# Gestion de production à EDF et symétries structurées en PLNE

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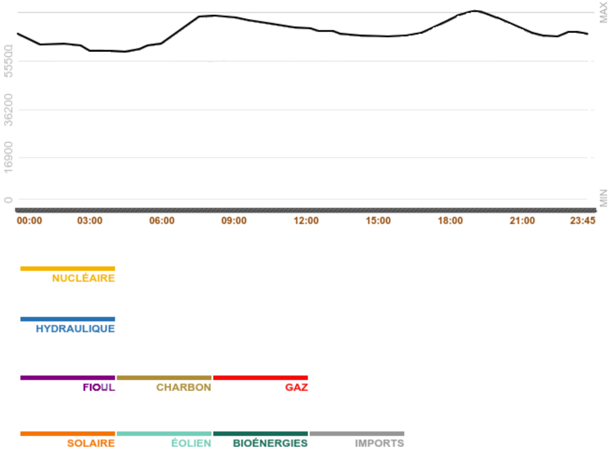
MAOA - 8 janvier 2019

# Outline

# The Unit Commitment Problem (UCP)

Demand for electric power on a one-day horizon

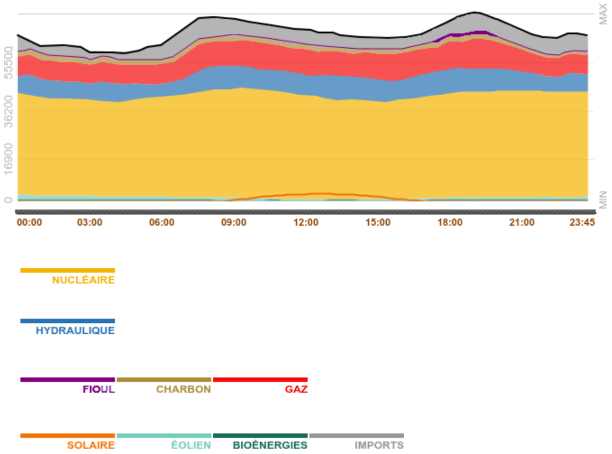



EDF power production units in France, including:

- 58 **nuclear** reactors
- 500 **hydro-power** plants dispatched in 50 valleys
- 20 **fossil-fuel** power plants
  fuel oil/gas turbines
  combined gas cycle
  coal-fired power plants

# The Unit Commitment Problem (UCP)

Power production satisfying the demand

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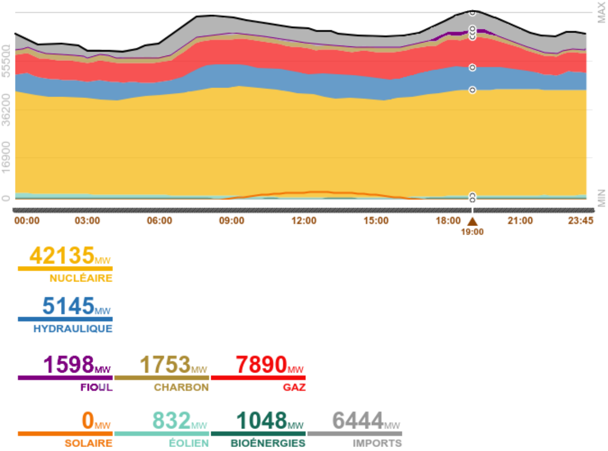


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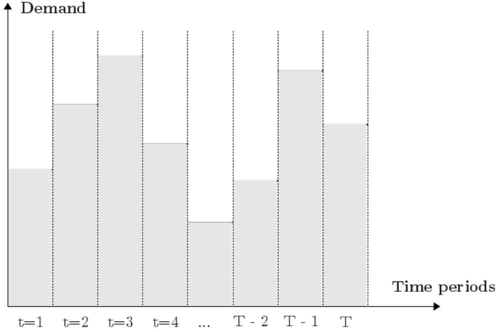
- **Discretized** time horizon $T$
- **Demand** for electric power $D_t$ at time $t$



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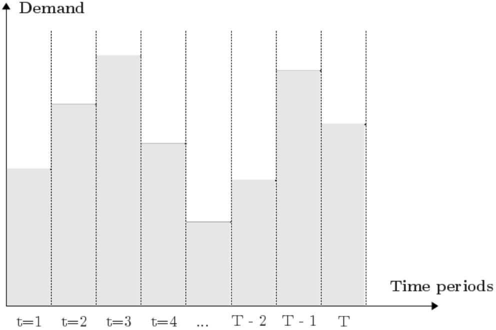


$n$ production units

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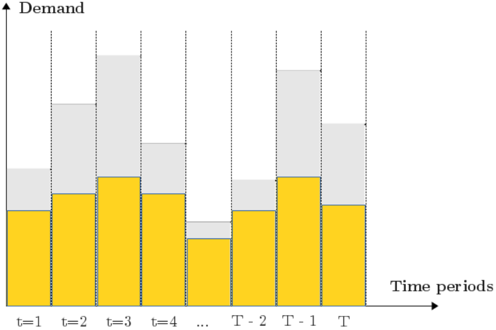


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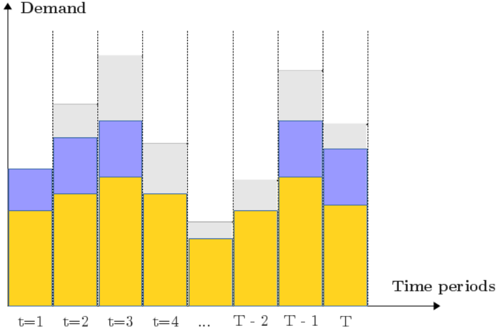


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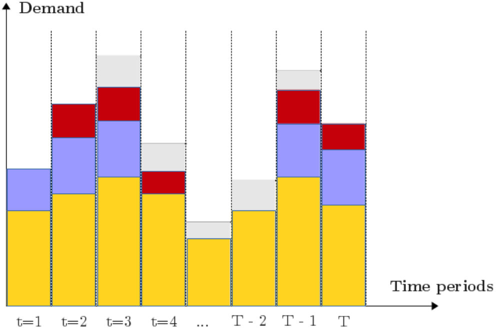


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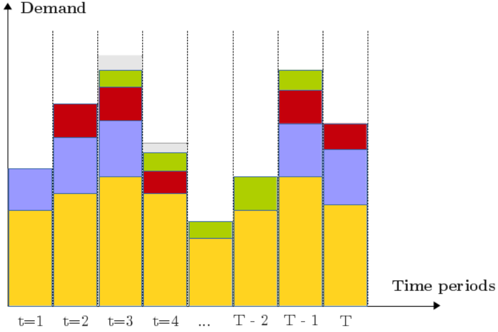


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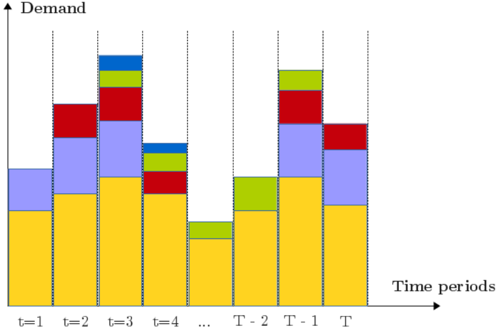



$n$ production units

# The Unit Commitment Problem (UCP)

**Production units must satisfy various technical constraints:**

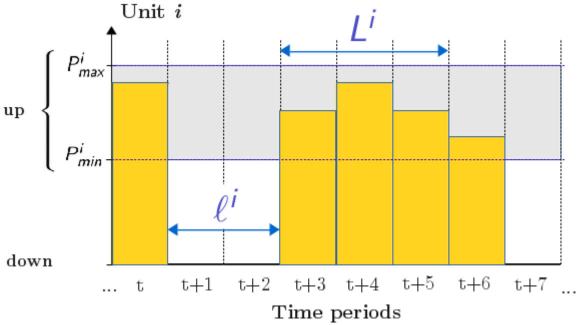
**Thermal units (nuclear, gas, fuel oil, coal) :**

- Minimum up and down times
- Production limits
- Ramp constraints, *i.e.*, production variation limits
- ...

