Tileability I
CIPMA Research School – Tiling and Tessellations – August 24 – September 4 2015 – Isfahan, IRAN

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Tiling a Square with Similar Rectangles

Exercise 1 (Determination of the ration)

a) In order to tile a square with rectangles similar to $1 \times x$ (up to rotation) as described in the figure below, which ratio x can you choose?

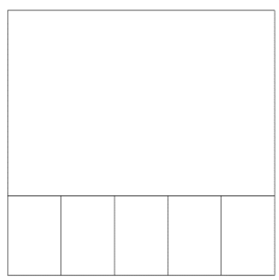


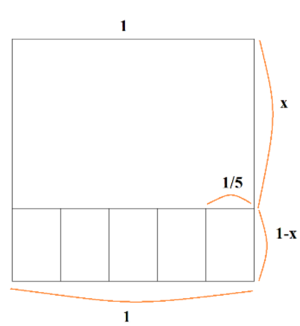
Figure 0

Solution:

"Similar rectangles" are rectangles with the same "width to length ratio" which can be also be stated as "if we have rectangles $a \times b$ and $c \times d$, they are similar if and only if $\frac{a}{b} = \frac{c}{d}$ or $\frac{a}{b} = \frac{d}{c}$"

The big bounding square, in Figure 0, is 1×1. The sum of the widths (the shorter edge of the rectangles) of bottom rectangles has to be 1.

There are 5 of them so each should have a width of $\frac{1}{5}$. And look at the picture below for more information:



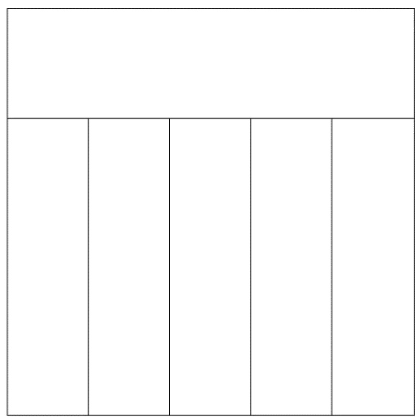
The necessary and sufficient condition for rectangles $(1/5) \times (1-x)$ and $x \times 1$ to be similar is $\frac{\frac{1}{5}}{1-x} = \frac{x}{1}$ (because we know visually that width of rectangles are $1/5$ and x respectively).

Then the desired x are the solutions of $x \times (1-x) = 1/5 \times 1$ which in fact is $x^2 - x + 1/5 = 0$ and its solutions are $x_\pm = \frac{1 \pm \sqrt{1/5}}{2} = \frac{5 \pm \sqrt{5}}{10}$.

As it is clear visually from Figure 0, the x is greater than $1/2$ so x in Figure 1, is in fact $x_+ = \frac{5+\sqrt{5}}{10}$.

b) What happens now if you choose $x = \frac{5-\sqrt{5}}{10}$?

For the case $x = x_- = \frac{5-\sqrt{5}}{10}$, following picture gives the right visual intuition with the upper rectangle being $x \times 1$:



Simple Proofs of a Rectangle Tiling Theorem

Theorem 1: *If a finite number of rectangles, every one of which has at least one* integer *side, perfectly tile a big rectangle, them the big rectangle also has at least one* integer *side*.

Fourteen proofs of theorem 1 were published by Wagon:
Wagon, S. (1987) Fourteen proofs of a result about tiling a rectangle. The American Mathematical Monthly 94 (7) : 601-617

In the following we establish two proofs of this theorem. These proofs generalize to other situations.

In particular Theorem 1 generalizes to the two following statements.

Theorem 2: *If a finite number of rectangles, every one of which has at least one* rational *side, perfectly tile a big rectangle, them the big rectangle also has at least one* rational *side*.

Theorem 3: *If a finite number of rectangles, every one of which has at least one* algebra *side, perfectly tile a big rectangle, them the big rectangle also has at least one* algebra *side*.

➢ Exercise 2 (Rational sides vs. integer sides)

In order to prove the second theorem why is it enough to prove the first one?

