Lecture

Solving combinatorial optimization problems using mathematical programming

Section 0 : Introduction

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A **Combinatorial Optimization Problem**

is to

find a greatest (smallest) element
within a valuated finite set.

Combinatorial Optimization Problem

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Given :

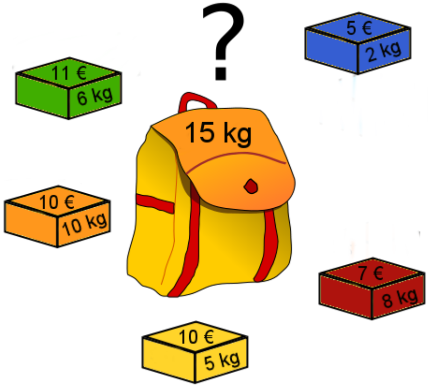
- a finite subset of elements $E = \{e_1, \ldots, e_n\}$
- a **solution set** $\mathcal{F}$ of subsets of E
- a weight $c = (c(e_1), \ldots, c(e_n))$

a **Combinatorial Optimization Problem** is to find a solution $F \in \mathcal{F}$ whose weight $c(F) = \sum_{e \in F} c(e)$ is maximum (or min.),

$$\textit{i.e.} \quad \max \{c(F) \mid F \in \mathcal{F}\}.$$

The knapsack problem

Which boxes to chose
to maximize the profit
without exceeding 15kg ?



Définition

Let us consider n **objects**.
Each object $i \in \{1, \ldots, n\}$
- with a **profit** g_i
- with a **weight** p_i
to be put inside a knapsack of **maximum total weight** P.

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its weight $\sum_{i \in S} p_i$ is non-greater than P *(Knapsack Constraint)*

its profit $\sum_{i \in S} g_i$ is maximum. *(Objective function)*

How to incode a knapsack solution ?

An instance is given by :

i	1	2	3	4	5	
g_i	5	7	10	11	10	
p_i	2	8	10	6	5	≤ 15

A solution $S \subset \{1, \ldots, n\}$ is corresponding to **an incidence vector** χ^S such that

$$\chi^S[i] = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

$S_1 = \{1, 2, 5\}$

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1	1	0	0	1

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profit :	5	7			10	$= 22$
weight :	2	8			5	$= 15$

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Recognition Algorithm :

$S_2 = \{1, 4, 5\}$

$\chi^{S_2} =$

1	0	0	1	1

$pds \leftarrow 0$

For i from 1 to n

$pds \leftarrow pds + p_i * \chi^S[i]$

EndFor

If $pds \leq P$ **Then** True

Else False

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Solution of profit 26

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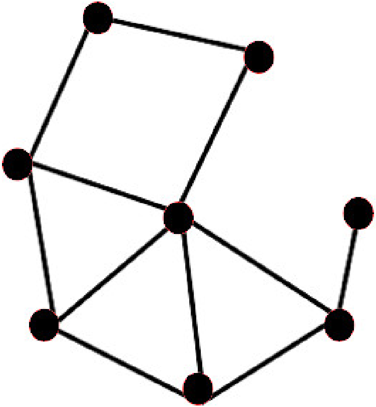
weight : 2 8 6 $= 16$

Not a solution.

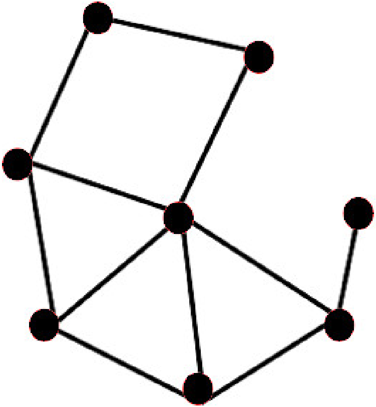
Huge knapsacks...

Graph



A **graph** $G = (V, E)$ is a pair where
- V is the node set
- $E \subseteq V \times V$ is the edge set.