# Example: constraints of thermal units

**Each thermal unit $i$ must satisfy:**

- Production limits $P^i_{min}$, $P^i_{max}$
- Minimum up time $L^i$
- Minimum down time $\ell^i$
- Ramp constraints



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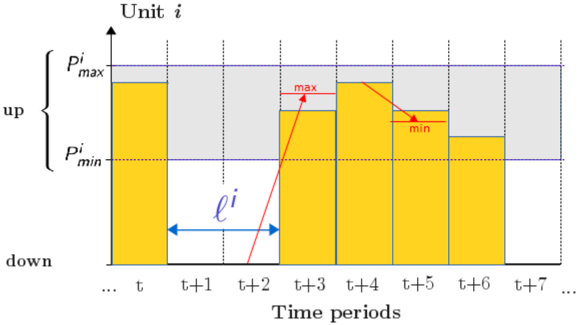
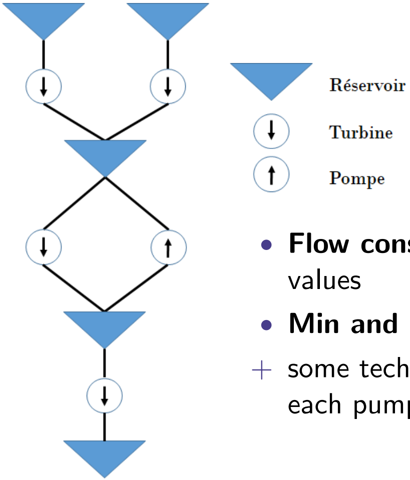


# The Unit Commitment Problem (UCP)

**Constraints in hydro-valleys :**

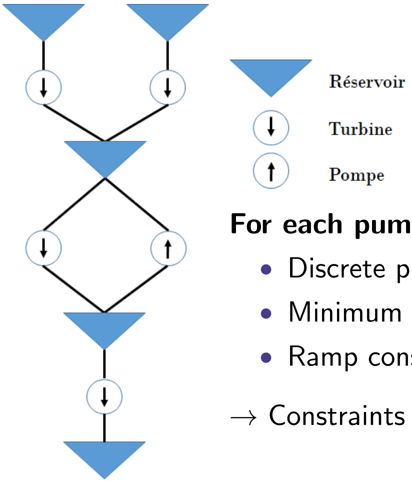
Hydro-valley = Network of turbines, pumps and reservoirs




- **Flow constraints** : conservation and min/max values
- **Min and max volume** in each reservoir

\+ some technical constraints for each turbine and each pump

# Technical constraints of pumps and turbines




**For each pump and each turbine:**

- Discrete power levels
- Minimum time on each power level
- Ramp constraints...

$\rightarrow$ Constraints similar to thermal units.

# Production costs

- **Costs** for each unit $i$:
  - **proportional** production cost $c_p^i$
  - **fixed** production cost $c_f^i$
  - **start-up** cost $c_0^i$

**Unit Commitment Problem** (UCP)

**Find a production plan** that

- satisfies technical constraints of all units
- meets the **demand** at each time
- minimizes the total production cost

# UCP resolutions at EDF

At EDF :

**The UCP is solved for various time horizons :**

- Mid-term horizons (from one year ahead to one week ahead)
- Day to day (for the next day)
  $\rightarrow$ **DAY-AHEAD UCP**
- Hour to hour (for the next hour to the end of the day)
  $\rightarrow$ **INTRA-DAY UCP**

**The UCP is solved for various areas :**

- Metropolitan France
- Insular regions (Corse, Guadeloupe, Réunion...)
  $\rightarrow$ **INSULAR UCP**

# UCP resolutions at EDF

**DAY-AHEAD UCP** : **Lagrangian relaxation of demand constraints**

- **Decomposition**

Each production unit (nuclear, coal, gas, fuel-oil, hydro-valley) corresponds to a subproblem:

  - **Thermal** subproblems solved by **dynamic programming**
  - **Hydro-valley** subproblems solved by **IP techniques**

- **Coordination** of the subproblems

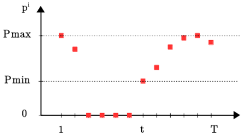
→ *Historic resolution technique at EDF*
→ *Solves large scale instances with heterogeneous units on a daily basis*

**INTRA-DAY UCP** and **INSULAR SYSTEMS UCP** : **Branch&Bound**

# ILP formulation

For each unit $i \in \{1, ..., n\}$ and each time $t \in \{1, ..., T\}$

- **production** variables $p_t^i \in \mathbb{R}$




$$\min_p \quad \sum_{i=1}^{n} cost(p^i)$$

$$\text{s. t.} \quad p^i \in \Pi^i \qquad \forall i \in \{1, ..., n\} \qquad (1)$$

$$\sum_{i=1}^{n} p_t^i \geq D_t \; \forall t \in \{1, ..., T\} \qquad (2)$$

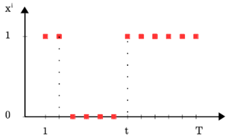
$$p_t^i \in \mathbb{R} \qquad \forall i \in \{1, ..., n\}, \; \forall t \in \{1, ..., T\}$$

- Operating domain of unit $i$ : $\Pi^i$

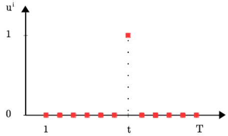
# Example of an operating domain (thermal unit)

For each unit $i \in \{1, ..., n\}$ and each time $t \in \{1, ..., T\}$

- **up/down status** variables $x_t^i \in \{0, 1\}$




- **start-up** variables $u_t^i \in \{0, 1\}$



# Example of an operating domain (thermal unit)

$$\Pi^i = \left\{ \quad p \quad \middle| \quad \begin{array}{ll} \displaystyle\sum_{t'=t-L^i+1}^{t} u^i_{t'} \leq x^i_t & \forall i \in \{1,...,n\},\ \forall t \in \{L^i,...,T\} \\ \displaystyle\sum_{t'=t-\ell^i+1}^{t} u^i_{t'} \leq 1 - x^i_{t-\ell^i} & \forall i \in \{1,...,n\},\ \forall t \in \{\ell^i+1,...,T \\ u^i_t \geq x^i_t - x^i_{t-1} & \forall i \in \{1,...,n\},\ \forall t \in \{2,...,T\} \\ P^i_{min} x^i_t \leq p^i_t \leq P^i_{max} x^i_t & \forall i \in \{1,...,n\},\ \forall t \in \{1,...,T\} \\ \widetilde{x}_{t,J},\ \widetilde{u}_{t,J} \in \{0,1,...,|J|\},\ \widetilde{p}_{t,J} \in \mathbb{R} & \forall i \in \{1,...,n\},\ \forall t \in \{1,...,T\} \end{array} \right.$$

# Lagrangian decomposition

$$\begin{aligned}\min_p \quad & \sum_{i=1}^{n} cost(p^i) \\ \text{s. t.} \quad & p^i \in \Pi^i \qquad \forall i \in \{1, ..., n\} \\ & \sum_{i=1}^{n} p^i_t \geq D_t \ \forall t \in \{1, ..., T\} \\ & p^i_t \in \mathbb{R} \qquad \forall i \in \{1, ..., n\}, \ \forall t \in \{1, ..., T\}\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\min_p \quad & \sum_{i=1}^{n} cost(p^i) + \sum_{t=1}^{T} \lambda_t (D_t - \sum_{i=1}^{n} p^i_t) \\ \text{s. t.} \quad & p^i \in \Pi^i \qquad \forall i \in \{1, ..., n\} \\ & p^i_t \in \mathbb{R} \qquad \forall i \in \{1, ..., n\}, \ \forall t \in \{1, ..., T\}\end{aligned}$$

# Lagrangian decomposition

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$$\begin{aligned} \min_{p} \quad & \sum_{i=1}^{n} \left( cost(p^i) - \sum_{t=1}^{T} \lambda_t p_t^i \right) \\ \text{s. t.} \quad & p^i \in \Pi^i \qquad \forall i \in \{1, ..., n\} \\ & p_t^i \in \mathbb{R} \qquad \forall i \in \{1, ..., n\}, \ \forall t \in \{1, ..., T\} \end{aligned}$$

# Decomposition in subproblems

$$\begin{aligned} \min_p \quad & \sum_{i=1}^{n} \left( cost(p^i) - \sum_{t=1}^{T} \lambda_t p_t^i \right) \\ \text{s. t.} \quad & p^i \in \Pi^i \qquad \forall i \in \{1, ..., n\} \\ & p_t^i \in \mathbb{R} \qquad \forall i \in \{1, ..., n\}, \ \forall t \in \{1, ..., T\} \end{aligned}$$

$$\downarrow \qquad \dots \qquad \downarrow$$

$$\begin{aligned} \min_p \quad & \left( cost(p^1) - \sum_{t=1}^{T} \lambda_t p_t^1 \right) \\ \text{s. t.} \quad & p^1 \in \Pi^1 \\ & p_t^1 \in \mathbb{R} \qquad \forall t \in \{1, ..., T\} \end{aligned}$$

$$\begin{aligned} \min_p \quad & \left( cost(p^n) - \sum_{t=1}^{T} \lambda_t p_t^n \right) \\ \text{s. t.} \quad & p^n \in \Pi^n \\ & p_t^n \in \mathbb{R} \qquad \forall t \in \{1, ..., T\} \end{aligned}$$

$\rightarrow$ $n$ **subproblems : 1 per unit**

# Dantzig-Wolfe reformulation

For each unit $i$, $\Pi^i$ set of feasible production plans.

