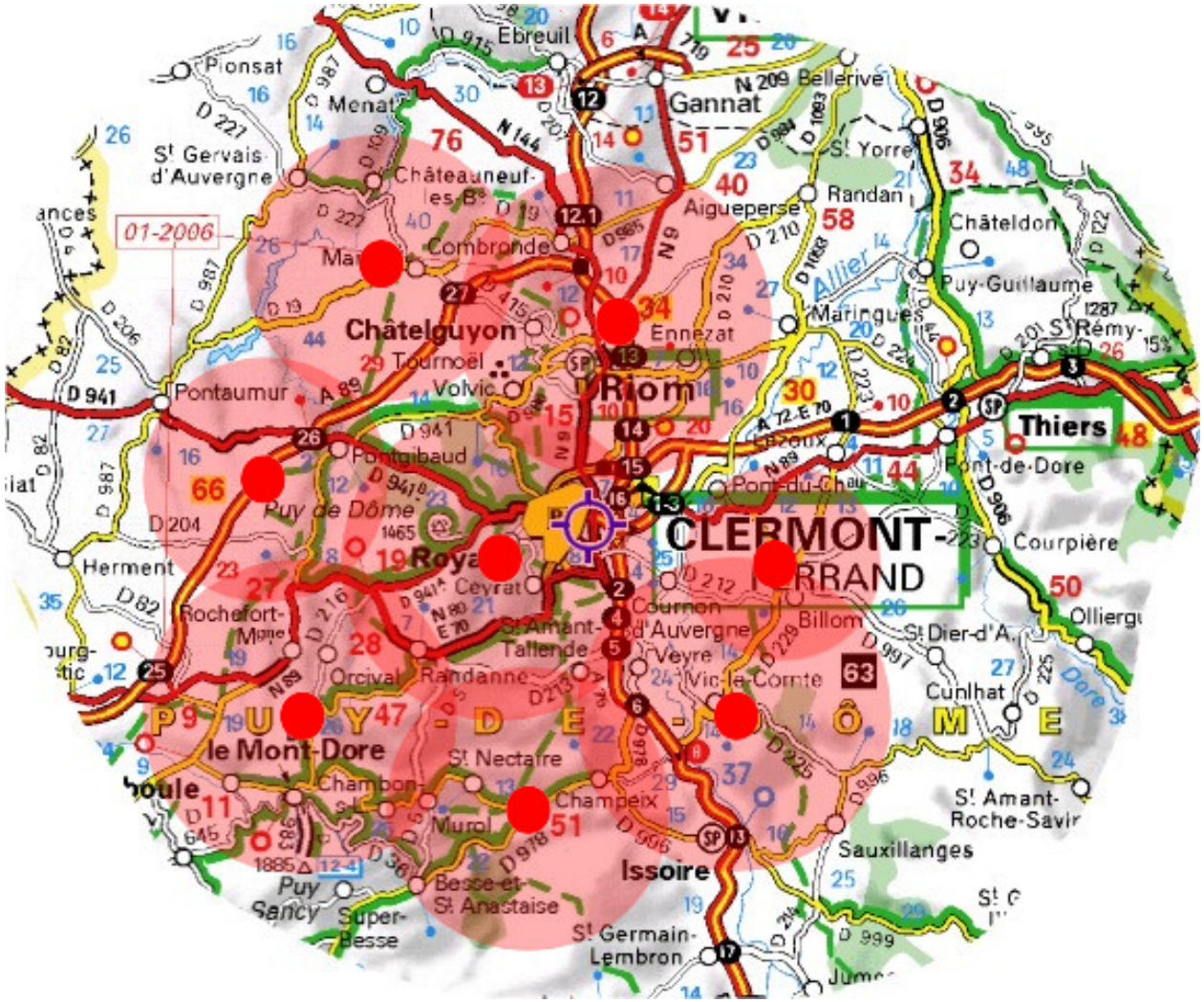
A set of antennas



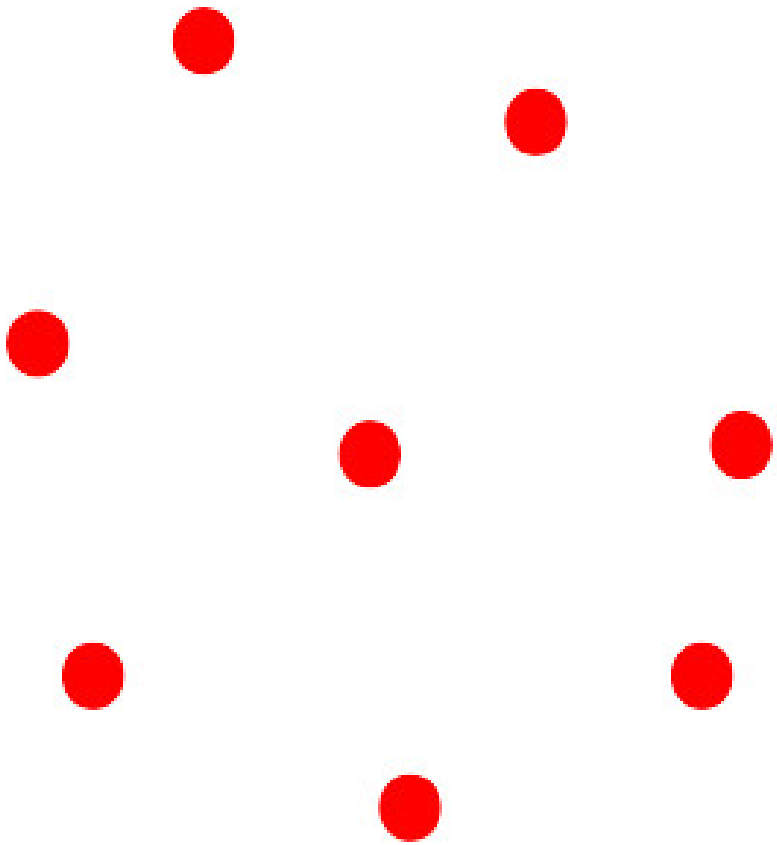
A set of antennas with their interference disks



How to find a subset of antennas with no interference ?