Graph



A **graph** $G = (V, E)$ is a pair where
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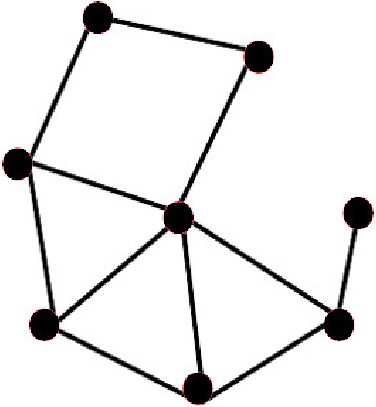
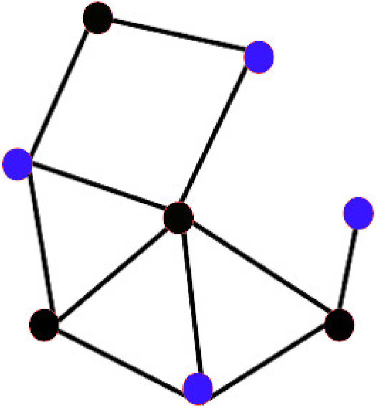
Two nodes are **adjacent** if they are linked by an edge.

Stable Set problem



A **stable set**
(or independent set)
is a pairwise non-adjacent node subset.

Stable Set problem



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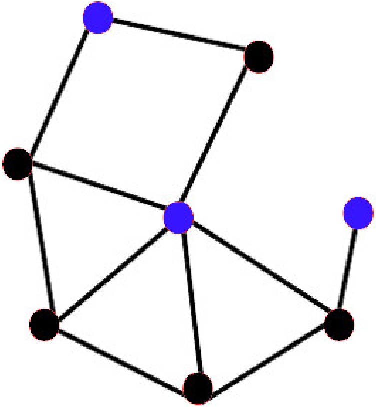

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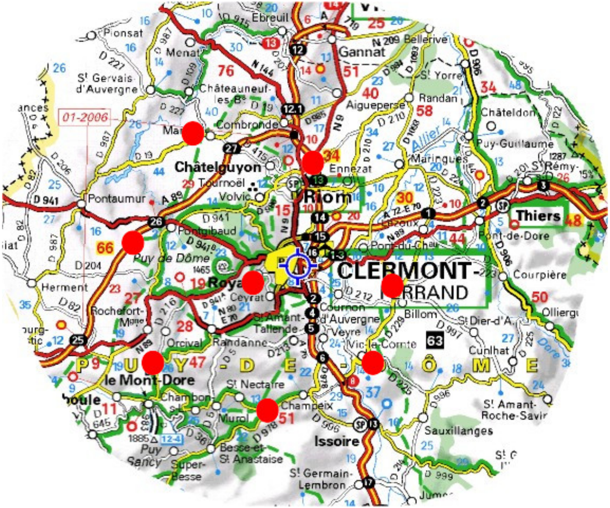
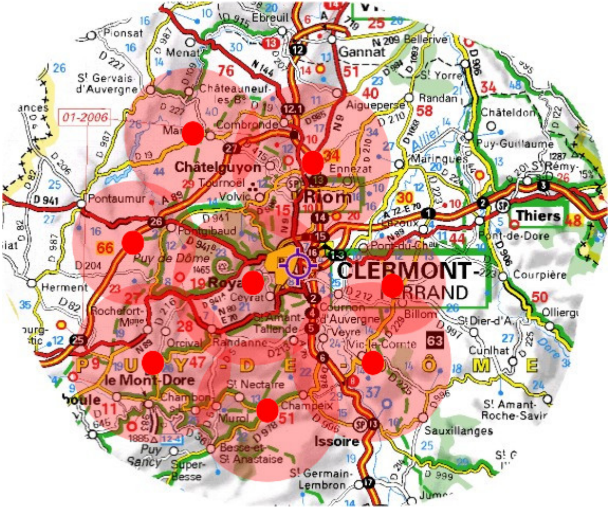
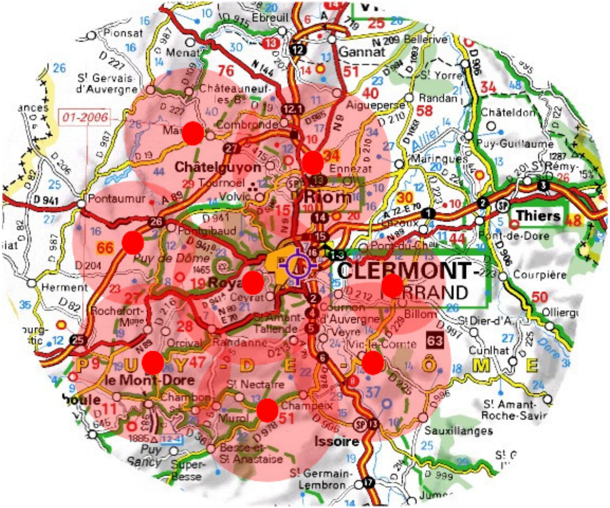
A set of antennas