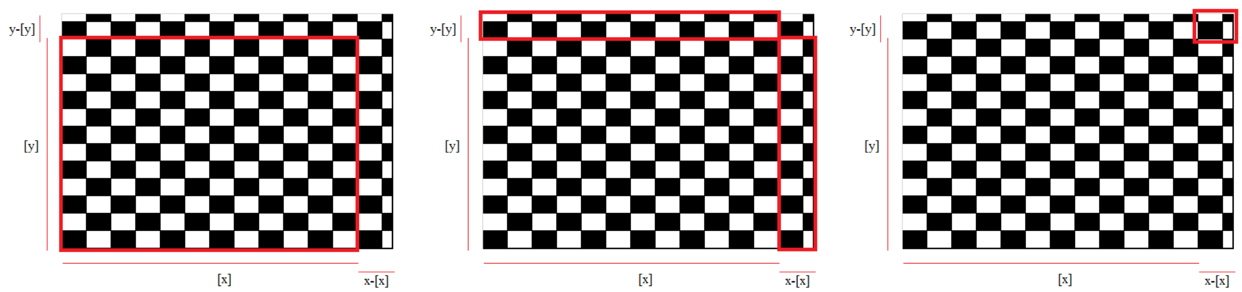
Solution: We are tiling the big (bounding rectangle) by finitely many small rectangles. So we can find the lcm (least common multiple) of all the dominators of all the rational numbers and scale the whole picture by that factor so all the rational numbers scale to integers. This way Theorem 2 is reduced to Theorem 1 (they are equivalent for finite tilings).

➢ Exercise 3 (Chessboard proof of Theorem 1)

Take the big rectangle and align its bottom left corner with a half-integer chessboard, that is, a chessboard whose squares have side $1/2$. Let the bottom left corner be black.

1. Prove that if the upper corner of the big rectangle is at (x, y) the excess black area is
$$|x - r(x)|.|y - r(y)|$$
where $r(x)$ denotes the integer nearest to x.

Solution: Define $[a] \coloneqq \{greatest\ lower\ bound\ of\ all\ integers\ less\ than\ or\ equal\ to\ a\} = floor[a]$. There are 3 types of regions in general as depicted below:

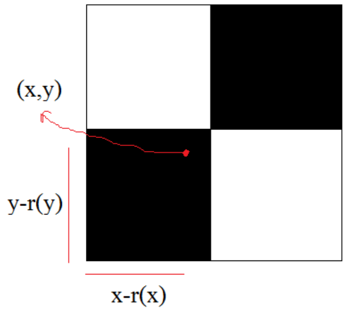


(a) (b) (c)

a) Region bounded by integer-side red rectangle. It is a rectangular chessboard with even number of columns and rows (since small cells have half-integer sides) so the total number of black and white cells in this region is equal and there is no excess black area.
b) Region bounded by red rectangle with one integer side. For example for the upper (horizontal red rectangle in the picture) since the length (horizontal dimension) is an integer, we have even number of column and if we consider them in pairs starting from left side, we can see that each 2-column-pairs has excess black area equal to zero. And it shows that (b) does not contribute to excess black areas.
c) The only region in which there is a possibility of having difference between black and white area, is (c)

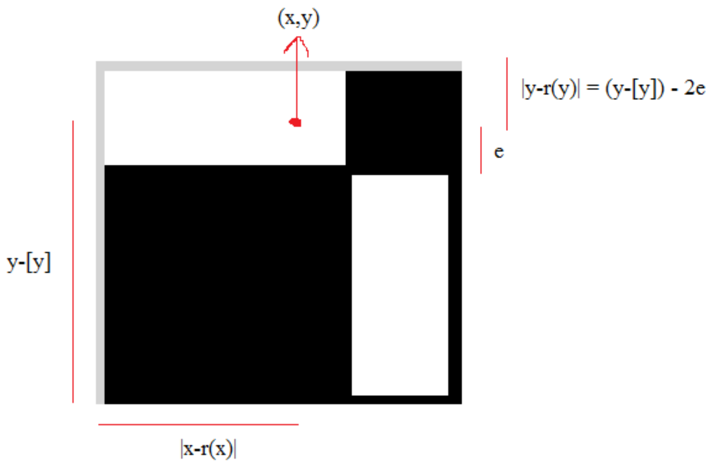
Apart from marginal cases (where x or y are integers or half-integers) there are 3 cases for (c):

I. $x - [x] < \frac{1}{2}$ *and* $y - [y] < 1/2$: so $r(x) = [x]$ and also it is the case for y.



So the excess black area is just calculated as stated.