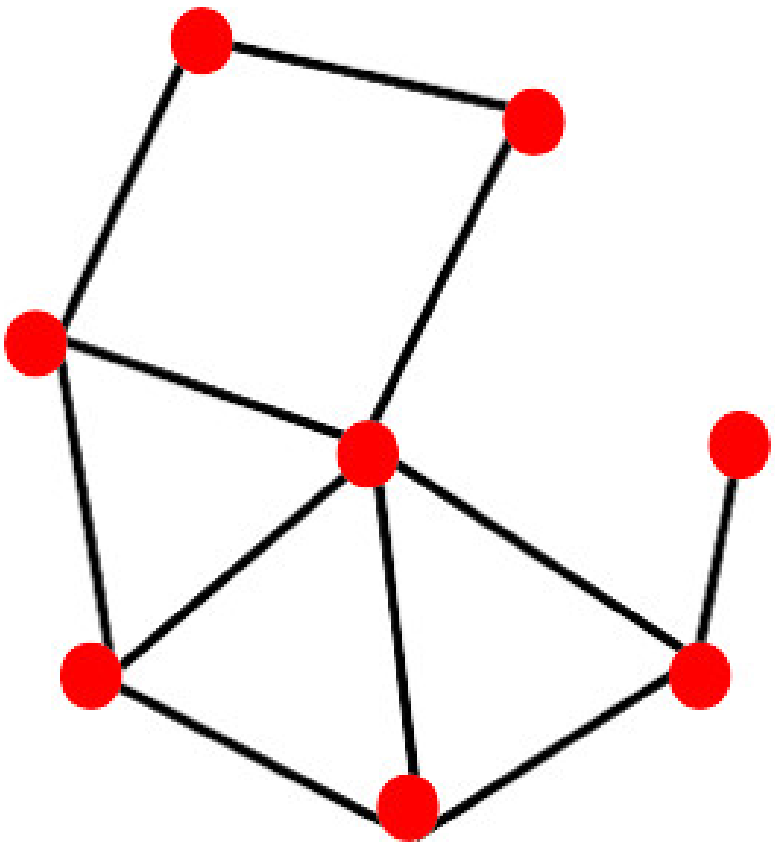
A model graph



Un graphe $G = (S, V)$
avec
- V : a node by antenna
- E : an edge between two antennas if their interference disks are in conflict.

Finding a subset of antennas with no interference reduces to the stable set problem.

A model graph



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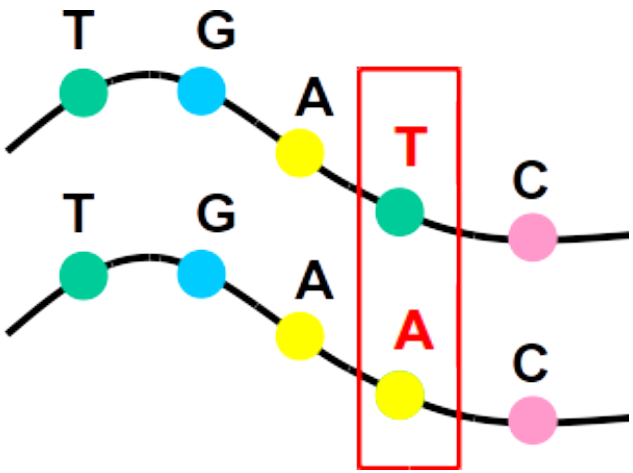
Combinatorics within genomic



Genome sequencing involves sorting through small fragments of DNA (called SNPs) that have been sequenced using bio-mechanical processes. In the process, false SNPs are created.

We want to sort the fragments, omitting as few SNPs as possible.

Its a **Combinatorial optimisation problem**

which also reduces to the **stable set problem !**.

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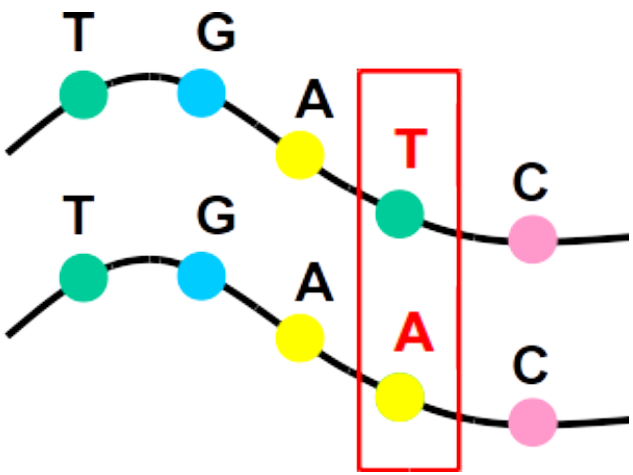

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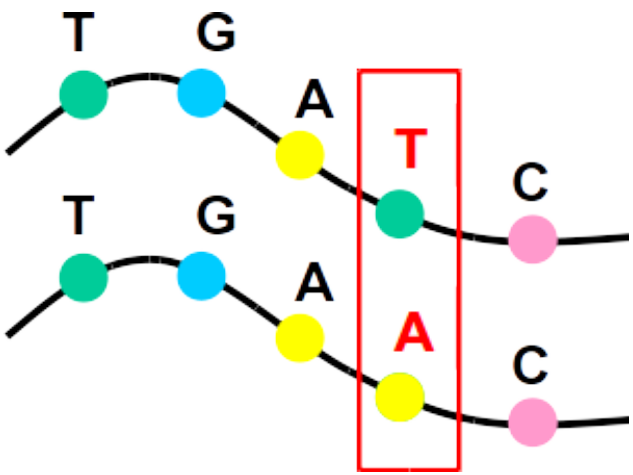

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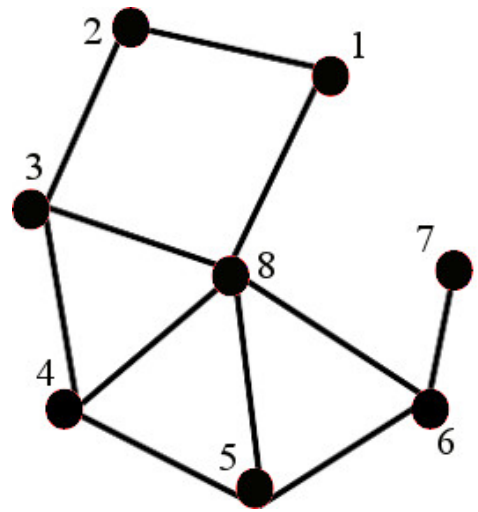
How to incode a stable set solution ? ?

Instance given by
a graph with
n nodes.