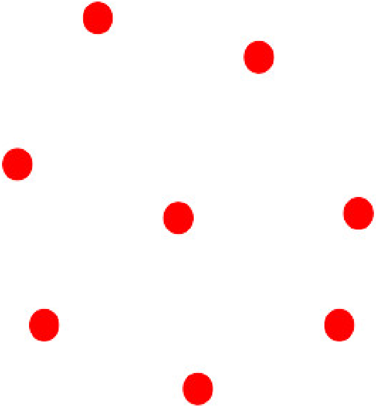
A set of antennas with their interference disks



How to find a subset of antennas with no interference ?



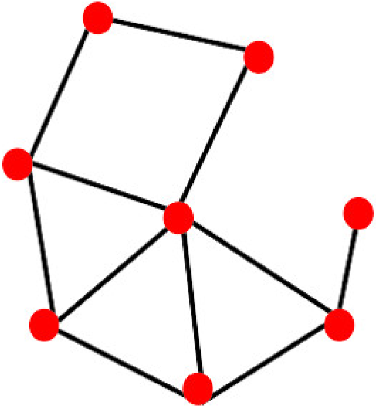
A model graph



Un graphe $G = (S, V)$ avec
- V : a node by antenna
- E : an edge between two antennas if their interference disks are in conflict.

Finding a subset of antennas with no interference reduces to the stable set problem.

A model graph



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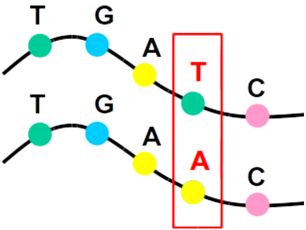
Combinatorics within genomic



Genome sequencing involves sorting through small fragments of DNA (called SNPs) that have been sequenced using bio-mechanical processes. In the process, false SNPs are created.

We want to sort the fragments, omitting as few SNPs as possible.

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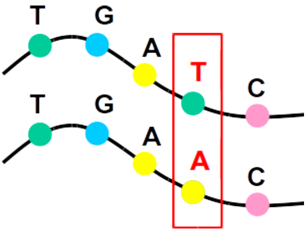

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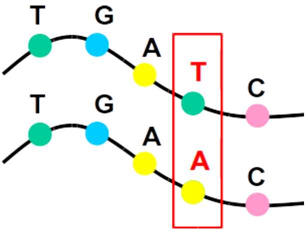

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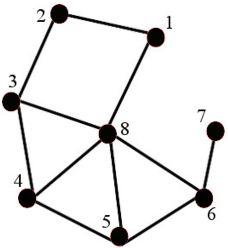
Its a **Combinatorial optimisation problem**

which also reduces to the **stable set problem !**.

Lippert, R., Schwartz, R., Lancia, G., Istrail, S. : Algorithmic strategies for the single nucleotide polymorphism haplotype assembly problem. Brief. Bioinform 3, 23–31 (2002)

How to incode a stable set solution ? ?

Instance given by a graph with n nodes.



A solution $S \subset \{1, \ldots, n\}$ is corresponding to an **vecteur d'incidence** χ^S such that

$$\chi^S[i] = \begin{cases} 1 & \text{si } i \in S \\ 0 & \text{sinon.} \end{cases}$$

$S_1 = \{1, 7, 8\}$

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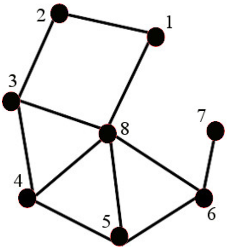
How to test whether a vector is a solution ?