**New variable space :** for each $\pi \in \Pi^i$,

$$\lambda_\pi = 1 \qquad \text{iff} \qquad \textbf{plan } \pi \textbf{ chosen for unit } i$$

$$(DW_{UCP}) \qquad \begin{array}{lll} \min\limits_{\lambda} & \sum\limits_{i=1}^{n} \sum\limits_{\pi \in \Pi^i} c^\pi \lambda^\pi & \\ \text{s. t.} & \sum\limits_{\pi \in \Pi^i} \lambda^\pi = 1 & \forall i \in \{1, ..., n\} \\ & \sum\limits_{i=1}^{n} \sum\limits_{\pi \in \Pi^i} p_t^\pi \lambda^\pi \geq D_t & \forall t \in \{1, ..., T\} \\ & \lambda^\pi \in \{0, 1\} & \forall i \in \{1, ..., n\} \quad \forall \pi \in \Pi^i \end{array}$$

$p_t^\pi$ : power produced by unit $i$ at time $t$ in plan $\pi \in \Pi^i$

# Outline

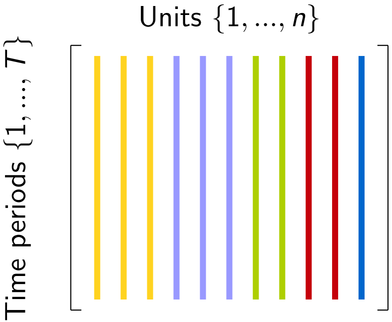
# Symmetries in the UCP

→ **Symmetries in the UCP come from the existence of identical units.**

- Variables $x_t^i \in \{0,1\}$: unit $i$ up at time $t$.
  → matrix $x = (x_t^i)$

- Groups of identical units, *i.e.*, units with same characteristics & costs

$x =$



# Example : 3 identical units with horizon $T = 3$

$\rightarrow$ given one solution $x_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$\rightarrow$ 5 symmetric solutions :

$$x_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad x_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$x_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad x_6 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

# Symmetry in integer programming - Definitions

$$\min \left\{ c(x) \mid x \in \mathcal{X} \right\} \qquad (ILP)$$

$\mathcal{X}$ subset of $m \times n$ binary matrices

- A **symmetry** is a permutation $\pi$ of the variables s.t.

$$\forall x \in \mathcal{X},\ \pi(x) \in \mathcal{X} \text{ and } c(x) = c(\pi(x))$$

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad x' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- The **symmetry group** $\mathcal{G}$ of $(ILP)$ is the set of all symmetries.
- Symmetry group $\mathcal{G}$ partitions $\mathcal{X}$ into **orbits**.

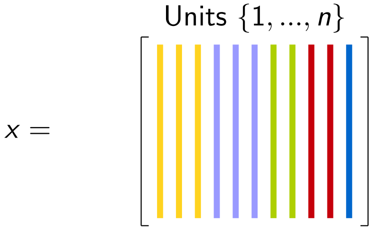
# In this presentation...

We consider problems with **structured symmetries**, *i.e.*

**any column permutation is a symmetry.**

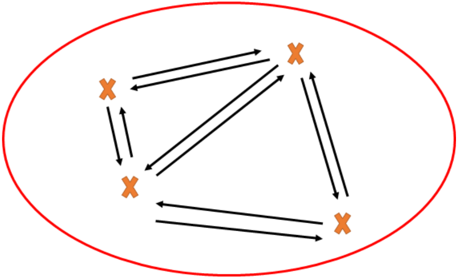
Thus, the symmetry group $\mathcal{G}$ is the set of all column permutations.

In the UCP, not exactly **all** column permutations but:




$\rightarrow$ everything we do in the "all column permutation" case will also apply to this more general case.

# Orbits



**Orbits :** for any solution $x$, the orbit of $x$ contains :

**every solution obtained by arbitrarily permuting $x$'s columns**

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$\rightarrow$ **orbit of $x_1$ = orbit of $x_2$ = ... = orbit of $x_6$ =** $\{x_1, x_2, x_3, x_4, x_5, x_6\}$

# Example : 3 identical units with horizon $T = 3$

→ for another solution $y_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

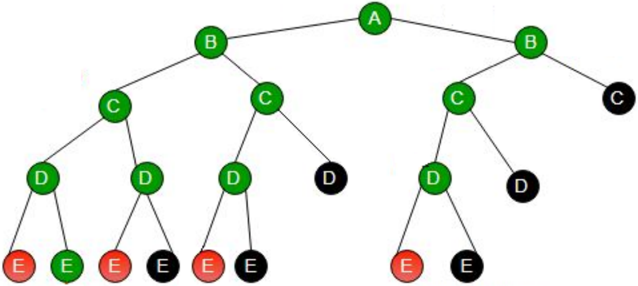
→ 2 symmetric solutions :

$$y_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad y_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

→ **orbit of $y_1$ = orbit of $y_2$ = orbit of $y_3$ =** $\{y_1, y_2, y_3\}$

# Symmetries in ILP

Many symmetric solutions lead to many symmetric branches in the B&B tree that cannot be pruned.



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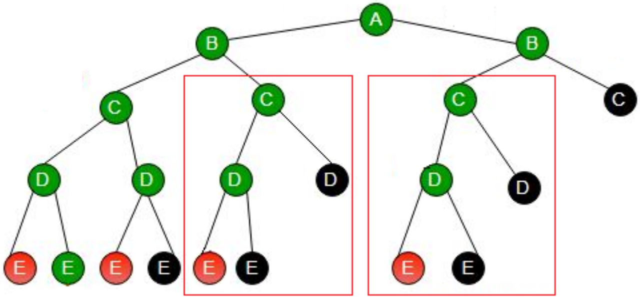
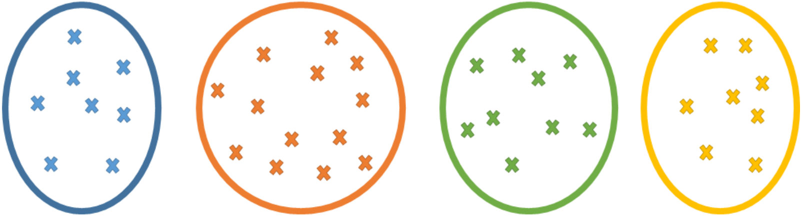


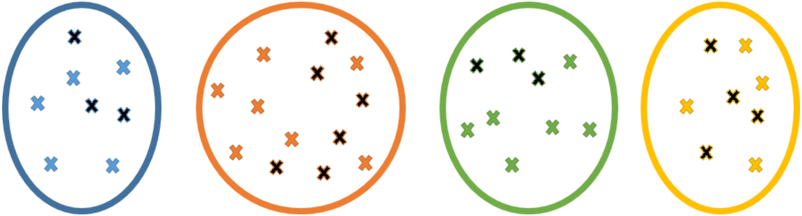

→ the B&B tree might get very big !

# How to break symmetries?



**How to restrict the search space to non-symmetric solutions only ?**

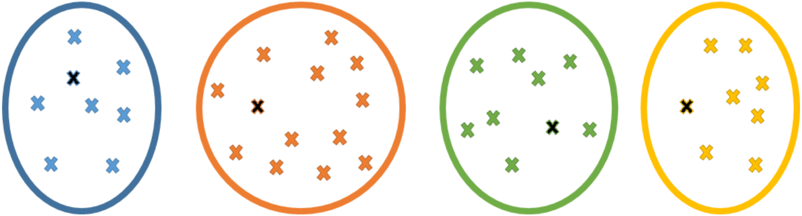
# How to break symmetries?