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$S_1 = \{1, 7, 8\}$

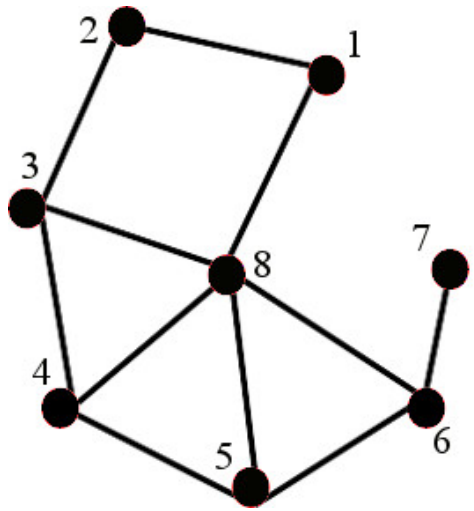
$\chi^{S_1} =$

i	1	2	3	4	5	6	7	8
	1	0	0	0	0	0	1	1

How to test whether a vector is a solution ?

Recognition Algorithm :

For i from 1 to n
  For j from 1 to n
    If $\{i, j\}$ is an edge
      and if $\chi^S[i] = 1$ and $\chi^S[j] = 1$
      Then STOP : False
  EndFor
EndFor
STOP : True



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Solution (with 3 nodes

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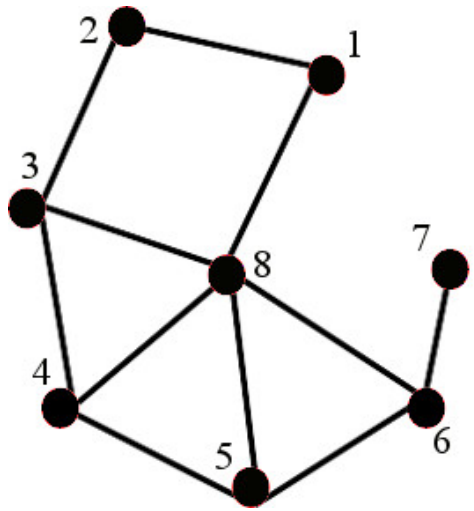

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Recognition algorithms

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If $pds \leq P$ **Then** True
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Execution time proportionnal to n.

Stable set

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Execution time proportionnal to n^2.

Polynomial Algorithms
which can be fast ($\leq n^4$ for instance).

Solving algorithms

Which is the complexity of the best known "Generic Algorithm" so find the optimal solution of combinatorial optimization problems ?

Only **very very slow** algorithms with an execution time proportional to
2^n, 3^n, $!n$

Exponential algorithms.

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Enumerate ?

Consider an optimization problem with n elements.
Assuming that we know a **polynomial** (and. fast) algorithm for recognizing a solution,
can we finde the best solution through enumeration ?

For each subset S of elements
  $\chi^S \leftarrow$ the incidence vector of S.
  Recogntion algorithm for χ^S.
  If χ^S is a solution
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STOP : the stocked solution.

There are 2^n **subsets** : **exponential** execution time !

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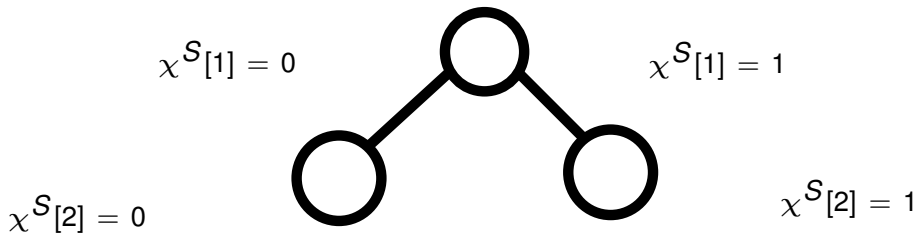

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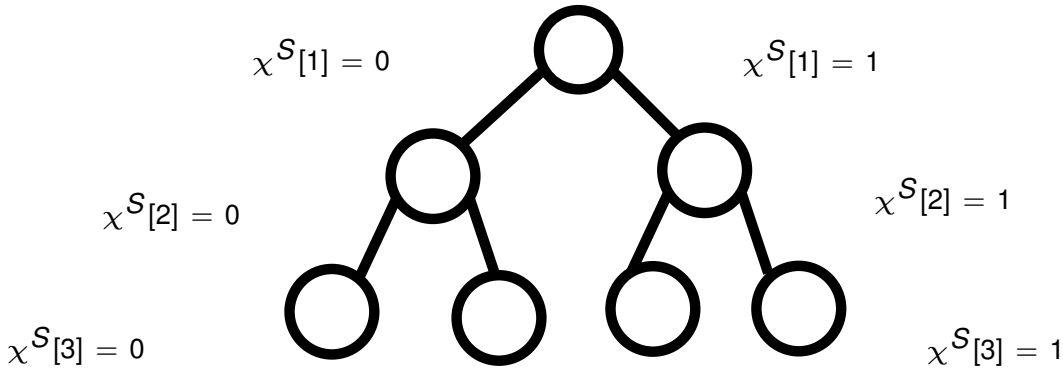

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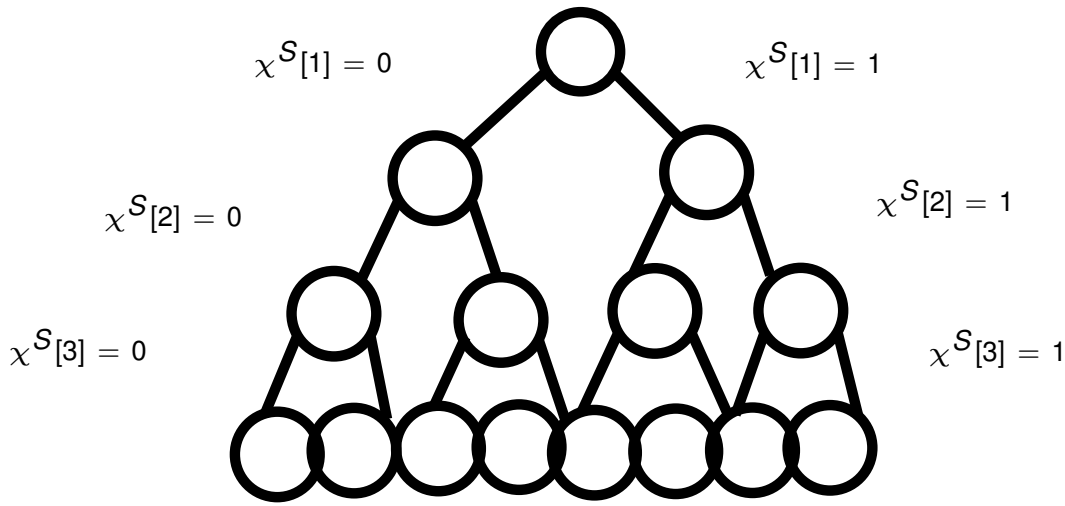