Recognition Algorithm :

For i from 1 to n

For j from 1 to n

If $\{i, j\}$ is an edge

and if $\chi^S[i] = 1$ and $\chi^S[j] = 1$

Then STOP : False

EndFor

EndFor

STOP : True

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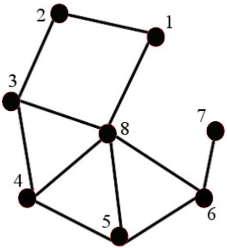

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Solution (with 3 nodes

Recognition algorithms

Knapsack

$pds \leftarrow 0$
For i from 1 to n
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If $pds \leq P$ **Then** True
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Execution time proportionnal to n.

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Execution time proportionnal to n^2.

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Polynomial Algorithms
which can be fast ($\leq n^4$ for instance).

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Which is the complexity of the best known "Generic Algorithm" so find the optimal solution of combinatorial optimization problems ?

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Solving algorithms

Which is the complexity of the best known "Generic Algorithm" so find the optimal solution of combinatorial optimization problems ?

Only **very very slow** algorithms with an execution time proportional to 2^n, 3^n, $!n$

Exponential algorithms.

Enumerate ?

Consider an optimization problem with n elements.
Assuming that we know a **polynomial** (and. fast) algorithm for recognizing a solution,
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Recogntion algorithm for χ^S.
If χ^S is a solution
Stock S as the best known solution encountered yet.
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STOP : the stocked solution.

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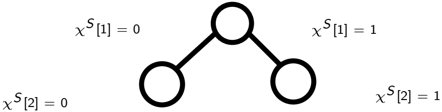
There are 2^n **subsets** : **exponential** execution time !

Combinatorial explosion



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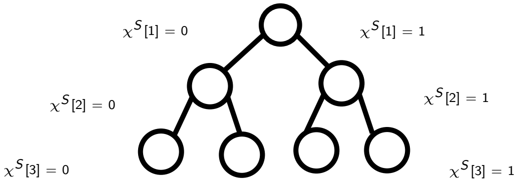

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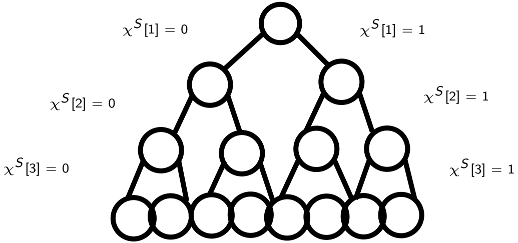

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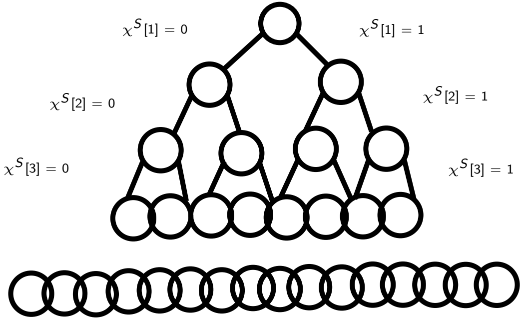

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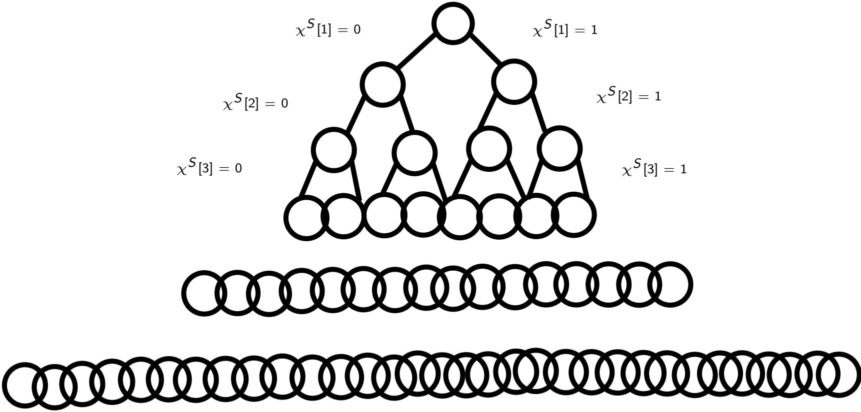

Combinatorial explosion

Let's take one of the most powerful computers in 2015

Tianhe-2 (China)

33.86 petaflops where 1 petaflop represents the processing of 10^{15} operations per second (one million billion).

Assume that the recognition algorithm takes $100n$ elementary operations. For $n = 10$, we can process 33 860 billion subsets in 1 second !

