Lecture "Solving combinatorial optimization problems using mathematical programming"

Exercises

1 Compact MIP formulations

Exercice* 1 : *Transportation problem

A car manufacturing company has 3 plants in three place A, B and C. It needs to transport the necessary metals from two ports P and Q. Each plant requires 400 tons per week for A, 300 tons for B and 200 tons for C. The ports of P and Q can supply 550 tonnes and 350 tonnes respectively. Transport costs between these cities are given in kilo-euro per tonne in the following table.

	A	B	C
P	5	6	3
Q	3	5	4

Propose a model of this transport problem to satisfy demand, based on available quantities and minimizing transport costs.

Exercice* 2 : *Leisure centers

A region is divided into six zones (zones 1,...,6). The municipality wants to build leisure centers in some of these zones. And it wants to build a minimum number of centers so that, for each zone, there is at least one center that is no more than 15 minutes (by car) from that zone. The time needed to get from one zone to another is given in the following table:

de	zone 1	zone 2	zone 3	zone 4	zone 5	zone 6
zone 1	0	10	20	30	30	20
zone 2	10	0	25	35	20	10
zone 3	20	25	0	15	30	20
zone 4	30	35	15	0	15	25
zone 5	30	20	30	15	0	14
zone 6	20	10	20	25	14	0

2.1) Formulate the problem of determining the minimum number of centers to be built, and the zones in which they must be built, as a linear integer program.

2.2) Modify the program to meet the following constraint: if a center is built in zone 1, then a center must be built in zone 4.

2.3) Which inequality models the following constraint: at least one zone among zones 1, 2 and 3 must have at least one center no more than 15 minutes away. Is it necessary to add it to the program?

2.4) Using the ideas from the previous question, can we reduce the formulation by removing inequalities?

Exercice* 3 : *Decision in industrial production

A company producing plastic beads wants to expand into a new geographical area. These plastic beads are the raw material for many industrial objects (seats, tool handles, cans, etc.). The company has approached n customers and plans to sell d_i tons of plastic beads to each $i \in \{1, \ldots, n\}$ customer over the next 5 years. The company has m potential sites $s_1, \ldots, s_m$ for its factories. The cost of installing a plant on site s_j has been estimated at c_j euros, $j = 1, \ldots, m$. The planned plants are not all of the same production capacity: one site s_j will have a capacity of M_j tonnes of logs over the next 5 years, $j = 1, \ldots, m$. Production costs are assumed to be independent of production location. Finally, we calculate the transport costs per tonne c_{ij} between a customer i and a site s_j, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

The company wishes to determine the sites on which to establish its factories in order to satisfy customer demand while minimizing the total cost (installation, production and delivery) over the next 5 years.

Give a MIP model of this problem.

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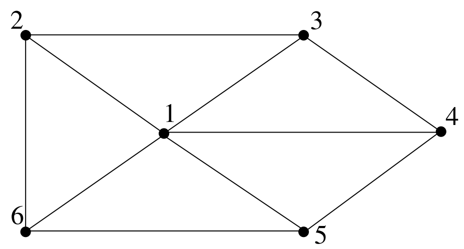
2 Extrem points and cutting inequalities

Exercice* 4 : *Wheel inequalities for the stable set problem

Consider the maximum-weight stable problem on the following graph $H_1 = (V, E, c)$ (where c is a weight of 1 on each vertex):



Consider the edge formulation:

$$
\begin{aligned}
\text{Max } \sum_{u \in V} c(u)x(u) & \\
x(u) + x(v) \leq 1 \quad & \forall uv \in E, && (1)\\
0 \leq x(u) \leq 1 \quad & \forall u \in E, && (2)\\
x(u) \text{ integer}, \quad & \forall u \in V.
\end{aligned}
$$

4.1) Give all the inequalities of the formulation for H_1.

4.2) Give a fractional solution of the linear relaxation of the formulation that satisfies at least 6 inequalities to equality. What would need to be proved for it to be an extreme point of the formulation?