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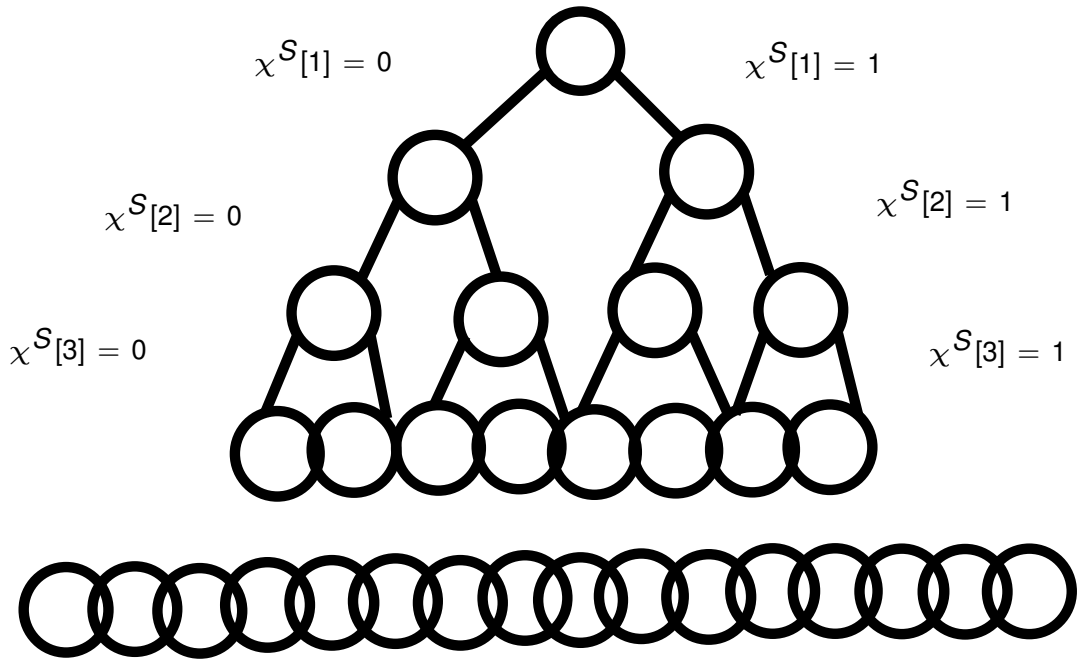

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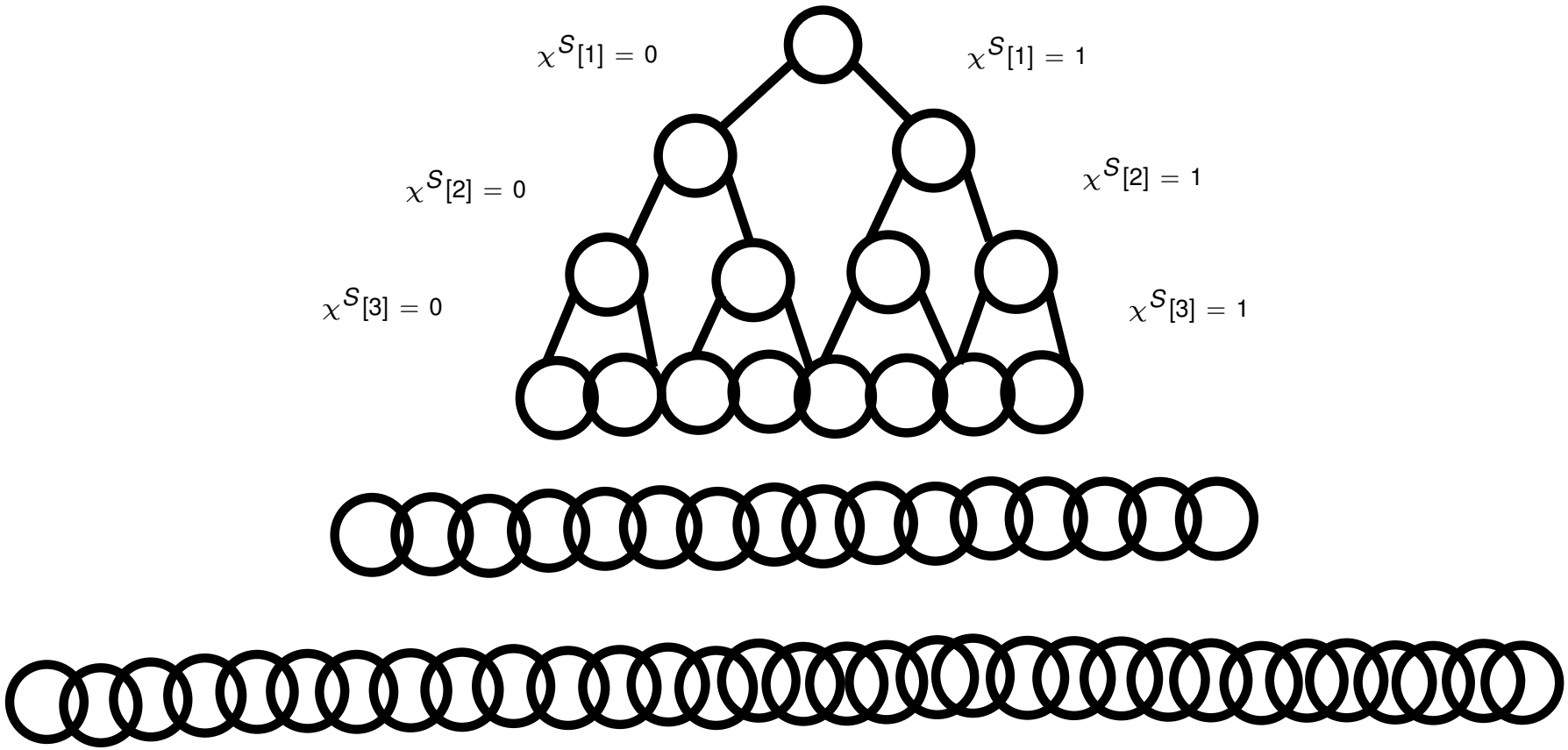

Combinatorial explosion

Let's take one of the most powerful computers in 2015
Tianhe-2 (China)
33.86 petaflops where 1 petaflop represents the processing of 10^{15} operations per second (one million billion).