	Tianhe-2			Futuristic Computer
n	n^3	n^5	$100n2^n$	$100n2^n$
10	0,00...001 sec	0,00...001 sec		
20	0,00...001 sec	0,00...001 sec		
50	0,00...001 sec	0,00...001 sec		
60	0,00...001 sec	0,00000002 sec		
80	0,00...001 sec	0,0000009 sec		
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50	0,00...001 sec	0,00...001 sec	168,9 sec	
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100	0,00...001 sec	0,0000003 sec	$1,21.10^{10}$ years	
1000	0,00000003 sec	0,03 sec	1.10^{282} years	

Age of the universe $13,7.10^9$ years

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Combinatorial Explosion

Enumerating 2^n solutions is not a technical problem
that very powerful computers could sweep away.

It's necessary to **circumvent the combinatorial explosion** with mathematical and algorithmic tools.

Problem complexity

Just because we know an exponential algorithm for solving a problem doesn't mean it's difficult !

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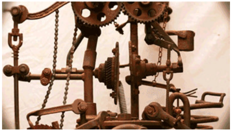

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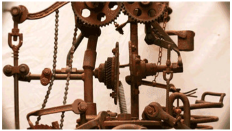

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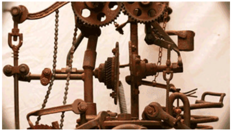


We're looking for the **fastest** algorithm to solve a problem !

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if an exponential algorithm exists to solve it.
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Question : In what complexity classes are Combinatoriel Optimization problems ?

$\mathcal{NP}$ **problems**

A particular classe have been created...
Combinatorial optimization problems for which we know **how to recognize a solution** with a polynomial algorithm

$\Rightarrow$ the problem class $\mathcal{NP}$ "Nondeterministic polynomial time"

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A particular classe have been created...
Combinatorial optimization problems for which we know **how to recognize a solution** with a polynomial algorithm

$\Rightarrow$ the problem class $\mathcal{NP}$ "Nondeterministic polynomial time"

Under this assumption, we've seen that it's possible to use an exponential enumeration algorithm, then

$\Rightarrow \mathcal{NP} \subset \mathcal{EXP}$

In addition, a polynomial problem is in $\mathcal{NP}$

$\Rightarrow \mathcal{P} \subset \mathcal{NP} \subset \mathcal{EXP}$

And then ?

Is $\mathcal{P}$ equal to $\mathcal{NP}$?

i.e.

"Is there a polynomial algorithm for solving combinatorial optimization problems ?"

Answer :

And then ?

Is $\mathcal{P}$ equal to $\mathcal{NP}$?

i.e.

"Is there a polynomial algorithm for solving combinatorial optimization problems ?"

Answer : We don't know !

On a human scale, we only know this enumeration algorithm !

It's one of the 7 problems in the Clay Mathematics Institute of Cambridge's million-dollar Millennium Prize Problems !

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a knapsack with all objects of the same weight.

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- sort the *n* objects from the most expensive to the least expensive
- take the objects one by one in that order as long as they fit in the bag !

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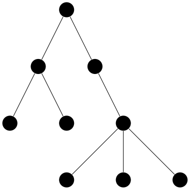
- sort the n objects from the most expensive to the least expensive
- take the objects one by one in that order as long as they fit in the bag !

This algorithm is polynomial (of the order of n^2).

A very simple stable set subcase

Let's look at a very simple stable set instance :
finding a maximum cardinality stable set over a **tree**.



Tree property :
Given a leaf v, there exists a maximum cardinality stable set containing v.

(Proof : Let v a leaf and u its unique neighbour. Let S be a maximum stable set. Either $u \notin S$, then $S \cup \{v\}$; or $u \notin S$, then $S \cup \{u\} \setminus \{v\}$ is another maximum stable set).

A very simple stable set subcase

A **greedy** algorithm :

$S \leftarrow \emptyset$
While G has at least one edge
    Let v be a leaf and u its neighbour
    $S \leftarrow S \cup \{v\}$
    Delete from G nodes u and v and all their incident edges
EndWhile
Add to S all the remaining nodes.
STOP : S

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This algorithm is polynomial (of the order of n).

$\mathcal{NP}$-hard

In the current state of scientific knowledge, it is not known whether or not there is a polynomial algorithm for solving the knapsack or the stable set problem in general !