**How to restrict the search space to non-symmetric solutions only ?**

- Use symmetry-breaking **heuristics** removing some symmetric solutions (but not all)
  **NB : black crosses = solutions not removed**

# How to break symmetries?



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- Use symmetry-breaking **heuristics** removing some symmetric solutions (but not all)
- **Choose one representative** per orbit and **restrict** the search to the representative set

# How to break symmetries?

**How to restrict the search space to non-symmetric solutions only ?**

- Use symmetry-breaking **heuristics** removing some symmetric solutions (but not all)
- **Choose one representative** per orbit and **restrict** the search to the representative set
- **Reformulate** (i.e., change variable space)
  $\rightarrow$ indeed, symmetries depend on how the problem is formulated

# Outline

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# 1-counting inequalities

Idea : restrict the search to solutions such that :

**number of 1's in column $j \geq$ number of 1's in column $j+1$**

$$\sum_{i=1}^{m} x_{i,j} \geq \sum_{i=1}^{m} x_{i,j+1}, \quad \forall j \in \{1, ..., n-1\}$$

(Lima and Novais, 2016)

# Example : 3 identical units with horizon $T = 3$

$\rightarrow$ 6 symmetric solutions :

$$x_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \rightarrow \text{ these solutions remain}$$

$$x_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad x_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \rightarrow \text{ these solutions are removed}$$

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# Outline

# Orbital branching

**At node $a$ of the B&B tree:** instead of branching on $x_{i,j} = 0 \vee x_{i,j} = 1$

- look for variables symmetric to $x_{i,j}$ at node $a$

At node $a$, some variables are fixed: $\begin{bmatrix} 1 & \times & 1 & 1 & 1 \\ \times & 0 & \times & \times & \times \\ \times & 1 & \times & \times & \times \\ 0 & 1 & \times & 0 & 0 \end{bmatrix}$

(Ostrowski, Anjos and Vannelli, 2015)

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  Set of symmetric variables $O = \{x_{i,j_1},\ x_{i,j_2},\ \ldots,\ x_{i,j_k}\}$

- branch on :

$$\text{at least } \alpha \text{ variables of } O \text{ are equal to } 1$$
$$\vee$$
$$\text{at most } \alpha - 1 \text{ variables of } O \text{ are equal to } 1$$

(Ostrowski, Anjos and Vannelli, 2015)

# Orbital branching

at least $\alpha$ variables of $O$ are equal to 1
$\vee$
at most $\alpha - 1$ variables of $O$ are equal to 1

$\downarrow$

at least $\alpha$ variables of $O$ are equal to 1
$\vee$
at least $k - \alpha$ var. of $O$ equal 0

$\downarrow$ (By symmetry)

first $\alpha$ variables of $O$ are equal to 1
$\vee$
last $k - \alpha$ variables of $O$ are equal to 0

# Outline

# Lexicographic order

$\rightarrow$ **Order on the columns.**

Consider two columns $X, Y \in \{0,1\}^m$.

## Lexicographic order

$Y$ is **lexicographically greater than** $X$ (denoted $Y \succeq X$) if :
$\exists i \in \{1, ...., m-1\}$ s. t.

- $\forall i' \leq i,\ Y_{i'} = X_{i'}$
- $Y_{i+1} = 1$ and $X_{i+1} = 0$

## Example

$$Y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \succeq \quad X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

# Full Orbitopes

- **When the symmetry group is the set of all column permutations:**

$x \in \mathcal{X}$ lex-order representative $\iff x(1) \succeq \ldots \succeq x(n)$

$\iff$ **Columns of $x$ are in non-increasing lexicographic order**

**Example**

$x_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is a lex-order representative.

**Full Orbitope**

$$\mathcal{P}_0 = \text{Conv}\left\{ \text{binary matrices } x \mid x(1) \succeq \ldots \succeq x(n) \right\}$$

Solving on $\mathcal{X} \iff$ Solving on $\mathcal{X} \cap \mathcal{P}_0$

# Outline

First idea to enforce that each $x$ is in the full orbitope:

**add new constraints in the model**

$\rightarrow$ For a given matrix $x = (x_{i,j}) = [\ x(1),\ x(2),\ ...,\ x(n)\ ]$,
where $x(1)$, $x(2)$, ... $x(n)$ are $x$'s columns.

**Find linear inequalities enforcing that :** $x(1) \succeq x(2) \succeq ... \succeq x(n)$

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**Find linear inequalities enforcing that :** $x(1) \succeq x(2) \succeq ... \succeq x(n)$

**One option :** for each pair of columns $j$, $j+1$, add inequalities

$$2^{m-1}x_{1,j} + 2^{m-2}x_{2,j} + ... + x_{m,j} \ \geq\ 2^{m-1}x_{1,j+1} + 2^{m-2}x_{2,j+1} + ... + x_{m,j+1}$$

$$\begin{matrix} j & & j+1 & & \\ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} & & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & & \begin{matrix} \rightarrow \text{ weight } 2^{m-1} \\ \rightarrow \text{ weight } 2^{m-2} \\ \dots \\ \rightarrow \text{ weight } 2^{0} \end{matrix} \end{matrix}$$

$\rightarrow$ **numerical difficulties** (exponential coefficients)

# Full orbitope inequalities

**Find linear inequalities enforcing that :** $x(1) \succeq x(2) \succeq \ldots \succeq x(n)$

**Other option:** add new variables

$$y_{i,j} = 1 \quad \text{iff} \quad \text{column } j \text{ and } j+1 \text{ are equal from row 1 to } i$$

## Example

$$\begin{array}{ccccc} & j & & j+1 & \\ i=1 & \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \end{array}$$

| | $j$ | $j+1$ | |
|---|---|---|---|
| $i=1$ | 1 | 1 | $y_{1,j} = 1$ |
| $i=2$ | 0 | 0 | $y_{2,j} = 1$ |
| $i=3$ | 1 | 0 | $y_{3,j} = 0$ |
| $i=4$ | 0 | 1 | $y_{4,j} = 0$ |
| $i=5$ | 0 | 0 | $y_{5,j} = 0$ |

(Loos, 2011)

(Bendotti, Fouilhoux and R., 2019)

# Full orbitope inequalities

$$y_{i,j} = 1 \quad \text{iff} \qquad \text{column } j \text{ and } j+1 \text{ are equal from row 1 to } i$$

**New constraints :**

- $y_{i,j} = 1 \iff \begin{cases} y_{i-1,j} = 1 \\ x_{i,j} = x_{i,j+1} \end{cases}$
- If $\begin{cases} y_{i-1,j} = 1 \\ y_{i,j} = 0 \end{cases}$ then $\begin{cases} x_{i,j} = 1 \\ x_{i,j+1} = 0 \end{cases}$

## Example

$$\begin{array}{lcccl} & j & & j+1 & \\ i=1 & 1 & & 1 & y_{1,j}=1 \\ i=2 & 0 & & 0 & y_{2,j}=1 \\ i=3 & 1 & & 0 & y_{3,j}=0 \\ i=4 & 0 & & 1 & y_{4,j}=0 \\ i=5 & 0 & & 0 & y_{5,j}=0 \end{array}$$

# Full orbitope inequalities

$y_{i,j} = 1$ iff column $j$ and $j+1$ are equal from row 1 to $i$

**Constraints :**

$$\begin{aligned}
&1 - y_{1,j-1} = x_{1,j-1} - x_{1,j} && \forall j \in \{2, ..., n\} \\
&y_{i-1,j-1} \leq y_{i,j-1} + x_{i,j-1} && \forall i \in \{2, ..., m\},\ j \in \{2, ..., n\} \\
&y_{i-1,j-1} + x_{i,j} \leq 1 + y_{i,j-1} && \forall i \in \{2, ..., m\},\ j \in \{2, ..., n\} \\
&x_{i,j-1} \leq 1 - y_{i,j-1} + x_{i,j} && \forall i \in \{2, ..., m\},\ j \in \{2, ..., n\} \\
&y_{i,j-1} \leq y_{i-1,j-1} && \forall i \in \{2, ..., m\},\ j \in \{2, ..., n\} \\
&x_{i,j} \leq (1 - 2y_{i-1,j} - y_{i,j}) + x_{i,j-1} &&
\end{aligned}$$

$\rightarrow$ complete linear description in variable space $(x, y)$
of the 2-column full orbitope

(Loos, 2011)

# Outline

# Fixing in the full orbitope

Another idea to enforce that each $x$ is in the full orbitope:

**fix variables during the B&B search**
**to ensure columns are in lex order**

- At each node $a$ of the B&B tree, some variables are already fixed to 0 (variables $I_0^a$) or already fixed to 1 (variables $I_1^a$)
  $\rightarrow$ defines a set

$$F = \{ \text{ matrices } x \mid x_{i,j} = 1 \ \forall (i,j) \in I_1^a \quad x_{i,j} = 0 \ \forall (i,j) \in I_0^a \}$$

- At node $a$, we want to enumerate solutions $x$ such that

$$x \in F \cap P \cap \{0,1\}^{(m,n)}$$

  where $P$ is the full orbitope.

