Rhombus Tilings

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Moscow, Spring 2011
1. Dualization of multigrids
2. Projection of higher dimensional lattices
3. Matching rules: basics
4. Matching rules: results
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Pentagrids

Penrose tiling $\equiv$ pentagrid with integer-sum shift (de Bruijn).
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Pentagrids

Different integer-sum shifts yield different Penrose tilings.
Playing with the shift

Forgetting the integer-sum condition still yields rhombus tilings.
Playing with the shift

Not Penrose tilings, but so-called generalized Penrose tilings.
One can actually consider any number $n \geq 2$ of grids (here, $n = 7$).
Playing with the grid number

This yields rhombus tilings with arbitrary point-symmetry.
Playing with the grid spacing

Uniform grid spacing (different grids can have a different spacing).
Uniform grid spacing (different grids can have a different spacing).
Playing with the grid spacing

Quasiperiodic grid spacing.
Playing with the grid spacing

Quasiperiodic grid spacing.
Playing with the grid spacing

General grid spacing.
Playing with the grid spacing

General grid spacing.
Formally

**Definition (Grid)**

Let $\vec{g}$ be a unit vector of $\mathbb{R}^2$ and $C$ be a discrete subset of $\mathbb{R}$. The $\vec{g}$-directed and $C$-spaced grid is $G := \{\vec{x} \in \mathbb{R}^2 \mid \langle \vec{x} | \vec{g} \rangle \in C\}$.

Let $K_G$ index by integer the strips of $G$ (in the direction of $\vec{g}$).

**Definition (Dual of a multigrid $(G_1, \ldots, G_d)$)**

To a mesh containing $\vec{x} \in \mathbb{R}^2$ is associated the point $\sum_i K_{G_i}(\vec{x})\vec{g}_i$, and segments connect points associated to edge-adjacent meshes.

This defines tilings of the plane with at most $\binom{d}{2}$ different rhombi.
Formally

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**Theorem (de Bruijn, 1986)**

*The dualization of a multigrid is a quasiperiodic rhombus tiling if and only if each grid has a quasiperiodic spacing.*
Can any rhombus tiling be obtained as the dual of some multigrid?
Given a rhombus tiling, draw \textit{pseudolines} in parallel ribbons of tiles.
Given a rhombus tiling, draw *pseudolines* in parallel ribbons of tiles.
This yields a *pseudogrid*. Is it topologically equivalent to a multigrid?
If yes, then the dualization yields back the original rhombus tiling.
Pseudogrids

But this does not always holds (Ringel, 1956 – Grünbaum, 1972).
Pseudogrids

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Let there be light!

Consider a rhombus tiling defined by three grids (here, 3-fold).
Let there be light!

Shadowing $\leadsto$ kind of digital plane of the Euclidean space!
Consider a rhombus tiling where edges can take at most $d$ directions.
Map an arbitrary vertex onto an arbitrary vector of $\mathbb{Z}^d$. 
Modify $\pm 1$ the $k$-th entry when moving along the $k$-th direction.
Rhombus vertices are mapped onto vertices of unit $d$-dim. squares.
The whole tiling is mapped onto a stepped surface of $\mathbb{R}^2$: its *lift.*
Plane tilings

**Definition (Plane tiling)**

A rhombus tiling is said to be *plane* if its lift lies inside a “slice” $V + [0, 1)^d$, where $V$ is an affine plane of $\mathbb{R}^d$.

The plane $\vec{V}$ is sometimes called *physical* or *real* space, while its orthogonal $\vec{V}^\perp$ is called *reciprocal*, *internal* or *perp-* space.

Parameters of $\vec{V}$ are called *slope* or *phason-strain* of the tiling.

**Proposition (Gähler and Rhyner, 1986)**

Plane tilings exactly correspond to uniformly spaced multigrids.
Almost plane tilings

**Definition (Almost plane tiling)**

A rhombus tiling is said to be *almost plane* if its lift lies inside a “slice” $V + [0, t)^d$, where $V$ is an affine plane of $\mathbb{R}^d$ and $t \in \mathbb{R}$.

The smallest possible $t$ is the *thickness* or *fluctuation* of the tiling.
Almost plane tilings

Definition (Almost plane tiling)

A rhombus tiling is said to be almost plane if its lift lies inside a “slice” \( V + [0, t)^d \), where \( V \) is an affine plane of \( \mathbb{R}^d \) and \( t \in \mathbb{R} \).

The \( t = 1 \) case corresponds to plane tilings.
Long-range order of plane tilings yields Bragg peaks.
Almost plane tilings still have this long-range order!
Rhombus tilings are projection of $d$-dim. unit squares (remind lift).
Select three edge directions and emphasize rhombi defined by them.
Shadows

Rotate in $\mathbb{R}^d$ until all the remaining edges orthogonally project.
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Rotate in $\mathbb{R}^d$ until all the remaining edges orthogonally project.
This yields a rhombus tiling, called a *shadow*, whose lift is in $\mathbb{R}^3$. 
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Local rules

Terminology:
- set $T$ of tiles $\mapsto$ set $X_T$ of tilings;
- $r$-pattern of a tiling: tiles lying inside a ball of radius $r > 0$;
- $r$-atlas of $X \subset X_T$: $r$-patterns of tilings in $X$ (up to isometry).