Assume that the recognition algorithm takes $100n$ elementary operations. For $n = 10$, we can process 33 860 billion subsets in 1 second !

	Tianhe-2			Futuristic Computer
n	n^3	n^5	$100n2^n$	$100n2^n$
10	0,00...001 sec	0,00...001 sec	0,00...001 sec	0,00...001 sec
20	0,00...001 sec	0,00...001 sec	0,00000006 sec	0,00...001 sec
50	0,00...001 sec	0,00...001 sec	168,9 sec	0,000001 sec
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Age of the universe $13,7.10^{9}$ years

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Combinatorial Explosion

Enumerating 2^n solutions is not a technical problem
that very powerful computers could sweep away.

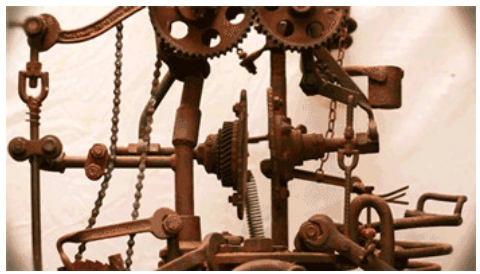
It's necessary to **circumvent the combinatorial explosion** with
mathematical and algorithmic tools.

Problem complexity

Just because we know an exponential algorithm for solving a problem doesn't mean it's difficult !

To crack a nut, you can :



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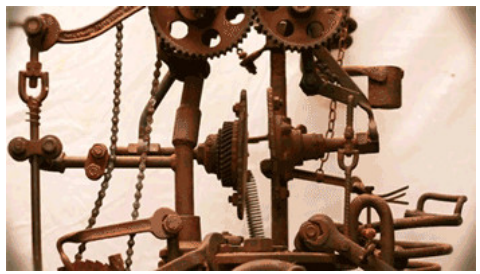


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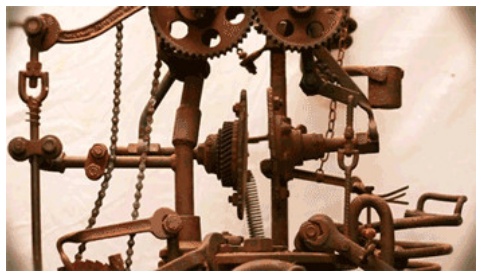


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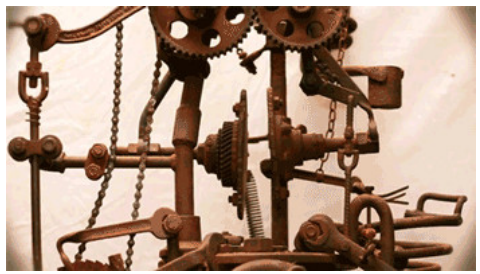


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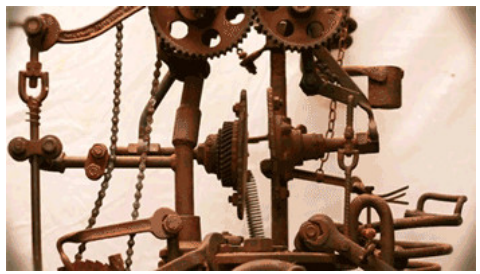


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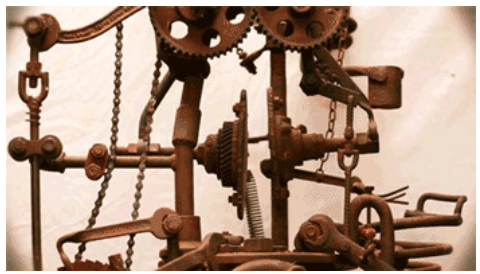


We're looking for the **fastest** algorithm to solve a problem !

Problem Complexity

A problem is **of exponential complexity**
if an exponential algorithm exists to solve it.
$\Rightarrow$ the problem class $\mathcal{EXP}$.

A problem is **of polynomial complexity**
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So $\mathcal{P}$ is **included** in $\mathcal{EXP}$, we note $\mathcal{P} \subset \mathcal{EXP}$

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$\mathcal{NP}$ problems

A particular classe have been created...
Combinatorial optimization problems for which we know **how to recognize a solution** with a polynomial algorithm

$\Rightarrow$ the problem class $\mathcal{NP}$ "Nondeterministic polynomial time"

Under this assumption, we've seen that it's possible to use an exponential enumeration algorithm, then

$\Rightarrow \mathcal{NP} \subset \mathcal{EXP}$

In addition, a polynomial problem is in $\mathcal{NP}$

$\Rightarrow \mathcal{P} \subset \mathcal{NP} \subset \mathcal{EXP}$

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And then ?

Is $\mathcal{P}$ equal to $\mathcal{NP}$?

i.e.

"Is there a polynomial algorithm for solving combinatorial optimization problems ?"

Answer : We don't know !

On a human scale, we only know this enumeration algorithm !

It's one of the 7 problems in the Clay Mathematics Institute of Cambridge's million-dollar Millennium Prize Problems !

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Is every Combinatorial Optimisation problem exponential ?

A very simple knapsack subcase

Let's look at a very simple knapsack instance :
a knapsack with all objects of the same weight.

A **greedy** algorithm :

- sort the n objects from the most expensive to the least expensive
- take the objects one by one in that order as long as they fit in the bag !

This algorithm is polynomial (of the order of n^2).

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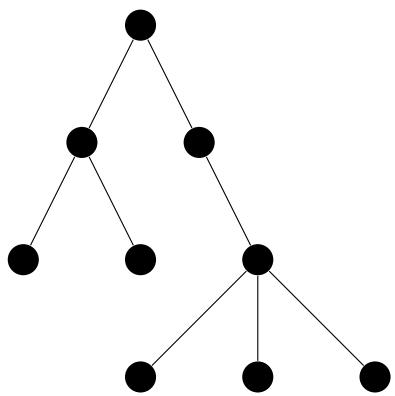
Let's look at a very simple stable set instance :
finding a maximum cardinality stable set over a **tree**.



Tree property :
Given a leaf v, there exists a maximum cardinality stable set containing v.

(Proof : Let v a leaf and u its unique neighbour. Let S be a maximum stable set. Either $u \notin S$, then $S \cup \{v\}$; or $u \notin S$, then $S \cup \{u\} \setminus \{v\}$ is another maximum stable set).