$\rightarrow$ **it may be possible to fix more variables to 0 or to 1 at node** $a$

# Example

At node *a*, there are some fixed variables :
("$\times$" stands for free variables i.e. variables not fixed yet)

$$x = \begin{bmatrix} \times & 1 \\ \times & 1 \\ 0 & \times \end{bmatrix}$$

We want $x$'s columns to be in lexicographic order.

# Example

At node $a$, there are some fixed variables :
("$\times$" stands for free variables i.e. variables not fixed yet)

$$x = \begin{bmatrix} {\color{red}1} & 1 \\ \times & 1 \\ 0 & \times \end{bmatrix}$$

We want $x$'s columns to be in lexicographic order.

# Example

At node $a$, there are some fixed variables :
("$\times$" stands for free variables i.e. variables not fixed yet)

$$x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & \times \end{bmatrix}$$

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# Example

At node $a$, there are some fixed variables :
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$$x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

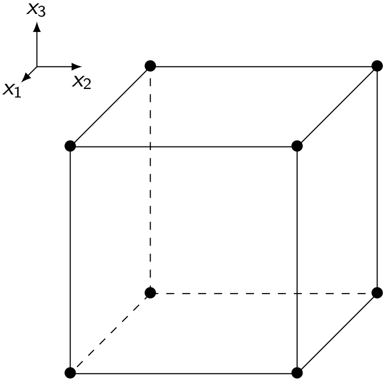
We want $x$'s columns to be in lexicographic order.

# Variable fixing - Intersection with a cube face

V. Kaibel, M. Peinhardt, M. Pfetsch. *Orbitopal fixing*, Proc. IPCO, 2007.

- Let $C = \text{Conv} \; \{ \; m \times n \text{ binary matrices } \}$
- A non-empty face $F$ of $C$ is given by two index sets $I_0$, $I_1$:

$$F = \{x \in C \quad | \quad x_{i,j} = 0 \; \forall (i,j) \in I_0, \qquad x_{i,j} = 1 \; \forall (i,j) \in I_1\}.$$



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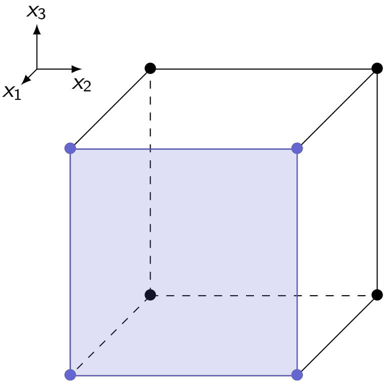


- **For a given face $F$ of $C$**

$F = \{x \in [0,1]^3 \mid x_1 = 1\}$

$I_1 = \{1\},\ I_0 = \varnothing$

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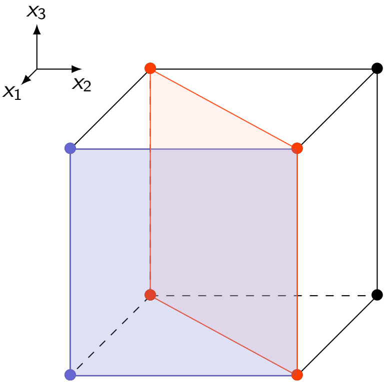


- **For a given face $F$ of $C$**
- **For a given polytope $P \subset C$**

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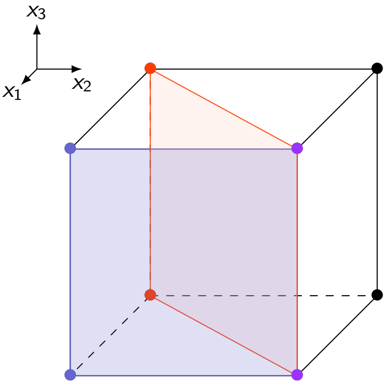


- **For a given face $F$ of $C$**
- **For a given polytope $P \subset C$**

$\rightarrow$ Solution set $S$

$S = P \cap F \cap \{0,1\}^{(m,n)}$

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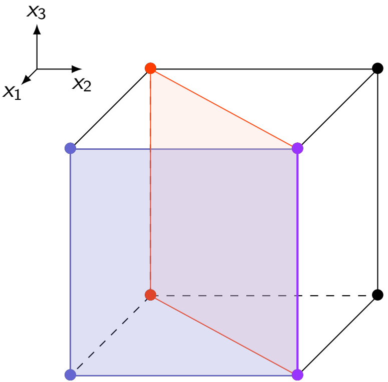


- **For a given face $F$ of $C$**
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$\rightarrow$ Solution set $S$

$S = P \cap F \cap \{0,1\}^{(m,n)}$

**Fixing of $P$ at $F$**

$\text{Fix}_F(P)$ = the smallest face of $C$ that contains $S$

**$I_0^\star$ and $I_1^\star$: index sets defining $\text{Fix}_F(P)$** (if non-empty)

# Variable fixing - Intersection with a cube face

V. Kaibel, M. Peinhardt, M. Pfetsch. *Orbitopal fixing*, Proc. IPCO, 2007.

- Let $C$ = Conv { $m \times n$ binary matrices }
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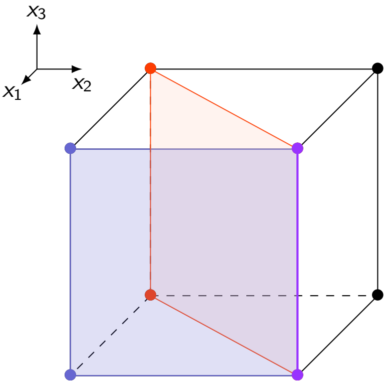


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Fixing $\text{Fix}_F(P)$ : $I_0^\star = \emptyset$, $I_1^\star = \{1,2\}$, *i.e.*, $x_1 = x_2 = 1$

# Orbitopal fixing in the B&B tree

V. Kaibel, M. Peinhardt, M. Pfetsch. *Orbitopal fixing*, Proc. IPCO, 2007.

We want to restrict the solution set $\mathcal{X}$ to a polytope $P$,
$\rightarrow$ **we want to enumerate only solutions in $\mathcal{X} \cap P$.**

## Orbitopal fixing algorithm

At each node $a$ of the B&B tree:

- (INPUT) $I_0^a$: variables set to 0, $I_1^a$: variables set to 1
  $\rightarrow$ $F$ cube face defined by $I_0^a$, $I_1^a$
- **Compute** $I_0^\star$ and $I_1^\star$: index sets defining $\mathrm{Fix}_F(P)$
- (OUTPUT) Fixed variables:
  $\rightarrow$ **Fix** to 0 variables in $I_0^\star$ and to 1 variables in $I_1^\star$

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  $\rightarrow$ **Fix** to 0 variables in $I_0^\star$ and to 1 variables in $I_1^\star$

# Orbitopal fixing for the full orbitope

**Let $P = \mathcal{P}_0$ (the full orbitope).**

**For a given face $F$:**

- $\text{Fix}_F(P)$: cube face containing the intersection $F \cap P$.
- $I_0^\star$ and $I_1^\star$: index sets defining $\text{Fix}_F(P)$

**Theorem**

*$\forall F$, $I_0^\star$ and $I_1^\star$ can be computed in linear time.*

Idea:

Find binary matrices $M_{min}$ and $M_{max}$ in $F \cap P$ s.t.
$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}$ :

$$i.e. \quad \forall \text{ column } j, \quad M_{min}(j) \preceq x(j) \preceq M_{max}(j)$$

(Bendotti, Fouilhoux and R., 2018)

# Computing $I_0^\star$ and $I_1^\star$ - Example

*Example: face $F$ defined by* $I_0 = \{(4,2),(3,3),(5,3)\}$ *and* $I_1 = \{(1,1),(2,1),(4,1),(2,2),(5,2),(4,3),(1,4),(2,4)\}$

$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

$$\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}$$

$$M_F$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

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$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

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$I_1 = \{(1,1),(2,1),(4,1),(2,2),(5,2),(4,3),(1,4),(2,4)\}$

$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}$, $x$ of the form

$$\underbrace{\begin{bmatrix} 1 & \times & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \times & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \times & 1 & 0 & 0 \end{bmatrix}}_{M_{min}} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_F} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_{max}}$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

*Example: face $F$ defined by* $I_0 = \{(4,2),(3,3),(5,3)\}$ *and*
$I_1 = \{(1,1),(2,1),(4,1),(2,2),(5,2),(4,3),(1,4),(2,4)\}$

$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}$, $x$ of the form

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$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \times & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \times & 1 & 0 & 0 \end{bmatrix}}_{M_{min}} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_F} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_{max}}$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

*Example: face F defined by* $I_0 = \{(4,2),(3,3),(5,3)\}$ *and*
$I_1 = \{(1,1),(2,1),(4,1),(2,2),(5,2),(4,3),(1,4),(2,4)\}$