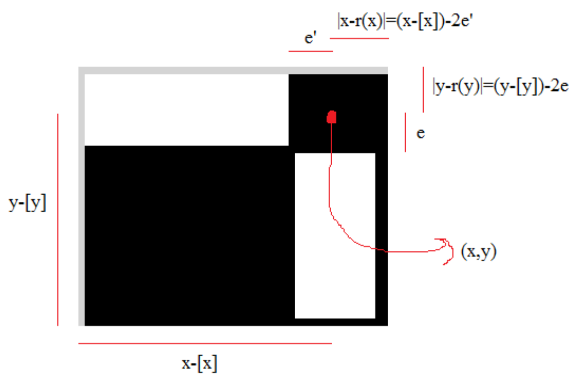
II. $x - [x] < \frac{1}{2}$ *and* $y - [y] > 1/2$ (or just exchange the role of x and y): so $r(x) = [x]$ and $r(y) = [y] + 1$.



So the excess black area should be calculated by calculating the black area minus the white area which is: $Excess = (y - [y]) \times |x - r(x)| - 2e \times |x - r(x)| = (y - [y] - 2e) \times |x - r(x)| = |y - r(y)| \times |x - r(x)|$

III. $x - [x] > \frac{1}{2}$ *and* $y - [y] > \frac{1}{2}$: so $r(x) = [x] + 1$ and it's the same case for y.

This case is just the same as the above one and you can derive it yourself easily by adding black areas and subtracting white areas.



$$Excess: (y - [y]) \times (x - [x]) - 2e \times (x - [x]) - 2e' \times (y - [y]) + 2e \times 2e'$$

$$= ((y - [y]) - 2e) \times ((x - [x]) - 2e') = |x - r(x)| \times |y - r(y)|$$

2. Conclude that if the big rectangle can be tiled by a finite number of rectangles, every one of which has at least one integer side, the big rectangle must have an integer side.

Solution: If $|x - r(x)|.|y - r(y)| = 0$ then $x = r(x)$ or $y = r(y)$ which means that at least one of $x\ or\ y$ is an integer .If a rectangle has one integer side then there is no excess black (or white) area (with respect to the way we defined the chessboard). If we put some rectangles together where there is no excess black area in each, the resulting shape (regardless of the shape) will not have any excess black area. Now the statement of theorem is that "we put some rectangles which have at least one integer side, to make a big rectangle" so the big rectangle has no excess black area (if we choose $(x, y) = coordinates\ of\ the\ top - right\ vertex\ of\ big\ rectangle$ and $|x - r(x)|.|y - r(y)| = 0$) which concludes the theorem.

➢ Exercise 4 (Proof of Theorems 1, 2 and 3 and walks in a graph)

In the three theorems, at least one of the big rectangle's sides must have a special property. More precisely depending whether we wish to prove Theorem 1, 2 or 3, we will call a number special if it is integer, rational or algebraic, respectively. We define a point (x, y) in the plane to be special if both coordinates x and y are special.

We associate with each small rectangle four vertices and two edges (see Figure 1). The four vertices are the four vertices are the four corners of the rectangle. The two edges are two parallel sides of the rectangle that are both special in length. (Every small rectangle has two such special edges, by the statement of the theorem.)

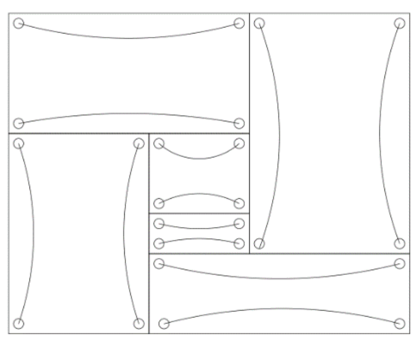


FIGURE 1 – Vertices and edges associated with rectangles

1. Prove that if a vertex is special, then its neighboring vertex is also special.

Solution: We suppose that "neighboring vertex" is just in the graph theory sense which means "there is a special edge between two vertices". According to the definition of being special (being integer, rational and algebraic), since integer numbers, rational numbers and algebraic numbers are close under summation (or subtraction) and moving from one vertex along a special edge to another vertex means "add/subtract two special numbers" so whenever we move from one special vertex along a special edge, we will get to another vertex which is also special.

We now define a graph by identifying all the vertices that have identical coordinates. In this graph there may be double edges connecting two vertices. Such double edges are not merged into single edges.

2. Prove that apart from the four vertices at the corners of the big rectangle, which have degree 1, all other vertices have degree two or four.