$\mathcal{NP}$-hard

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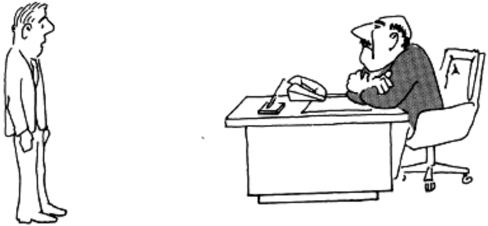
In fact, we were able to prove that knapsack or the stable set problems are just as difficult as all the problems in the $\mathcal{NP}$ class !

A problem is said to be $\mathcal{NP}$-**hard**
if it is as difficult as any problem in the $\mathcal{NP}$ class.
$\Rightarrow$ the problem class $\mathcal{NP}$-hard.

$\mathcal{NP}$-hardness



Boss, I can't find a polynomial algorithm for the stable set problem

Figure from "The Garey et Johnson"

$\mathcal{NP}$-hardness

But if you think I'm just an ungifted searcher

Figure from "The Garey et Johnson"

$\mathcal{NP}$-hardness

...neither are all the others !

Figure from "The Garey et Johnson"

Strongly or weakly $\mathcal{NP}$-hard

A computational problem may have numerical parameters : like the weight of a knapsack.

A $\mathcal{NP}$-hard problem is said to be **weakly $\mathcal{NP}$-hard**
if there is an algorithm for the problem whose running time is polynomial in the dimension of the problem and magnitudes of its data.

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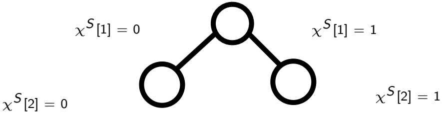
For weakly $\mathcal{NP}$-hard, it's often the case, that there exists a **dynaming programming scheme** whose complexity depends on the magnitudes of the data.

Dynamic Programming scheme

Some subcases are the same within the enumeration : this puts the brakes on the combinatorial explosion.



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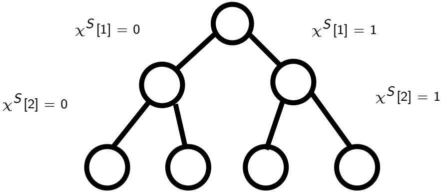

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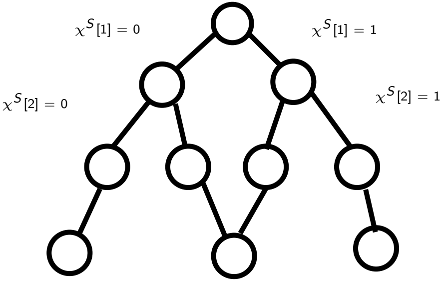

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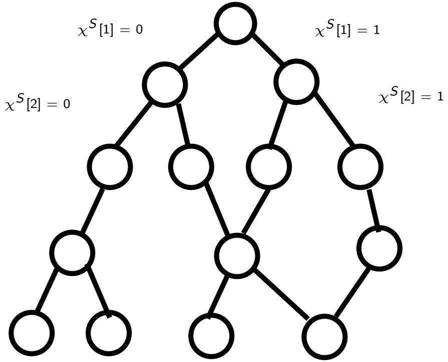

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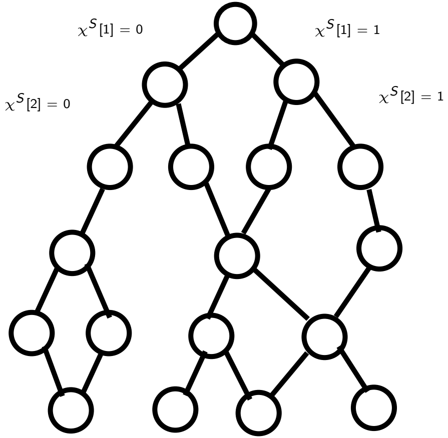

1. Definition and complexity

2. List of OC problems

Knapsack problem

Input : n objects
profit g_i $\forall i \in \{1, \ldots, n\}$
weight p_i $\forall i \in \{1, \ldots, n\}$
maximum total weight P.

