Exercises – session II

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Exercice 1 : Cellular automata and percolation.

In this exercise we consider only cellular automata of dimension 2 with states $\mathcal{A} = \{0, 1\}$, neighborhood $\mathbb{U} = \{(0,0), (1,0), (0,1), (-1,0), (0,-1)\}$ and having the property : $x(\mathbf{i}) = 1 \Rightarrow F(x)(\mathbf{i}) = 1$. For such a cellular automaton F, we are interested in the set

$$X_F = \{x \in \mathcal{A}^{\mathbb{Z}^2} : \forall \mathbf{i}, \exists t, F^t(x)(\mathbf{i}) = 1\}$$

and its measure $\mu_p(X_F)$ where μ_p is the product measure (Bernoulli measure) where 1 has probability p and 0 probability $1 - p$.

Question 1. *Consider F such that a 0 becomes 1 if there is at least one 1 in its neighborhood.*
For which p do we have $\mu_p(X_F) = 1$?

Question 2. *A cellular automaton admits a finite obstacle if there is a finite pattern of 0s such that, whatever the context, these 0 stays 0 forever.*
Show that if F admits a finite obstacle then $\mu_p(X_F) = 0$ for any p.

Question 3. *Consider F such that a 0 becomes 1 if and only if there is at least three 1 in its neighborhood.*
For which p do we have $\mu_p(X_F) = 1$?

Now we consider the cellular automaton G such that a 0 becomes 1 if and only if there is at least two 1 in its neighborhood.

Question 4. *Does G admits a finite obstacle ?*

Question 5. *Suppose you have a $n \times m$ rectangle of 1s in a configuration. What property of the rest of the configuration is necessary and sufficient to prevent this rectangle from growing and invading all the configuration ? Show that if we suppose n and m large enough, the probability of having this property is strictly less than 1.*

Question 6. *Conclude about $\mu_p(X_F)$.*

Exercice 2 : Randomization with XOR cellular automaton.

In this exercise we consider (again) the cellular automaton defined by :

$$d = 1, \mathcal{A} = \{0, 1\}, \mathbb{U} = \{-1, 1\}, f(a, b) = a + b \bmod 2$$

and therefore $F(x)(\mathbf{i}) = x(\mathbf{i} - 1) + x(\mathbf{i} + 1) \bmod 2$

Take any $0 < p < 1$ and consider the Bernoulli measure μ_p that choose 1s with probability p and 0s with probability $1 - p$.

We are interested in the sequence $\left(F^t(\mu_p)\right)_{t \in \mathbb{N}}$.

Question 7. *What is the value of $F^{2^n}(\mu_p)([1]_0)$ (the probability to have a 1 after 2^n steps at position 0) ?*

Question 8. *$\mu_{\frac{1}{2}}$ is the uniform probability measure (probability $1/2$ for 1s and $1/2$ for 0s). Can we have $\lim_t F^t(\mu_p) = \mu_{\frac{1}{2}}$?*

Question 9. *For any $\epsilon > 0$, find some n such that*

$$F^n(\mu_p)([1]_0) \in [\frac{1}{2} - \epsilon; \frac{1}{2} + \epsilon]$$

(probability to have 1 at position 0 at step n is close to $1/2$).

Question 10. *Same as the previous question but instead of considering the cylinder $[1]$, do it for any cylinder $[u]$ and show :*

$$F^n(\mu_p)([u]_0) \in [\frac{1}{2^{|u|}} - \epsilon; \frac{1}{2^{|u|}} + \epsilon]$$

Exercice 3 : CA and subshifts.

Here we consider any CA in any dimension if not specified otherwise.

Question 11. *Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a subshift of finite type and F a cellular automaton. Show that $F^{-1}(X)$ is a subshift of finite type.*

Question 12. *Show that for any $N \geq 0$ the set of configuration $F^t(\mathcal{A}^{\mathbb{Z}^d})$ is a sofic subshift.*

Question 13. *In this question we focus on dimension 1. The limit set of a cellular automaton is $\Omega_F = \cap_{t \geq 0} F^t\left(\mathcal{A}^{\mathbb{Z}^d}\right)$. Construct a cellular automaton such that Ω_F is not sofic.*

Hint. If the alphabet contains symbols M,B,→,←,T, and if L is the language of Ω_F, and if you can show that

$$L \cap \{MB^i \to T \leftarrow B^j M : i,j \in \mathbb{N}\} = \{MB^n \to T \leftarrow B^n M : n \in \mathbb{N}\}$$

then Ω_F is not sofic.

Exercice 4 : The firing squad synchronization problem.

The goal of this exercise is to find a cellular automaton F that solves the following synchronization problem.

The (stupid) military analogy is the following : for some (stupid) reason, a line of soldiers is ready to fire and waits the order of the general. But the line is so long that the soldier at the end of the line can not here him. Moreover for another (stupid) reason, the general wants that all soldier fire exactly at the same time. Can you help the (stupid) soldiers to solve this problem with a simple method that works whatever the length of the line ?

Formally, the dimension is 1 and we suppose that the alphabet contains the symbols : G (general),S (soldier), E (end), F (fire) and B (blank). We want that, for any $n \geq 2$, if we start from the configuration

$$\cdots BBB\underbrace{GS\cdots SE}_{n-2}BBB\cdots$$

then we reach at some time t_n the configuration

$$\cdots BBB\underbrace{F\cdots F}_{n}BBB\cdots$$

with the constraint that the state F never appears before time t_n.
Hint. Starts with n a power of two.