We now add the following odd-cycle inequality to the formulation.

$$
\sum_{u \in V(C)} x(u) \leq \left\lfloor \frac{|C|}{2} \right\rfloor \quad \forall \text{ odd cycle } C, \tag{3}
$$

$$
\tag{4}
$$

Note: we've seen clique inequalities in class: we can see that H_1 contains no cliques of sizes greater than 3 and that all the inequalities associated with cliques of sizes 2 (edges) and 3 (odd cycles of size 3) are now in the formulation.

4.3) Give all formulation inequalities for H_1.

4.4) Show that a solution that exactly satisfies the odd cycle inequalities corresponding to $(1, 2, 3)$, $(1, 3, 4)$, $(1, 4, 5)$, $(1, 5, 6)$, $(1, 6, 2)$ et $(2, 3, 4, 5, 6)$ is fractionnal.

4.5) Give the arguments proving that this point is a fractional extreme point of the polyhedron defined by the trivial (2), edge (1) and odd-cycle (1) inequalities for this graph H_1.

4.6) Sum the triangle inequalities (size 3 cycle) from the formulation for H_1 and a trivial inequality to obtain a new inequality according to the Chvátal rounding principle.

4.7) Prove a second time the validity of this inequality withour Chvátal sum.

4.8) Is this inequality cut the extrem points of the first question?

Exercice 5 *: Cover inequalities for the stable set problem*

Let us Consider this instance of the knapsack problem.

$$
\begin{aligned}
Max \quad c_1x_1 + c_2x_2 + ... + c_nx_n \\
a_1x_1 + a_2x_2 + ... + a_nx_x \leq b \\
x_i \in \{0, 1\} \text{ pour } i = 1, ..., n
\end{aligned}
$$

Assume $a_i > 0$ for $i = 1, ..., n$ and $b > 0$. Consider the polyhedron resulting from the linear relaxation of this formulation.

A subset R of $\{1, ..., n\}$ is said to be a cover set for P if $\sum_{i \in R} a_i > b$.

5.1) Show that if R is a cover set of P then the following constraint is valid for the knapsack problem:

$$\sum_{i \in R} x_i \leq |R| - 1.$$

5.2) Let the instance, called Q

$$\begin{aligned} Max \quad z = 10x_1 + 19x_2 + 12x_3 + 12x_4 + x_5 + 3x_6 + x_7 \\ 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \\ x_i \in \{0,1\} \text{ pour } i = 1, ..., 7. \end{aligned}$$

Are the following sets are cover set for the knapsack problem Q:
$R_1 = \{1,2,3\}$, $R_2 = \{2,3,4\}$ and $R_3 = \{3,4,5,6\}$? Give the corresponding cover onstraints for those that are.

5.3) For each constraint given in 2), give an extreme point of the $tildeQ$ domain (the linear relaxation of Q) that can be cut by this constraint. Justify your answer.

5.4) *Separation of overlap inequalities.* Since the number of cover inequalities is potentially exponential, we'd like to propose a separation algorithm. Let $tildex$ be a fractional point resulting from the linear relaxation of the knapsack problem (or from a relaxation reinforced with valid inequalities): the separation problem thus comes down to determining whether or not there exists a cover set R such that $sum_{i \in R} \tilde{x}_i > |R| - 1$ and, if there exists one, to exhibit it.
a) Let $tildex'_i = 1 - \tilde{x}_i$ for all $i \in \{1, \ldots, n\}$. Show that, for a given R, the inequality can then be re-written $\sum_{i \in R} \tilde{x}'_i \geq 1$.
b) Show that this separation problem can be reduced to a problem where $\tilde{x}'$ is the coefficient of the objective function. To which "family" of problems does this problem belong?
c) Assuming that this reduction is in fact an equivalence, what is the complexity of this separation problem?
d) Explain how to use these inequalities in practice.

Exercice 6 : *Polyhedral study of the acyclic subgraph problem*

Let $G = (V, A)$ be a directed graph, where V is the set of vertices and A is the set of arcs. Let uv be an arc of A from vertex u to vertex v. A sequence of arcs $P = (u_0u_1, u_1u_2, ..., u_{k-1}u_k)$ is called *path* in G, and P is said to be of size k. A *circuit* C in G is a path such that $u_0 = u_k$. We denote $V(P)$ (resp. $V(C)$) the set of vertices involved in a path (resp. a circuit).