A very simple stable set subcase

A **greedy** algorithm :

$S \leftarrow \emptyset$
While G has at least one edge
    Let v be a leaf and u its neighbour
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    Delete from G nodes u and v and all their incident edges **EndWhile**
Add to S all the remaining nodes.
STOP : S

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$\mathcal{NP}$-hard

In the current state of scientific knowledge, it is not known whether or not there is a polynomial algorithm for solving the knapsack or the stable set problem in general !

In fact, we were able to prove that knapsack or the stable set problems are just as difficult as all the problems in the $\mathcal{NP}$ class !

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$\mathcal{NP}$-hardness

Boss, I can't find a polynomial algorithm for the stable set problem

Figure from "The Garey et Johnson"

$\mathcal{NP}$-hardness

But if you think I'm just an ungifted searcher

Figure from "The Garey et Johnson"

$\mathcal{NP}$-hardness

...neither are all the others !

Figure from “The Garey et Johnson”

Strongly or weakly $\mathcal{NP}$-hard
A computational problem may have numerical parameters : like the weight of a knapsack.

A $\mathcal{NP}$-hard problem is said to be **weakly $\mathcal{NP}$-hard**
if there is an algorithm for the problem whose running time is polynomial in the dimension of the problem and magnitudes of its data.

And otherwise, it is called **strongly $\mathcal{NP}$-hard**.

For weakly $\mathcal{NP}$-hard, it's often the case, that there exists a **dynaming programming scheme** whose complexity depends on the magnitudes of the data.

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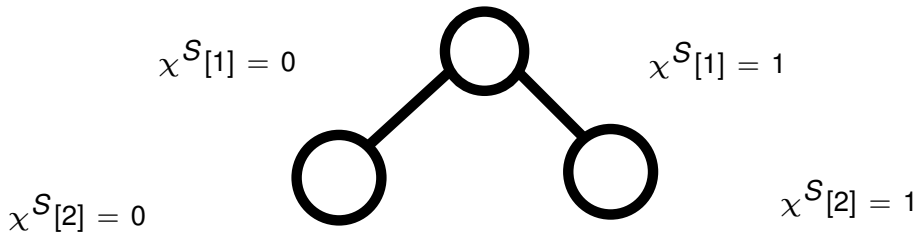
Some subcases are the same within the enumeration :
this puts the brakes on the combinatorial explosion.



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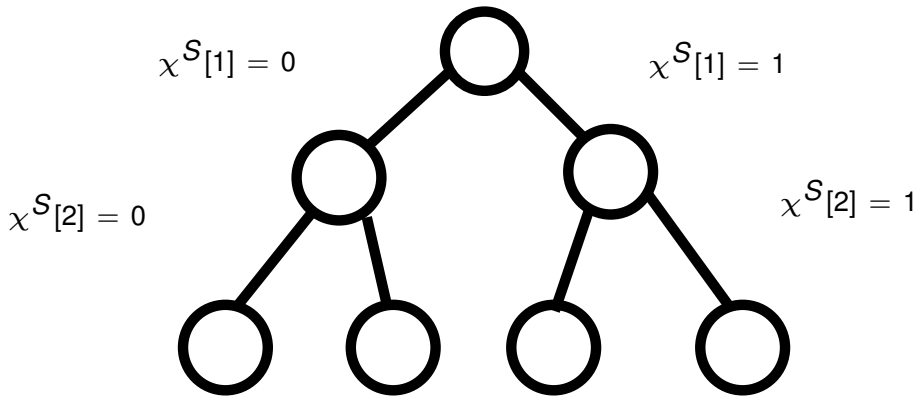

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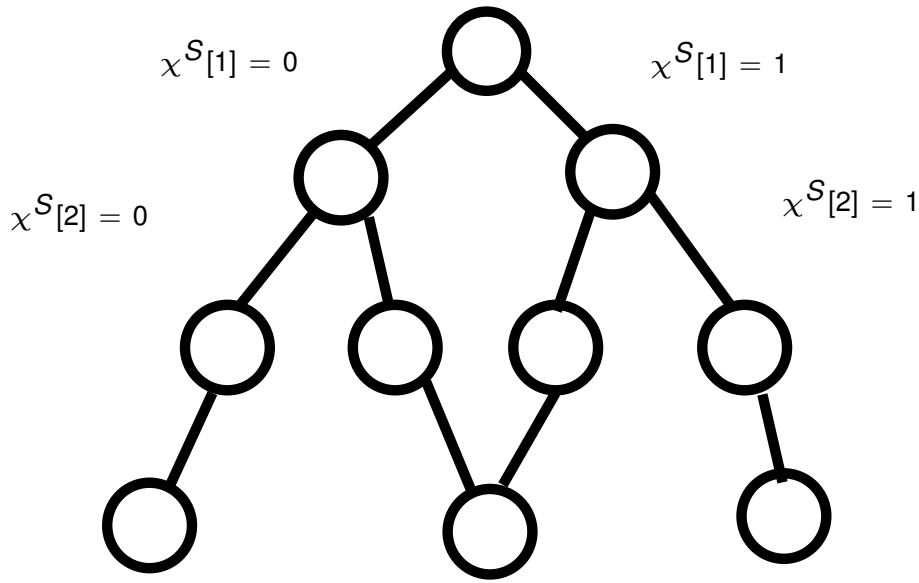

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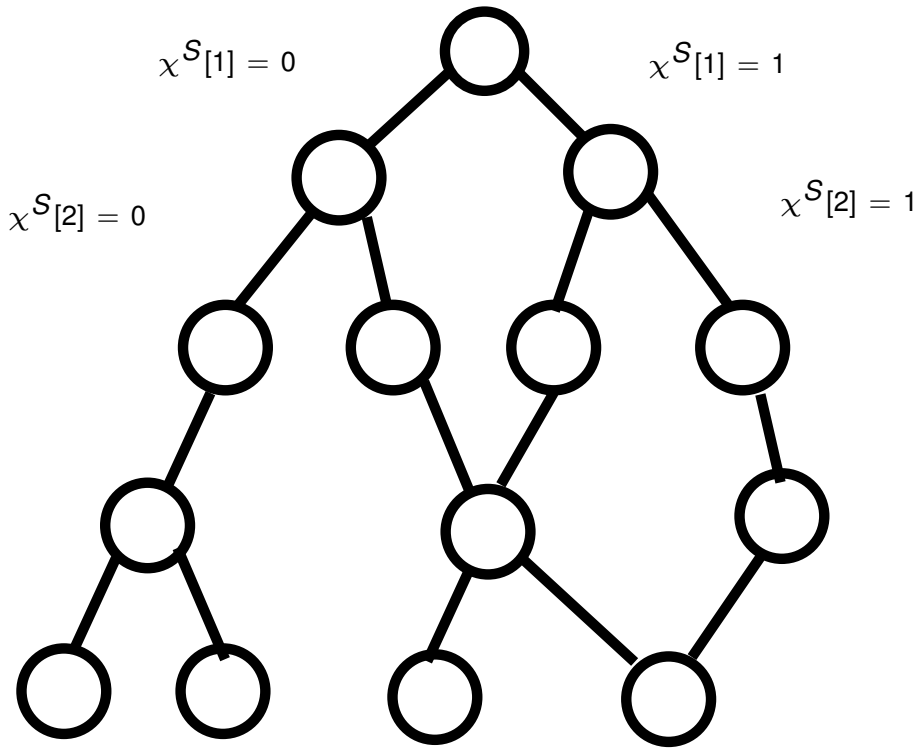