$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}$, $\quad x$ of the form

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \times & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \times & 1 & 0 & 0 \end{bmatrix}}_{M_{min}} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_F} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_{max}}$$

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$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

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# Computing $I_0^\star$ and $I_1^\star$ - Example

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$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{M_{min}} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_F} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_{max}}$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

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$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{M_{min}} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_F} \preceq \underbrace{\begin{bmatrix} 1 & \color{red}{1} & \times & 1 \\ 1 & 1 & \times & 1 \\ 1 & \color{red}{1} & 0 & \times \\ 1 & 0 & 1 & \times \\ 1 & 1 & 0 & \times \end{bmatrix}}_{M_{max}}$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

*Example: face $F$ defined by* $I_0 = \{(4,2),(3,3),(5,3)\}$ *and* $I_1 = \{(1,1),(2,1),(4,1),(2,2),(5,2),(4,3),(1,4),(2,4)\}$

$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

$$\underset{M_{min}}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \preceq \underset{M_F}{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}} \preceq \underset{M_{max}}{\begin{bmatrix} 1 & 1 & \color{red}{1} & 1 \\ 1 & 1 & \color{red}{1} & 1 \\ 1 & 1 & 0 & \times \\ 1 & 0 & 1 & \times \\ 1 & 1 & 0 & \times \end{bmatrix}}$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

*Example: face F defined by* $I_0 = \{(4,2),(3,3),(5,3)\}$ *and* $I_1 = \{(1,1),(2,1),(4,1),(2,2),(5,2),(4,3),(1,4),(2,4)\}$

$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}$, $x$ of the form

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{M_{min}} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_F} \preceq \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{M_{max}}$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

*Example: face $F$ defined by* $I_0 = \{(4,2),(3,3),(5,3)\}$ *and*
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$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

$$\underset{M_{min}}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \preceq \underset{M_F}{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}} \preceq \underset{M_{max}}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

*Example: face F defined by* $I_0 = \{(4,2),(3,3),(5,3)\}$ *and* $I_1 = \{(1,1),(2,1),(4,1),(2,2),(5,2),(4,3),(1,4),(2,4)\}$

$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{M_{min}} \preceq \underbrace{\begin{bmatrix} 1 & \times & \times & 1 \\ 1 & 1 & \times & 1 \\ \times & \times & 0 & \times \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix}}_{M_F} \preceq \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{M_{max}}$$

$$\Longrightarrow I_0^\star \text{ and } I_1^\star$$

# Computing $I_0^\star$ and $I_1^\star$ - Example

*Example: face F defined by* $I_0 = \{(4,2),(3,3),(5,3)\}$ *and*
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$\forall x \in F \cap P \cap \{0,1\}^{(m,n)}, \quad x$ of the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & \times \\ \times & 1 & 0 & \times \end{bmatrix} \preceq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$M_{min} \qquad\qquad M_F \qquad\qquad M_{max}$$

$$\Longrightarrow I_0^\star \text{ and } I_1^\star$$

# Outline

# Dantzig-Wolfe reformulation: UCP example

For each unit $i$, $\Pi^i$ set of feasible production plans.

**New variable space :** for each $\pi \in \Pi^i$,

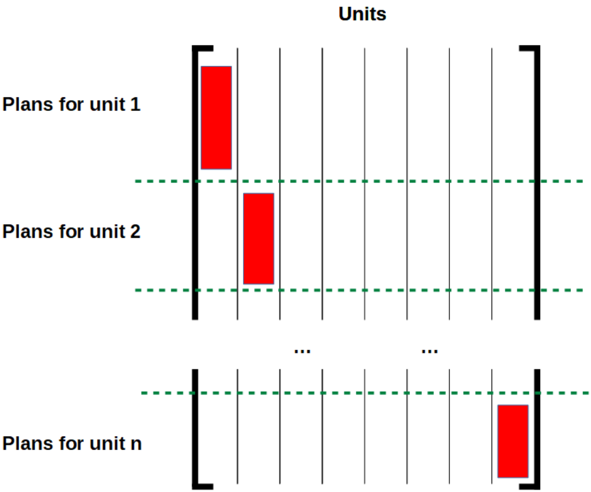
$$\lambda_\pi = 1 \quad \text{iff} \quad \textbf{plan } \pi \textbf{ chosen for unit } i$$

$$(DW_{UCP}) \qquad \begin{aligned} \min_{\lambda} \quad & \sum_{i=1}^{n} \sum_{\pi \in \Pi^i} c^\pi \lambda^\pi \\ \text{s. t.} \quad & \sum_{\pi \in \Pi^i} \lambda^\pi = 1 & \forall i \in \{1, ..., n\} \\ & \sum_{i=1}^{n} \sum_{\pi \in \Pi^i} p_t^\pi \lambda^\pi \geq D_t & \forall t \in \{1, ..., T\} \\ & \lambda^\pi \in \{0, 1\} & \forall i \in \{1, ..., n\} \quad \forall \pi \in \Pi^i \end{aligned}$$

$p_t^\pi$ : power produced by unit $i$ at time $t$ in plan $\pi \in \Pi^i$

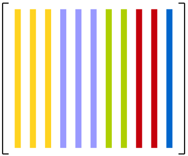
# Dantzig-Wolfe reformulation with identical units

→ non-symmetric solution space!



# Outline

# Aggregation of symmetric variables

$x =$ 

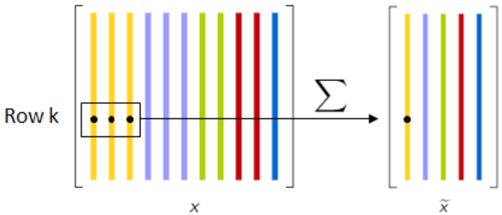
For set $J$ of "permutable" columns, each row $k$, **define aggregated variables :**

$$\widetilde{x}_{k,J} = \sum_{j \in J} x_{k,j}$$

$$\widetilde{x}_{k,J} = \text{number of variables to 1 among variables } x_{k,j},\ j \in J$$

(Fischetti, Liberti, Salvagnin and Walsh, 2017)

# Aggregation of symmetric variables




For set $J$ of "permutable" columns, each row $k$, **define aggregated variables :**

$$\widetilde{x}_{k,J} = \sum_{j \in J} x_{k,j}$$

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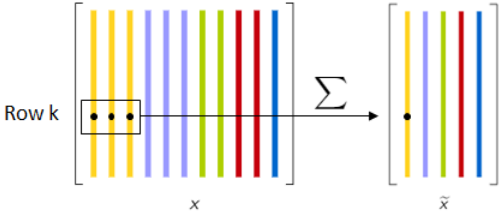
# Aggregation of symmetric variables




For set $J$ of "permutable" columns, each row $k$, **define aggregated variables :**

$$\widetilde{x}_{k,J} = \sum_{j \in J} x_{k,j}$$

$$\widetilde{x}_{k,J} = \text{number of variables to 1 among variables } x_{k,j},\ j \in J$$

**Replace every occurrence of variable** $x_{k,j}$ **by** $\frac{\widetilde{x}_{k,J}}{|J|}$ in the integer program, for each $j \in J$.

(Fischetti, Liberti, Salvagnin and Walsh, 2017)

$PC = \{$ set of permutable columns $\}$

**Original ILP**

$$(P) \qquad v = \min_x \quad \sum_{k=1}^{m} \sum_{j=1}^{n} c_{k,j} x_{k,j}$$

$$\text{s. t.} \quad Ax \leq b$$

$$x_{k,j} \in \{0,1\} \quad \forall j \in \{1,...,n\},\ \forall k \in \{1,...,m\}$$

$$\Big\downarrow \text{(relaxation)}$$

**Agregated ILP**

$$(\widetilde{P}) \qquad \widetilde{v} = \quad \min_{\widetilde{x}} \sum_{k=1}^{m} \sum_{J \in PC} \widetilde{c}_{k,J} \widetilde{x}_{k,J}$$

$$\text{s. t.} \quad \widetilde{A}\widetilde{x} \leq b$$

$$\widetilde{x}_{k,J} \in \{0,1,...,|J|\} \quad \forall j \in PC,\ \forall k \in \{1,...,m\}$$

(Fischetti, Liberti, Salvagnin and Walsh, 2017)

$PC = \{$ set of permutable columns $\}$

**Original ILP**

$$(P) \qquad v = \min_x \sum_{k=1}^{m} \sum_{j=1}^{n} c_{k,j} x_{k,j}$$

$$\text{s. t.} \quad Ax \leq b$$

$$x_{k,j} \in \{0, 1\} \quad \forall j \in \{1, ..., n\}, \ \forall k \in \{1, ..., m\}$$

$\downarrow$ (relaxation)

**Agregated ILP**

$$(\widetilde{P}) \qquad \widetilde{v} = \min_{\widetilde{x}} \sum_{k=1}^{m} \sum_{J \in PC} \widetilde{c}_{k,J} \widetilde{x}_{k,J}$$

$$\text{s. t.} \quad \widetilde{A}\widetilde{x} \leq b$$

$$\widetilde{x}_{k,J} \in \{0, 1, ..., |J|\} \quad \forall j \in PC, \ \forall k \in \{1, ..., m\}$$

Reduces the number of variables

But replaces binary variables by integer variables

(Fischetti, Liberti, Salvagnin and Walsh, 2017)

# General case

In general: $(\widetilde{P})$ **is a relaxation of** $(P)$, $i.e.$ $\widetilde{v} \leq v$.