Definition (Local rules)

$X \subset X_T$ admits local rules if it is characterized by a $r$-atlas, $r > 0$.

Dynamical systems terminology:
- $X_T$: fullshift over $T$;
- $X \subset X_T$ translation-invariant and closed: shift;
- $X$ admits local rules $\equiv X$ is a shift of finite type.
Decorated local rules

Terminology:

- Decorated tiling: tiles can be colored, labelled, notched etc.;
- locally derivable from \( \equiv \) image under a local map of.

**Definition (Decorated local rules)**

\( X \subset X_T \) admits *decorated local rules* if it is locally derivable from a set of decorated tilings which admits local rules.

Dynamical systems terminology:

- locally derivable from \( \equiv \) *topological factor* of;
- \( X \) admits decorated local rules \( \equiv \) \( X \) is a *sofic* shift.
One-dimensional examples

Consider the fullshift \( \{a, b\}^\mathbb{Z} \).

Local rules that admit these subshifts?

1. the sequences with no more than 10 consecutive \( b \);
2. the sequences with at most one \( b \)-run;
3. the centro-symmetric sequences;
4. the non-periodic sequences.
Strong and weak local rules

Distinction introduced by Levitov for rhombus tilings:

**Definition (Strong and weak local rules)**

Local rules which define a set of rhombus tilings are said to be
- *strong* if the tilings are all parallel plane tilings;
- *weak* if the tilings are parallel almost plane tilings.

Remind: bounded fluctuations do not destroy long-range order!
One-dimensional examples

Fullshift over \( \{a, b\} \equiv \) one-dimensional rhombus tilings.

Type of these local rules (and subshifts they define)?

1. \( \{aba, bab\} \)
2. \( \{aa, ab, ba\} \)
3. \( \{aabb, abba, bbaa, baab\} \)
4. \( \{a_ia_{i+1}, a_ib_i, b_ia_{i-p}\}_{1 \leq i \leq q} \)
Two-dimensional examples

Consider this decorated rhombus.
Two-dimensional examples

Two rhombi match if they form an arrow on their common edge.
Two-dimensional examples

This allows only one plane tiling $\leadsto$ strong (decorated) rules.
Two-dimensional examples

This allows only one plane tiling $\rightsquigarrow$ strong (decorated) rules.
Two-dimensional examples

This allows only one plane tiling \( \rightsquigarrow \) strong (decorated) rules.
Two-dimensional examples

Consider now this decorated rhombus.
Two-dimensional examples

Matching are free on empty edges, as before on arrowed ones.
Two-dimensional examples

Matching are free on empty edges, as before on arrowed ones.
Two-dimensional examples

Matching are free on empty edges, as before on arrowed ones.
Two-dimensional examples

This allows only small fluctuations on tile ribbons.
Two-dimensional examples

The same thus holds for the whole tiling \( \rightsquigarrow \) weak decorated rules.
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Shifting the cut of a fully periodic shadow

Consider a plane tiling obtained by a rational cut in $\mathbb{R}^3$. 
Shifting the cut of a fully periodic shadow

Shifting (in $\mathbb{R}^3$) the cut just shifts (in $\mathbb{R}^2$) the tiling.
Shifting the cut of a fully periodic shadow

This corresponds to local rearrangements (*flip*) on a 2-dim. lattice.
Consider a plane tiling obtained by an irrational cut in $\mathbb{R}^3$. 
Shifting the cut of a non-periodic shadow

Shifting the cut modifies the tiling but not the finite patterns.
Shifting the cut of a non-periodic shadow

Modifications are quasiperiodically spaced flips.
Shifting the cut of a non-periodic shadow

The smaller is the shift, the sparser are these flips.
Shifting the cut of a non-periodic shadow

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Shifting the cut of a non-periodic shadow

Removing a single flip increases the thickness $\sim$ non-plane tiling.
Shifting the cut of a non-periodic shadow

To forbid this, strong rules should be larger than the flip-spacing...
Shifting the cut of a semi-periodic shadow

Consider now the intermediary case.
Shifting the cut of a semi-periodic shadow

Shifting the cut modifies the tiling but not the finite patterns.
Shifting the cut of a semi-periodic shadow

Modifications are quasiperiodically spaced periodic lines of flips.
Shifting the cut of a semi-periodic shadow

The smaller is the shift, the sparser are these lines of flips.
Shifting the cut of a semi-periodic shadow

For similar reasons, this is incompatible with strong rules.
Theorem (Levitov, 1988)

If a rhombus tiling has strong rules, then its shadows are periodic.

Proof:
Assume that there are non-periodic shadows and strong rules.

1. by a sufficiently small shift on the cut (in $\mathbb{R}^n$):
   - fully periodic shadows are unchanged (for a suitable shift);
   - flips in non-periodic shadows are at dist. $\geq R$ from each other;
   - flip lines in semi-periodic shadows are sufficiently spaced to be at dist. $\geq R$, in the tiling, of a flip of non-periodic shadows.