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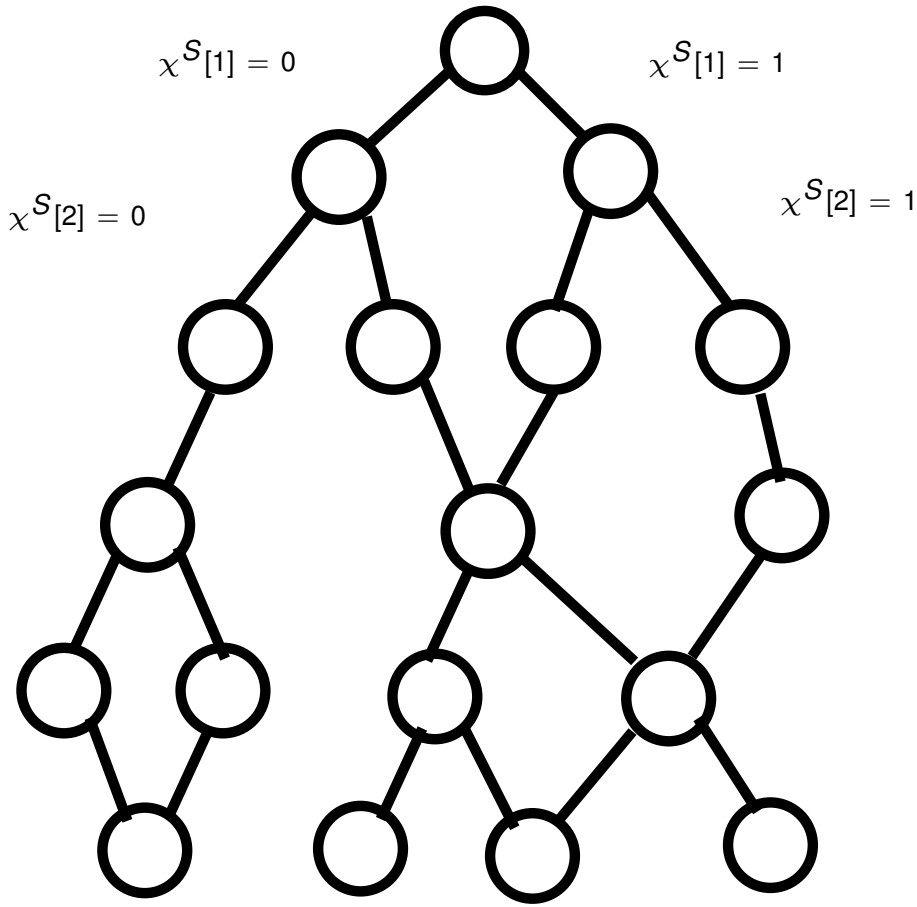

Knapsack problem

Input : n objects
profit $g_i \ \forall i \in \{1, \ldots, n\}$
weight $p_i \ \forall i \in \{1, \ldots, n\}$
maximum total weight P.

Output : Subset $S \subseteq \{1, \ldots, n\}$
s.t. $\sum_{i \in S} p_i \leq P$

Objective : Max $\sum_{i \in S} w_i$

Complexity : Weakly NP-hard
Dynamic programming scheme (in $O(nP)$ when $p_i \in \mathbb{N}$)

Polynomial cases : unit cost / unit weight

Difficulty : With MIP solver $n = 10^6$ in < 1 minutes.

Maximum weight stable set problem

Input : Undirected graph $G = (V, E)$
cost w_u $\forall u \in V$

Output : Subset $S \subseteq V$ of non-adjacent nodes

Objective : Max $\sum_{i \in S} w_i$

Complexity : Strongly NP-hard
Polynomial cases : perfect graphs (tree, planar, interval graphs...)
Difficulty : With MIP solver $n = 1000$ in often more than 1 hour
Some dedicated methodes (russian doll algo)

Shortest path problem

Input : Undirected (or directed) graph $G = (V, E)$
Two nodes $u_0, u_1 \in V$
Lenghts $l_e \; \forall e \in E$

Output : Path μ of G from u_0 to u_1

Objective : Min $\sum_{e \in \mu} l_e$

-

Complexity : Polynomial
Difficulty : With Dijkstra algorithm up to thousand of nodes in a few sec
With A^* algorithme, up to millions !

Shortest path problem

To quickly go from a point to another

The Traveling Salesman Problem (TSP)

Input : Undirected (or directed) graph $G = (V, E)$
lenght l_e $\forall e \in E$

Output : An hamiltonian cycle C of G (i.e. C goes once through each node)

Objective : Min $\sum_{e \in C} l_e$

Complexity : Strongly NP-hard
Polynomial cases : ?
Difficulty : Before 2003 : 200 nodes within several hours

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Concorde : a Branch-and-Cut algorithm and polyhedral results solves up to 200000 cities within one day !