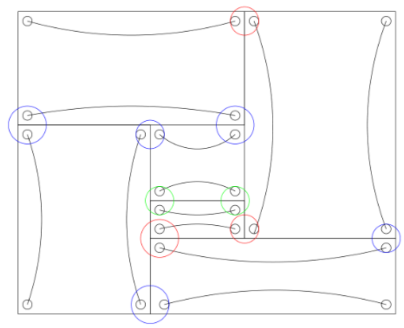
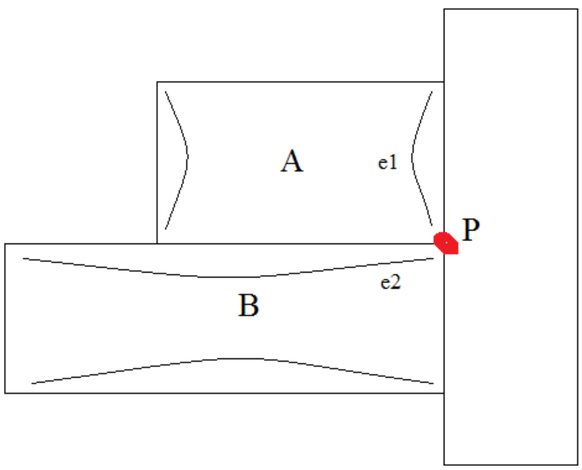
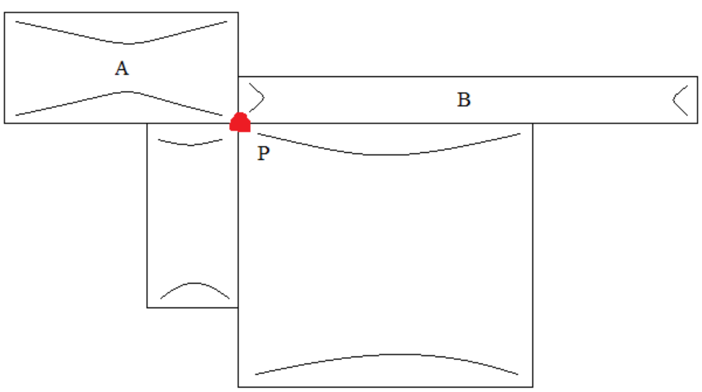


FIGURE 2 – The graph associated with rectangles

Solution: Considering rectangles in the plane, there are only 2 possibilities for two rectangles to be adjacent (have common edge) and have an edge in common (rectangles A and B are assumed adjacent at point P):



I. At vertex P, two edges are being identified so the number of edges at P is 2.



II. At vertex P, four edges are being identified so the number of edges at P is 4.

3. Aligning the coordinate system such that the bottom left vertex of the big rectangle is at the origin, prove that at least one side that is special in length.

Solution: Except at big rectangle corners (call one of the big rectangle's corners P), all other vertices have even degree. Starting from P, walk through the graph along edges and pass no edge more than once. Since the only edges with odd degree are the corners of the big rectangle (and number of edges is finite), we will get stuck at a point, call it T, after finite steps (no edges of T are left which have not been passed so we cannot go to another vertex from T). Since all vertices that have been passed, even edges of them has been traversed (except P which had only one vertex), when we get stuck at vertex T, it means that the degree of T must be odd. The proof is complete, since the only vertices with odd degree are the corners of the big rectangle (so T is a corner of the big rectangle and it cannot be P because P had only one edge which has traversed in the first step).

➢ Exercise 5 (Proof of Theorems1, 2 and 3 and higher dimensions)

Both proofs can be applied to the analogous problem in higher dimensions where each small hyperrectangle has one special dimension.

1. How can you generalize the chessboard proof? Then what is the excess black volume of a hyperrectangle that has one corner at the origin and one at x?