2. show that there is $k$ indep. from $R$ s.t. each diameter $R$ ball in the tiling contains at most $k$ flips of non-periodic shadows;

3. deduce that strong rules should have diameter $\frac{R}{2k}$, for any $R$. 
Periodic shadows yield \(\{3, 4, 5, 6, 8, 10, 12\}\)-fold tilings

\[ n\text{-fold tiling: plane tiling of slope } \mathbb{R}(u_1, \ldots, u_n) + \mathbb{R}(v_1, \ldots, v_n), \]

\[ u_k = \cos \left( \frac{2k\pi}{n} \right) \quad \text{and} \quad v_k = \sin \left( \frac{2k\pi}{n} \right). \]

Periodicity of shadows yields \(\cos(2\pi/n) \in \mathbb{Q}(\sqrt{D})\). Possible cases:

- \(\cos(2\pi/n) \in \mathbb{Q}\) if \(n = 3, 4, 6\)
- \(\cos(2\pi/n) \in \mathbb{Q}(\sqrt{2})\) if \(n = 8\)
- \(\cos(2\pi/n) \in \mathbb{Q}(\sqrt{3})\) if \(n = 12\)
- \(\cos(2\pi/n) \in \mathbb{Q}(\sqrt{5})\) if \(n = 5, 10\)
Periodic shadows yield \{3, 4, 5, 6, 8, 10, 12\}-fold tilings

\( n \)-fold tiling: plane tiling of slope \( \mathbb{R}(u_1, \ldots, u_n) + \mathbb{R}(v_1, \ldots, v_n) \),

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- \( \cos(2\pi/n) \in \mathbb{Q}(\sqrt{5}) \) if \( n = 5, 10 \)

These symmetries are exactly those yet experimentally observed!
Sufficient condition for weak rules

The \( ijk \)-shadow of a plane tiling of slope \( \mathbb{R} \mathbf{u} + \mathbb{R} \mathbf{v} \) is periodic iff:

\[
\exists \mathbf{p}_{ijk} \in \mathbb{Z}^3 \setminus \{0\}, \quad \det(\mathbf{u}_{ijk}, \mathbf{v}_{ijk}, \mathbf{p}_{ijk}) = (\mathbf{u}_{ijk} \land \mathbf{v}_{ijk}).\mathbf{p}_{ijk} = 0.
\]

This can be seen as an equation for three entries of \( \mathbf{u} \) and \( \mathbf{v} \).
The $ijk$-shadow of a plane tiling of slope $\mathbb{R}\vec{u} + \mathbb{R}\vec{v}$ is periodic iff:

$$\exists \vec{p}_{ijk} \in \mathbb{Z}^3 \setminus \{\vec{0}\}, \quad \det(\vec{u}_{ijk}, \vec{v}_{ijk}, \vec{p}_{ijk}) = (\vec{u}_{ijk} \wedge \vec{v}_{ijk}) \cdot \vec{p}_{ijk} = 0.$$  

This can be seen as an equation for three entries of $\vec{u}$ and $\vec{v}$.

**Theorem (Levitov-Socolar mix)**

*If periodic shadows of a plane tiling yield equations characterizing its slope, then this tiling does admit weak rules.*

**Proof:**

- the periodicity of a shadow can be enforced by local rules;
- the hypothesis ensure that this characterizes the tiling slope;
- no control on the intertwining of shadows $\rightsquigarrow$ only weak rules.
## Further results

<table>
<thead>
<tr>
<th>Tiling</th>
<th>undecorated rules</th>
<th>decorated rules</th>
</tr>
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<tbody>
<tr>
<td>5, 10-fold</td>
<td>strong</td>
<td>strong(^1)</td>
</tr>
<tr>
<td>8-fold</td>
<td>none(^2)</td>
<td>strong(^3)</td>
</tr>
<tr>
<td>12-fold</td>
<td>none(^3)</td>
<td>strong(^4)</td>
</tr>
<tr>
<td>((4 \not</td>
<td>n))-fold</td>
<td>weak(^5)</td>
</tr>
<tr>
<td>quadratic slope in (\mathbb{R}^4)</td>
<td>a.e. weak(^6)</td>
<td>strong(^7)</td>
</tr>
<tr>
<td>non-algebraic slope</td>
<td>none(^8)</td>
<td>?</td>
</tr>
</tbody>
</table>

(1): Penrose, 1974  
(2): Burkov, 1988  
(3): Le, 1992  
(4): Socolar, 1989  
(5): Socolar, 1990  
(6): Levitov, 1988  
(7): Le et al., 1992  
(8): Le, 1997
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<td>none&lt;sup&gt;3&lt;/sup&gt;</td>
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</tr>
<tr>
<td>(4 ∤ n)-fold</td>
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<td>strong?</td>
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### Conjecture

A plane tiling admits decorated rules iff its slope is computable.
Some references for this lecture:


These slides and the above references can be found there:

http://www.lif.univ-mrs.fr/~fernique/qc/