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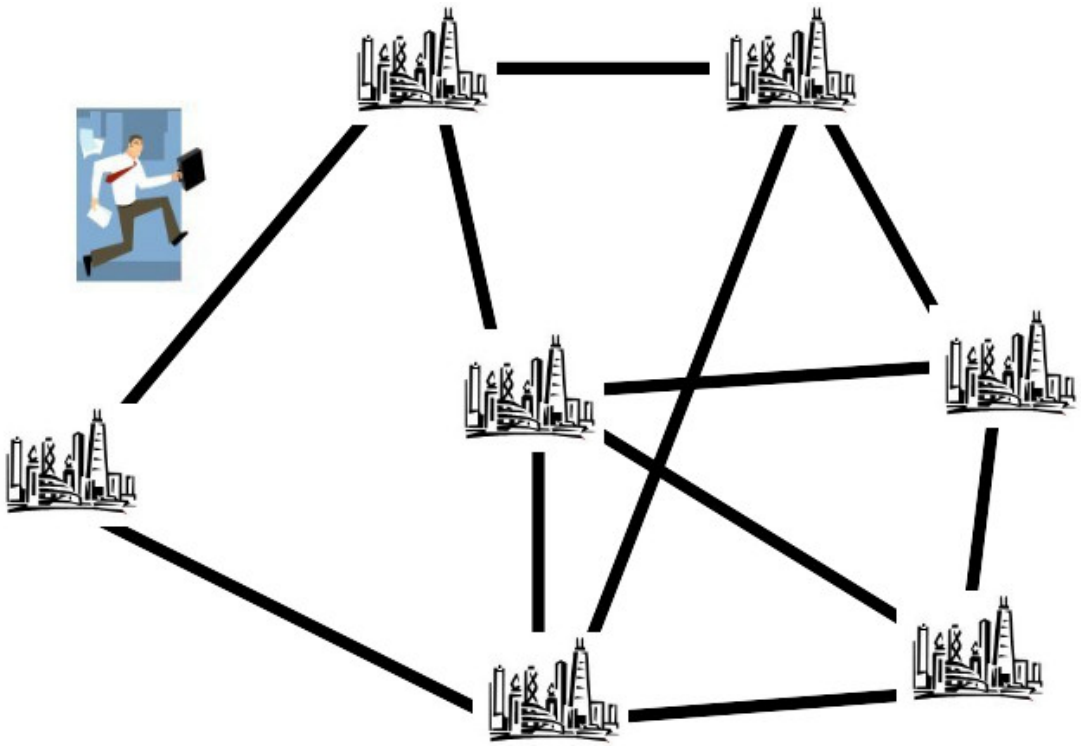

Starting from a city and going back to it
going once througt all the other cities
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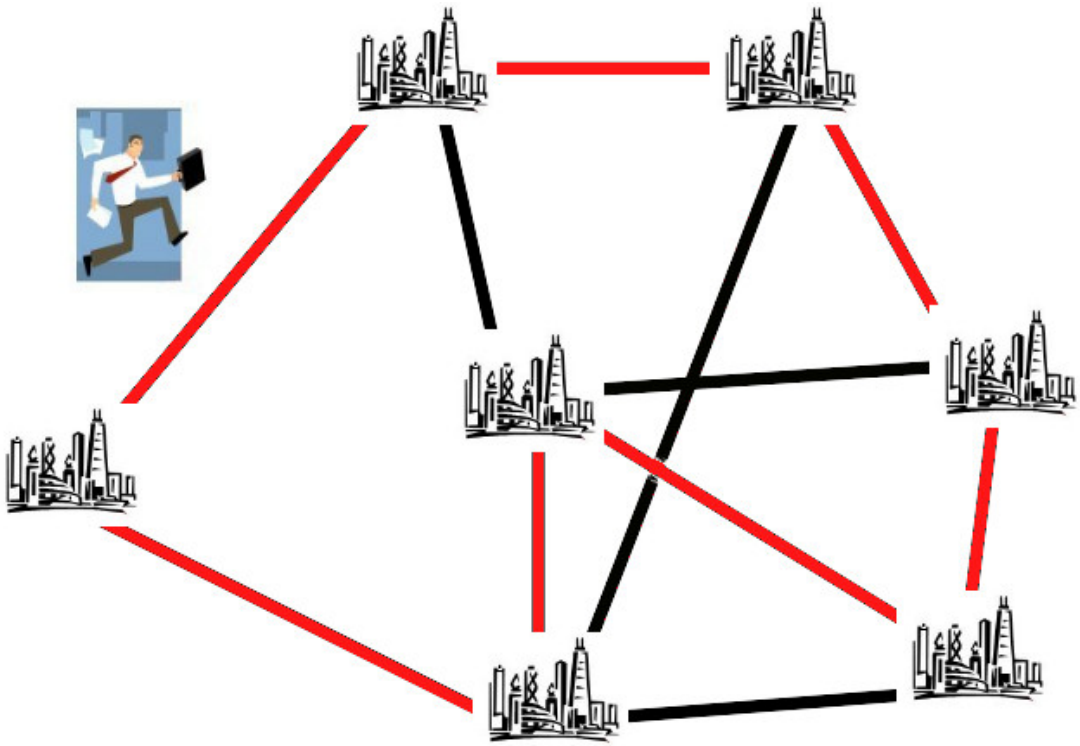

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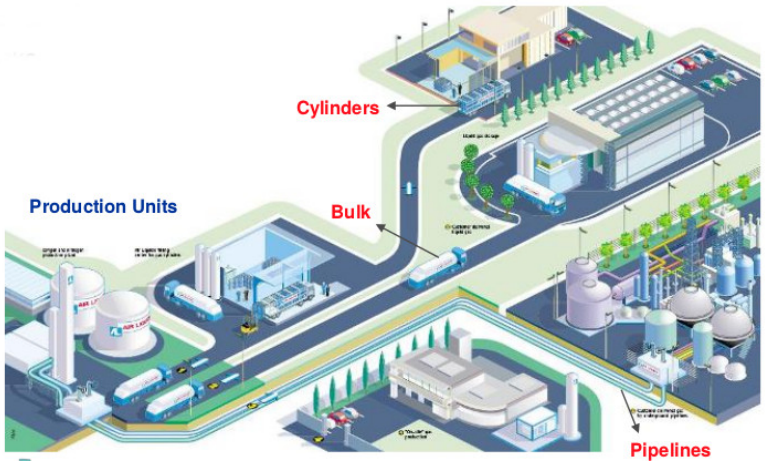
Real-world Operations Research problem



The **inventory routing problem** introduced by **Air Liquide** company with real daily data and questions to answer :

Determine routes for liquid oxygen trucks to deliver hospitals
so that

- hospital tanks are never empty (remote monitoring)
- rounds are feasible within the driver's working day
- costs are minimized !

Source : ROADEF Challenge 2016

How to solve Combinatorial Optimization Problem ?

• If the instances are very large and the problem very hard to solve
or if you do not have much time to spend on solving method

→ There exist methods to obtain “good” solutions (heuristics, meta-heuristics, machine learning...)

• If the problem is really important with a higly cost associated to solution
and if you have some time (several hours...)
The optimal solution is to be computed !

→ We will see in this lecture **how to circumvent the combinatorial explosion using mathematical programming !**

The Traveling salesman problem has been solved till 200 000 cicites !