Exercises -- session I

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Exercice 1 : The XOR cellular automaton.

In this exercise we consider the cellular automaton defined by :

$$d = 1, \mathcal{A} = \{0,1\}, \mathbb{U} = \{-1,1\}, f(a,b) = a + b \bmod 2$$

and therefore

$$F(x)(\mathbf{i}) = x(\mathbf{i}-1) + x(\mathbf{i}+1) \bmod 2$$

Question 1. *Define $x \in \mathcal{A}^{\mathbb{Z}^d}$ by*

$$x(\mathbf{i}) = \begin{cases} 1 & \text{if } \mathbf{i} = 0 \\ 0 & \text{else.} \end{cases}$$

Compute the space-time diagram of F up to time 8 starting from x.

Question 2. *Consider any configuration x. $F^{2^n}(x) =?$*

Question 3. *Let us define the group law $\oplus$ on $\mathcal{A}^{\mathbb{Z}}$ by*

$$x \oplus y = \mathbf{i} \mapsto x(\mathbf{i}) + y(\mathbf{i}) \bmod 2$$

$F^n(x \oplus y) =?$

Question 4. *A cellular automaton G on $\mathcal{A}^{\mathbb{Z}}$ is* linear *if $G(x \oplus y) = G(x) \oplus G(y)$. Let Λ be the set of linear CA over $\mathcal{A}^{\mathbb{Z}}$ and $\overline{\oplus}$ is the natural extension of $\oplus$ to maps :*

$$(F \mathbin{\overline{\oplus}} G)(x) = F(x) \oplus G(x)$$

1. *Show that $H \circ (F \mathbin{\overline{\oplus}} G) = H \circ F \mathbin{\overline{\oplus}} H \circ G$ for any $F, G, H \in \Lambda$, where $\circ$ is the composition of maps.*
2. *Quickly check that $\left(\Lambda, \overline{\oplus}, \circ\right)$ is a ring.*

Question 5. *Give an explicit formula for $F^n(x)$, for any n, using only the $x(\mathbf{i})$ for $\mathbf{i} \in \mathbb{Z}$.*

Hint. Show that $F = \mathfrak{S}_L \mathbin{\overline{\oplus}} \mathfrak{S}_R$ where $\mathfrak{S}_L$ (resp. $\mathfrak{S}_R$) is the translation to the left (resp. right).

Question 6. *We say that $F^n(x)(0)$ "depends on position $\mathbf{i}$" if the value $F^n(x)(0)$ changes when you change x at position $\mathbf{i}$. How many positions $F^n(x)(0)$ depends on ?*

Hint. Use Lucas Lemma (or, of your favorite professor, use the force)

Exercice 2 : Positive expansiveness.

Let's define the *trace map* $T_F : \mathcal{A}^{\mathbb{Z}} \to (\mathcal{A}^2)^{\mathbb{N}}$ by :

$$T_F(x) = (x_0, x_1), (F(x)_0, F(x)_1), \ldots, (F^n(x)_0, F^n(x)_1), \ldots$$

Question 7. *Show that $T_F(x) = (0,0),(0,0),(0,0),\ldots$ if and only if $x(\mathbf{i}) = 0$ for all $\mathbf{i}$.*

Question 8. *Show that $x \neq y$ implies $T_F(x) \neq T_F(y)$.*

Question 9. *Consider the Cantor metric on $\mathcal{A}^{\mathbb{Z}}$, i.e. the distance between x and y is*

$$2^{-min\{|\mathbf{i}| : x(\mathbf{i}) \neq y(\mathbf{i})\}}$$

Take two configurations $x \neq y$ that are very close, e.g. distant from less than $\frac{1}{1000000}$. Is it possible that $F^n(x)$ and $F^n(y)$ are also very close for all n ?

Question 10. *Show that $T_F\big(\mathcal{A}^{\mathbb{Z}}\big) = (\mathcal{A}^2)^{\mathbb{N}}$.*

Exercice 3 : Second order shift.

In this exercise we consider the CA defined by :

$$d = 1, \mathcal{A} = \{0,1\} \times \{0,1\}, \mathbb{U} = \{-1,0\}, f(a,b) = \big(\pi_1(a) + \pi_2(b) \bmod 2, \pi_1(b)\big)$$

and therefore

$$F(x)(\mathbf{i}) = \big(\pi_1(x(\mathbf{i}-1)) + \pi_2(x(\mathbf{i})) \bmod 2, \pi_1(x(\mathbf{i}))\big)$$

where π_1 and π_2 denote the projections on the first and second component of $\mathcal{A}$.

Question 11. *Show that F is reversible.*

Question 12. *Give an expression of $F^{2^{n+1}}$ using F^{2^n}, the shift maps and the identity map.*