A graph is said to be *acyclic* if it contains no circuits. Let $W \subset V$ be a subset of vertices, and let $A(W)$ be the set of arcs having both ends in W. The graph $(W, A(W))$ is then said to be the graph induced by W. Given a function $c : V \to \mathbb{R}$ which associates a weight $c(v)$ with any vertex $v \in V$, the *induced acyclic subgraph problem* (IASP) consists in determining an induced acyclic subgraph $(W, A(W))$ of G such that $c(W) = \sum_{v \in W} c(v)$ is maximum.

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The induced acyclic subgraph problem is equivalent to the following integer program (P).

$$
(\mathcal{P}) \begin{cases} Max & \sum_{u \in V} c(u)x(u) \\ & \sum_{u \in W} x(u) \leq |W| - 1, \quad \forall W \subset V \text{ s.t. } (W, A(W)) \text{ is a circuit} \quad (1) \\ & 0 \leq x(u) \leq 1, \quad \forall u \in V \quad (2) \\ & x(u) \text{ entier}, \quad \forall u \in V. \quad (3) \end{cases}
$$

General Polyhedral approach

Let $G = (V, A)$ be a directed graph with $n = |V|$. Let $P(G)$ be the polytope of induced acyclic subgraphs of G, i.e.

$$P(G) = conv\{\chi^W \in I\!R^n \mid (W, A(W)) \text{ est acyclique.}\}$$

6.1) Show that $P(G)$ is full dimensional.

6.2) Show that trivial inequalities $x(u) \geq 0$, $u \in V$ define facets of $P(G)$.

6.3) Let $G_0 = (V, A)$ be a graph that is a single circuit C_0. Show that inequlity inequality corresponding to C_0 defines a facet of $P(G_0)$.

6.4) Let be a graph $G_1 = (V, A)$ composed of a single circuit $C_1 = (v_1v_2, v_2v_3, ..., v_nv_1)$ such that there are two non-consecutive vertices v_i and v_j, $i \neq j$ are connected by an arc. Show that the circuit constraint corresponding to C_1 does not define a facet of $P(G_1)$. Deduce a reduced formulation of the PLNE (P).

Study on a diclique

Let $K_n = (W, A_W)$ be a di-complete directed graph (i.e. such that there is an arc connecting every vertex of W to every vertex of W). When K_n is a subgraph of a graph G, we call K_n a diclique of G. The graph K_4 is given in figure **??**.

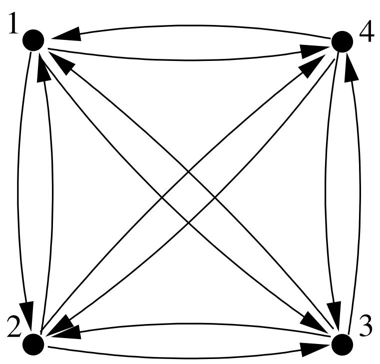

Figure 1: Diclique K_4

6.5) Consider the graph K_4 and the solution y^* given by $y^*(u_i) = \frac{1}{2}$, $i = 1, ..., 4$. Show that y^* satisfies all the constraints of the program (P) for K_4 when the integrity constraint is relaxed (3). Indicate which conditions prove that y^* is a vertex of the polyhedron defined by the constraints of this program for K_4.

6.6) We are now interested in any graph K_n. Prove that only one vertex of K_n can belong to a solution of PSAI. Deduce a valid inequality for $P(G)$ when G contains a diclique.

6.7) Let $G = (V, A)$ be a directed graph containing a diclique $K = (W, A_W)$. Assume that for any vertex u of $V \setminus W$, there exists a vertex v of W such that the arc uv or the arc vu does not exist in A (this is equivalent to saying that K is maximal in the sense of inclusion). Prove that the inequality in the previous question then describes a facet of $P(G)$. Deduce a new formulation for P.