Indeed, for a given solution $\widetilde{x}$, there may not exist $x$ such that $Ax \leq b$, $x_{kj} \in 0,1$ and $\widetilde{x}_{k,J} = \sum_{j \in J} x_{k,j}$ for each $J$ and $k$.

But: **in some particular cases** $\widetilde{v} = v$

**→ it is the case for what we call the Min-up/min-down Unit Commitment Problem (MUCP)**

$i.e.$ a restriction of the UCP with only the following constraints:

- Demand satisfaction at each time $t$
- Power limits $P_{min}$ and $P_{max}$ for each unit
- Minimum up and down time constraints

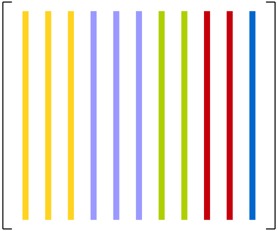
(B. Knueven, J. Ostrowski, J.P. Watson, 2017)

# Example: the MUCP

$$\widetilde{x}_{t,J} = \text{number of units of } J \text{ up at time } t$$

Units $\{1, \ldots, n\}$

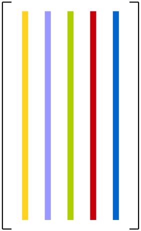
$x =$ Time periods $\{1, \ldots, T\}$



$\rightarrow$

Groups of identical units

$\widetilde{x} =$



$x_{t,i} \in \{0, 1\}$

$\widetilde{x}_{t,J} \in \{0, 1, \ldots, |J|\}$

(B. Knueven, J. Ostrowski, J.P. Watson, 2017)

# MUCP formulation

$$
\begin{aligned}
v = \quad \min_{x,u,p} \quad & \sum_{t=1}^{T} \sum_{i=1}^{n} c_f^i x_t^i + c_p^i p_t^i + c_0^i u_t^i \\
\text{s. t.} \quad & \sum_{t'=t-L^i+1}^{t} u_{t'}^i \leq x_t^i && \forall i \in \{1, ..., n\},\ \forall t \in \{L^i, ..., T\} \\
& \sum_{t'=t-\ell^i+1}^{t} u_{t'}^i \leq 1 - x_{t-\ell^i}^i && \forall i \in \{1, ..., n\},\ \forall t \in \{\ell^i + 1, ..., T\} \\
& u_t^i \geq x_t^i - x_{t-1}^i && \forall i \in \{1, ..., n\},\ \forall t \in \{2, ..., T\} \\
& \sum_{i=1}^{n} p_t^i \geq D_t && \forall t \in \{1, ..., T\} \\
& P_{min}^i x_t^i \leq p_t^i \leq P_{max}^i x_t^i && \forall i \in \{1, ..., n\},\ \forall t \in \{1, ..., T\} \\
& u_t^i,\ x_t^i \in \{0, 1\},\ p_t^i \in \mathbb{R} && \forall i \in \{1, ..., n\},\ \forall t \in \{1, ..., T\}
\end{aligned}
$$

# Construction of aggregated MUCP formulation

$\widetilde{x}_{t,J} = \sum_{i \in J} x_t^i$ $\qquad \widetilde{u}_{t,J} = \sum_{i \in J} u_t^i$ $\qquad \widetilde{p}_{t,J} = \sum_{i \in J} p_t^i$

First step: $\sum_{i=1}^n$ replaced by $\sum_{J \in PC} \sum_{i \in J}$

$$v = \min_{x,u,p} \sum_{t=1}^{T} \sum_{i=1}^{n} c_f^i x_t^i + c_p^i p_t^i + c_0^i u_t^i$$

$$\begin{aligned}
\text{s. t.} \quad & \sum_{t'=t-L^i+1}^{t} u_{t'}^i \leq x_t^i && \forall i \in \{1, ..., n\},\ \forall t \in \{L^i, ..., T\} \\
& \sum_{t'=t-\ell^i+1}^{t} u_{t'}^i \leq 1 - x_{t-\ell^i}^i && \forall i \in \{1, ..., n\},\ \forall t \in \{\ell^i + 1, ..., T\} \\
& u_t^i \geq x_t^i - x_{t-1}^i && \forall i \in \{1, ..., n\},\ \forall t \in \{2, ..., T\} \\
& \sum_{i=1}^{n} p_t^i \geq D_t && \forall t \in \{1, ..., T\} \\
& P_{min}^i x_t^i \leq p_t^i \leq P_{max}^i x_t^i && \forall i \in \{1, ..., n\},\ \forall t \in \{1, ..., T\} \\
& u_t^i,\ x_t^i \in \{0,1\},\ p_t^i \in \mathbb{R} && \forall i \in \{1, ..., n\},\ \forall t \in \{1, ..., T\}
\end{aligned}$$

# Construction of aggregated MUCP formulation

$\widetilde{x}_{t,J} = \sum_{i \in J} x_t^i$ $\widetilde{u}_{t,J} = \sum_{i \in J} u_t^i$ $\widetilde{p}_{t,J} = \sum_{i \in J} p_t^i$

First step: $\sum_{i=1}^n$ replaced by $\sum_{J \in PC} \sum_{i \in J}$

$$\widetilde{v} = \min_{\widetilde{x},\widetilde{u},\widetilde{p}} \sum_{t=1}^{T} \sum_{J \in PC} \sum_{i \in J} c_f^i x_t^i + c_p^i p_t^i + c_0^i u_t^i$$

$$\text{s. t.} \quad \sum_{t'=t-L^i+1}^{t} u_{t'}^i \leq x_t^i \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{L^i, ..., T\}$$

$$\sum_{t'=t-\ell^i+1}^{t} u_{t'}^i \leq 1 - x_{t-\ell^i}^i \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{\ell^i + 1, ..., T\}$$

$$u_t^i \geq x_t^i - x_{t-1}^i \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{2, ..., T\}$$

$$\sum_{J \in PC} \sum_{i \in J} p_t^i \geq D_t \qquad \forall t \in \{1, ..., T\}$$

$$P_{min}^i x_t^i \leq p_t^i \leq P_{max}^i x_t^i \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$$u_t^i,\ x_t^i \in \{0, 1\},\ p_t^i \in \mathbb{R} \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$\widetilde{x}_{t,J} = \sum_{i \in J} x_t^i$ $\widetilde{u}_{t,J} = \sum_{i \in J} u_t^i$ $\widetilde{p}_{t,J} = \sum_{i \in J} p_t^i$

Second step: $x_t^i$, $u_t^i$, $p_t^i$ replaced by $\frac{\widetilde{x}_{t,J}}{|J|}$, $\frac{\widetilde{u}_{t,J}}{|J|}$, $\frac{\widetilde{p}_{t,J}}{|J|}$

$$\widetilde{v} = \min_{\widetilde{x},\widetilde{u},\widetilde{p}} \sum_{t=1}^{T} \sum_{J \in PC} \sum_{i \in J} c_f^i x_t^i + c_p^i p_t^i + c_0^i u_t^i$$

$$\text{s. t.} \quad \sum_{t'=t-L^i+1}^{t} u_{t'}^i \leq x_t^i \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{L^i, ..., T\}$$

$$\sum_{t'=t-\ell^i+1}^{t} u_{t'}^i \leq 1 - x_{t-\ell^i}^i \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{\ell^i + 1, ..., T\}$$

$$u_t^i \geq x_t^i - x_{t-1}^i \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{2, ..., T\}$$

$$\sum_{J \in PC} \sum_{i \in J} p_t^i \geq D_t \qquad \forall t \in \{1, ..., T\}$$

$$P_{min}^i x_t^i \leq p_t^i \leq P_{max}^i x_t^i \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$$u_t^i,\ x_t^i \in \{0,1\},\ p_t^i \in \mathbb{R} \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$\widetilde{x}_{t,J} = \sum_{i \in J} x_t^i \qquad \widetilde{u}_{t,J} = \sum_{i \in J} u_t^i \qquad \widetilde{p}_{t,J} = \sum_{i \in J} p_t^i$