Solution: Before proceeding to generalize the previous proof, let's try to understand it a bit differently. If we keep the x coordinate constant and change the y coordinate, the contribution from vertical dimension is just $|y - r(y)|$. It means regardless of x, for calculating excess black area, vertical side is (after subtracting the possible white areas) $|y - r(y)|$. So for an n dimensional version, we just have to think of each dimension separately. So the formula below is concluded:

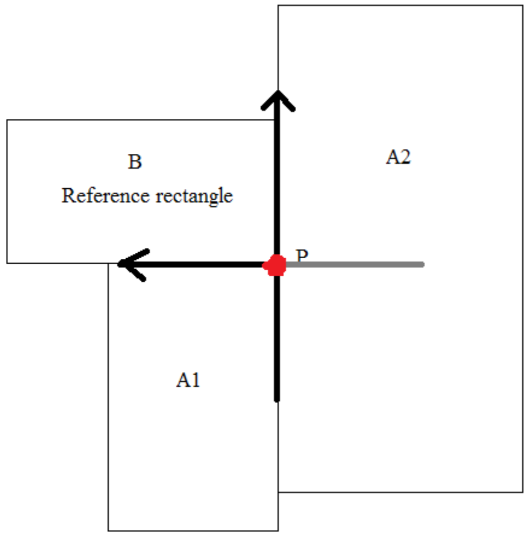
$$Excess\ n - dimensional\ black\ volume = |x_1 - r(x_1)| \times |x_2 - r(x_2)| \times \ldots \times |x_n - r(x_n)|$$

2. How do you define the graph associated with each hyperrectangle in higher dimension?

Solution: Before trying to solve the question, let's consider an N-dimensional hyperrectangle and recall some basic properties. A rectangle in plane is defined by 2 pairs of parallel lines (hyperplanes of $\mathbb{R}^2$), normal of each pair being parallel to one of the coordinate axes. Hyperrectangle is defined by d pairs of parallel hyperplanes, normal of each pair is parallel to one of the coordinate axes. You can see than each hyperrectangle has a total of 2^d vertices and 2^{d-1} edges in each hyperplane (defining the hyperrectangle).

By looking from another point of view, we can generalize the argument for the plane case.

Consider the two dimensional case: We associate a coordinate system to the vertex of intersection, e.g. one in the picture. Note that at least one rectangle (denoted by 'B') has a vertex at the intersection point, otherwise it was not a vertex of the final graph.



We can code each rectangle passing point P according to any arbitrary coordinate chosen, but we normalize this notation by choosing coordinate basis in a way that B lies in the first quadrant (all points belonging to it have all their coordinates positive). By this convention code the rectangles by 2-tuples and the axes they cover according to the rules below:
if a positive (negative) side of an axes is contained in the rectangle, we put + (-) in that coordinate.
if an axes (both positive side and negative side) is contained in the rectangle, then we put 0 in that coordinates.

Here are the codes for rectangles in the picture:

$$B = (+,+) \quad , \quad A1 = (+,-) \quad , \quad A2 = (-,0)$$

According to the convention above we should have following equations in each vertex (“vol(rectangle)” means “number of coordinate regions it covers” or “how many unit squares it covers at intersection points”)

$$\sum_{all\ rectangles\ R\ at\ P} vol(R) = 2^d \quad where\ d\ is\ the\ dimension\ of\ the\ space \qquad (*)$$

Volume of each rectangle is just calculated with the formula $vol(R) = 2^{count\ of\ 0s\ in\ its\ code}$. It is not hard to see that function $vol: Rectangles \to \mathbb{N}$ just calculates the “ordinary volume ($\mathbb{R}^n$) of intersection of Rectanlge with a $2 \times 2 \times \ldots \times 2$ Cube centered at P”.

According to $(*)$ and the formula for volume, if we remove a rectangle with volume greater than 1, the sum of the remaining volumes is still even. More clearly stated:

$$V = \sum_{rectangles\ R\ without\ zeros\ in\ their\ code} vol(R) = 2^d - \sum_{rectangles\ Q\ with\ zeros\ in\ their\ code} vol(Q)$$

The only thing which we care about in the above equality is that $V\ is\ even$. And also V only counts the vol of rectangles with only+ or – in their code which means that they have a corner at P. Let’s apply this to the dummy case we had in 2-dimensions:

$$vol(B) = 1, vol(A1) = 1, vol(A2) = 2 \Rightarrow \sum_{R\ is\ rectangle\ at\ P} vol(R) = 1 + 1 + 2 = 4 = 2^2$$

$$V = vol(B) + vol(A1) = 2^2 - vol(A2) = 2 \quad and \quad \#rectangles\ with\ vertex\ at\ P = 2$$

This kind of reasoning was independent of the dimension of the space (it means the conclusion is independent of "d"). So in any dimension, $V = \#number\ of\ vertices\ to\ be\ identified\ at\ P$ is even.