Output : Subset $S \subseteq \{1, \ldots, n\}$
s.t. $\sum_{i \in S} p_i \leq P$

Objective : Max $\sum_{i \in S} w_i$

Knapsack MIP

$$\mathrm{Max} \sum_{i=1}^{n} w_i x_i$$

$$\sum_{i=1}^{n} p_i x_i \leq P,$$

$$0 \leq x_i \leq 1, \text{ pour tout objet } i = 1, ..., n,$$

$$x_i \in \boldsymbol{N}, \text{ pour tout objet } i = 1, ..., n.$$

Maximum weight stable set problem

Input : Undirected graph $G = (V, E)$
cost $w_u\ \forall u \in V$

Output : Subset $S \subseteq V$ of non-adjacent nodes

Objective : Max $\sum_{i \in S} w_i$

Complexity : Strongly NP-hard
Polynomial cases : perfect graphs (tree, planar, interval graphs...)
Difficulty : With MIP solver $n = 1000$ in often more than 1 hour
Some dedicated methodes (russian doll algo)

Shortest path problem

Input : Undirected (or directed) graph $G = (V, E)$
Two nodes $u_0, u_1 \in V$
Lenghts $l_e \ \forall e \in E$

Output : Path μ of G from u_O to u_1

Objective : Min $\sum_{e \in \mu} l_e$

Complexity : Polynomial
Difficulty : With Dijkstra algorithm up to thousand of nodes in a few sec
With A^* algorithme, up to billions !

Shortest path problem

To quickly go from a point to another

The Traveling Salesman Problem (TSP)

Input : Undirected (or directed) graph $G = (V, E)$
lenght $l_e \ \forall e \in E$

Output : An hamiltonian cycle C of G (i.e. C goes once through each node)

Objective : Min $\sum_{e \in C} l_e$

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Concorde : a Branch-and-Cut algorithm and polyhedral results solves up to 200000 cities within one day !

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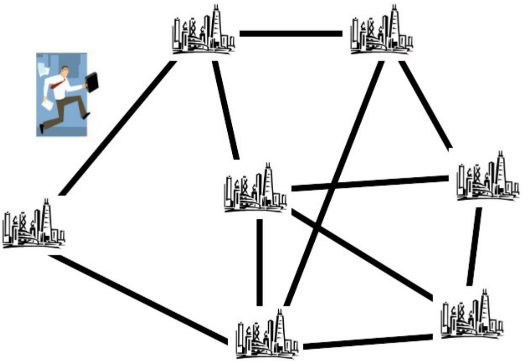

Starting from a city and going back to it
going once througt all the other cities
with a sorthest cycle.

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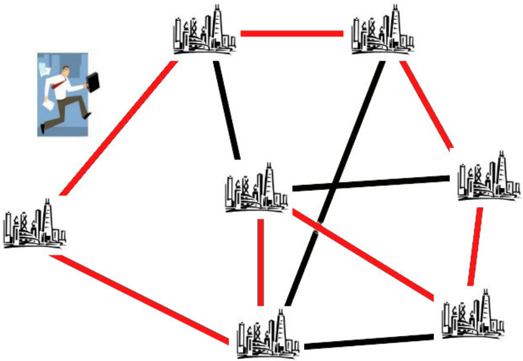

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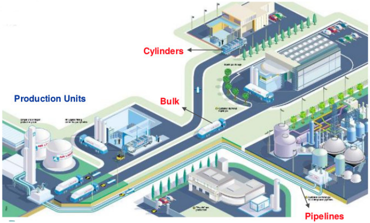
Real-world Operations Research problem



The **inventory routing problem** introduced by **Air Liquide** company with real daily data and questions to answer :

Determine routes for liquid oxygen trucks to deliver hospitals so that

- hospital tanks are never empty (remote monitoring)
- rounds are feasible within the driver's working day
- costs are minimized !

Source : ROADEF Challenge 2016

How to solve Combinatorial Optimization Problem ?

• If the instances are very large and the problem very hard to solve
or if you do not have much time to spend on solving method

→ There exist methods to obtain "good" solutions (heuristics, meta-heuristics, machine learning...)

• If the problem is really important with a higly cost associated to solution
and if you have some time (several hours...)
The optimal solution is to be computed !

→ We will see in this lecture **how to circumvent the combinatorial explosion using mathematical programming !**

The Traveling salesman problem has been solved till 200 000 cicites !