Second step: $x_t^i$, $u_t^i$, $p_t^i$ replaced by $\frac{\widetilde{x}_{t,J}}{|J|}$, $\frac{\widetilde{u}_{t,J}}{|J|}$, $\frac{\widetilde{p}_{t,J}}{|J|}$

$$\widetilde{v} = \min_{\widetilde{x},\widetilde{u},\widetilde{p}} \sum_{t=1}^{T} \sum_{J \in PC} \sum_{i \in J} c_f^i \frac{\widetilde{x}_{t,J}}{|J|} + c_p^i \frac{\widetilde{p}_{t,J}}{|J|} + c_0^i \frac{\widetilde{u}_{t,J}}{|J|}$$

$$\text{s. t.} \quad \sum_{t'=t-L^J+1}^{t} \frac{\widetilde{u}_{t',J}}{|J|} \leq \frac{\widetilde{x}_{t,J}}{|J|} \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{L^J, ..., T\}$$

$$\sum_{t'=t-\ell^J+1}^{t} \frac{\widetilde{u}_{t',J}}{|J|} \leq 1 - \frac{\widetilde{x}_{t-\ell^J,J}}{|J|} \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{\ell^J + 1, ..., T\}$$

$$\frac{\widetilde{u}_{t,J}}{|J|} \geq \frac{\widetilde{x}_{t,J}}{|J|} - \frac{\widetilde{x}_{t-1,J}}{|J|} \qquad \forall j \in J,\ \forall J \in PC,\ \forall t \in \{2, ..., T\}$$

$$\sum_{J \in PC} \sum_{i \in J} \frac{\widetilde{p}_{t,J}}{|J|} \geq D_t \qquad \forall t \in \{1, ..., T\}$$

$$P_{min}^i \frac{\widetilde{x}_{t,J}}{|J|} \leq \frac{\widetilde{p}_{t,J}}{|J|} \leq P_{max}^i \frac{\widetilde{x}_{t,J}}{|J|} \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$$\widetilde{x}_{t,J},\ \widetilde{u}_{t,J} \in \{0, 1, ..., |J|\},\ \widetilde{p}_{t,J} \in \mathbb{R} \qquad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$\widetilde{x}_{t,J} = \sum_{i \in J} x_t^i$ $\widetilde{u}_{t,J} = \sum_{i \in J} u_t^i$ $\widetilde{p}_{t,J} = \sum_{i \in J} p_t^i$

Simplify with $|J|$

$$\widetilde{v} = \min_{\widetilde{x},\widetilde{u},\widetilde{p}} \sum_{t=1}^{T} \sum_{J \in PC} \sum_{i \in J} c_f^i \frac{\widetilde{x}_{t,J}}{|J|} + c_p^i \frac{\widetilde{p}_{t,J}}{|J|} + c_0^i \frac{\widetilde{u}_{t,J}}{|J|}$$

s. t.

$$\sum_{t'=t-L^J+1}^{t} \frac{\widetilde{u}_{t',J}}{|J|} \leq \frac{\widetilde{x}_{t,J}}{|J|} \quad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{L^J, ..., T\}$$

$$\sum_{t'=t-\ell^J+1}^{t} \frac{\widetilde{u}_{t',J}}{|J|} \leq 1 - \frac{\widetilde{x}_{t-\ell^J,J}}{|J|} \quad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{\ell^J + 1, ..., T\}$$

$$\frac{\widetilde{u}_{t,J}}{|J|} \geq \frac{\widetilde{x}_{t,J}}{|J|} - \frac{\widetilde{x}_{t-1,J}}{|J|} \quad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{2, ..., T\}$$

$$\sum_{J \in PC} \sum_{i \in J} \frac{\widetilde{p}_{t,J}}{|J|} \geq D_t \quad \forall t \in \{1, ..., T\}$$

$$P_{min}^i \frac{\widetilde{x}_{t,J}}{|J|} \leq \frac{\widetilde{p}_{t,J}}{|J|} \leq P_{max}^i \frac{\widetilde{x}_{t,J}}{|J|} \quad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$$\widetilde{x}_{t,J},\ \widetilde{u}_{t,J} \in \{0, 1, ..., |J|\},\ \widetilde{p}_{t,J} \in \mathbb{R} \quad \forall i \in J,\ \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$\widetilde{x}_{t,J} = \sum_{i \in J} x_t^i$ $\widetilde{u}_{t,J} = \sum_{i \in J} u_t^i$ $\widetilde{p}_{t,J} = \sum_{i \in J} p_t^i$

Simplify with $|J|$

$$
\begin{aligned}
\widetilde{v} = \quad \min_{\widetilde{x},\widetilde{u},\widetilde{p}} \quad & \sum_{t=1}^{T} \sum_{J \in PC} c_f^i \widetilde{x}_{t,J} + c_p^i \widetilde{p}_{t,J} + c_0^i \widetilde{u}_{t,J} \\
\text{s. t.} \quad & \sum_{t'=t-L^J+1}^{t} \widetilde{u}_{t',J} \leq \widetilde{x}_{t,J} && \forall J \in PC,\ \forall t \in \{L^J, ..., T\} \\
& \sum_{t'=t-\ell^J+1}^{t} \widetilde{u}_{t',J} \leq |J| - \widetilde{x}_{t-\ell^J,J} && \forall J \in PC,\ \forall t \in \{\ell^J + 1, ..., T\} \\
& \widetilde{u}_{t,J} \geq \widetilde{x}_{t,J} - \widetilde{x}_{t-1,J} && \forall J \in PC,\ \forall t \in \{2, ..., T\} \\
& \sum_{J \in PC} \widetilde{p}_{t,J} \geq D_t && \forall t \in \{1, ..., T\} \\
& P_{min}^i \widetilde{x}_{t,J} \leq \widetilde{p}_{t,J} \leq P_{max}^i \widetilde{x}_{t,J} && \forall J \in PC,\ \forall t \in \{1, ..., T\} \\
& \widetilde{x}_{t,J},\ \widetilde{u}_{t,J} \in \{0, 1, ..., |J|\},\ \widetilde{p}_{t,J} \in \mathbb{R} && \forall J \in PC,\ \forall t \in \{1, ..., T\}
\end{aligned}
$$

# Aggregated MUCP formulation

$$\widetilde{v} = \min_{\widetilde{x},\widetilde{u},\widetilde{p}} \sum_{t=1}^{T} \sum_{J \in PC} c_f^i \widetilde{x}_{t,J} + c_p^i \widetilde{p}_{t,J} + c_0^i \widetilde{u}_{t,J}$$

$$\text{s. t.} \quad \sum_{t'=t-L^J+1}^{t} \widetilde{u}_{t',J} \leq \widetilde{x}_{t,J} \qquad \forall J \in PC,\ \forall t \in \{L^J, ..., T\}$$

$$\sum_{t'=t-\ell^J+1}^{t} \widetilde{u}_{t',J} \leq |J| - \widetilde{x}_{t-\ell^J,J} \qquad \forall J \in PC,\ \forall t \in \{\ell^J + 1, ..., T\}$$

$$\widetilde{u}_{t,J} \geq \widetilde{x}_{t,J} - \widetilde{x}_{t-1,J} \qquad \forall J \in PC,\ \forall t \in \{2, ..., T\}$$

$$\sum_{J \in PC} \widetilde{p}_{t,J} \geq D_t \qquad \forall t \in \{1, ..., T\}$$

$$P_{min}^i \widetilde{x}_{t,J} \leq \widetilde{p}_{t,J} \leq P_{max}^i \widetilde{x}_{t,J} \qquad \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

$$\widetilde{x}_{t,J},\ \widetilde{u}_{t,J} \in \{0, 1, ..., |J|\},\ \widetilde{p}_{t,J} \in \mathbb{R} \qquad \forall J \in PC,\ \forall t \in \{1, ..., T\}$$

Here $\widetilde{v} = v$: linked to the **total unimodularity** of blue constraints matrix.

(Baum and Trotter, 1979)

# Aggregation in practice

**For the MUCP:**

- **Aggregate** variables corresponding to identical units
- **Solve** the aggregated ILP
- **Disaggregate** the obtained aggregated solution (a simple algorithm exists)

Very efficient in practice!

But as soon as **ramp constraints** are taken into account:
$\rightarrow$ **disaggregation no longer possible.**

$\rightarrow$ other symmetry-breaking techniques must be used

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