Exercises -- session I

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Exercice 1 : The XOR cellular automaton.

In this exercise we consider the cellular automaton defined by:

\[ d = 1, \mathcal{A} = \{0, 1\}, \mathcal{U} = \{-1, 1\}, f(a, b) = a + b \mod 2 \]

and therefore

\[ F(x)(i) = x(i - 1) + x(i + 1) \mod 2 \]

**Question 1.** Define \( x \in \mathcal{A}^\mathbb{Z} \) by

\[ x(i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{else.} \end{cases} \]

Compute the space-time diagram of \( F \) up to time 8 starting from \( x \).

**Question 2.** Consider any configuration \( x \). \( F^{2^n}(x) = ? \)

**Question 3.** Let us define the group law \( \oplus \) on \( \mathcal{A}^\mathbb{Z} \) by

\[ x \oplus y = i \mapsto x(i) + y(i) \mod 2 \]

\( F^n(x \oplus y) = ? \)

**Question 4.** A cellular automaton \( G \) on \( \mathcal{A}^\mathbb{Z} \) is linear if \( G(x \oplus y) = G(x) \oplus G(y) \). Let \( \Lambda \) be the set of linear CA over \( \mathcal{A}^\mathbb{Z} \) and \( \oplus \) is the natural extension of \( \oplus \) to maps:

\[ (F \oplus G)(x) = F(x) \oplus G(x) \]

1. Show that \( H \circ (F \oplus G) = H \circ F \oplus H \circ G \) for any \( F, G, H \in \Lambda \), where \( \circ \) is the composition of maps.

2. Quickly check that \( (\Lambda, \oplus, \circ) \) is a ring.

**Question 5.** Give an explicit formula for \( F^n(x) \), for any \( n \), using only the \( x(i) \) for \( i \in \mathbb{Z} \).

Hint. Show that \( F = \mathcal{S}_L \oplus \mathcal{S}_R \) where \( \mathcal{S}_L \) (resp. \( \mathcal{S}_R \)) is the translation to the left (resp. right).

**Question 6.** We say that \( F^n(x)(0) \) “depends on position \( i \)” if the value \( F^n(x)(0) \) changes when you change \( x \) at position \( i \). How many positions \( F^n(x)(0) \) depends on ?

Hint. Use Lucas Lemma (or, of your favorite professor, use the force)

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Exercice 2 : Positive expansiveness.

Let’s define the trace map \( T_F : \mathcal{A}^\mathbb{Z} \to (\mathcal{A}^\mathbb{Z})^\mathbb{N} \) by:

\[ T_F(x) = (x_0, x_1), (F(x)_0, F(x)_1), \ldots, (F^n(x)_0, F^n(x)_1), \ldots \]
Question 7. Show that \( T_F(x) = (0, 0), (0, 0), (0, 0), \ldots \) if and only if \( x(i) = 0 \) for all \( i \).

Question 8. Show that \( x \neq y \) implies \( T_F(x) \neq T_F(y) \).

Question 9. Consider the Cantor metric on \( A^\mathbb{Z} \), i.e. the distance between \( x \) and \( y \) is
\[
2^{-\min\{|i|: x(i) \neq y(i)\}}
\]
Take two configurations \( x \neq y \) that are very close, e.g. distant from less than \( \frac{1}{1000000} \). Is it possible that \( F^n(x) \) and \( F^n(y) \) are also very close for all \( n \)?

Question 10. Show that \( T_F(A^\mathbb{Z}) = (A^2)^\mathbb{N} \).

Exercise 3: Second order shift.

In this exercise we consider the CA defined by:
\[
d = 1, \ A = \{0, 1\} \times \{0, 1\}, \ U = \{-1, 0\}, \ f(a, b) = (\pi_1(a) + \pi_2(b) \mod 2, \pi_1(b))
\]
and therefore
\[
F(x)(i) = (\pi_1(x(i - 1)) + \pi_2(x(i)) \mod 2, \pi_1(x(i)))
\]
where \( \pi_1 \) and \( \pi_2 \) denote the projections on the first and second component of \( A \).

Question 11. Show that \( F \) is reversible.

Question 12. Give an expression of \( F^{2n+1} \) using \( F^{2n} \), the shift maps and the identity map.