In this exercise, we focus on the calculation of Gröbner bases over fields and also over integers.

**Question 1.** Let us consider the following system:

\[
\begin{align*}
    x^2 + y^2 + z^2 &= 12 \\
    xyz &= 8 \\
    x^2 - y^2 - z^2 &= -4.
\end{align*}
\]

The aim of this exercise is to solve this system. Suppose that \( \mathbb{R} = \mathbb{K}[x, y, z] \) where \( \mathbb{K} \) is a field and \( z \prec_{\text{lex}} y \prec_{\text{lex}} x \). Let \( I = \langle x^2 + y^2 + z^2 - 12, xyz - 8, x^2 - y^2 - z^2 + 4 \rangle \subset \mathbb{R} \).

(a) Using Buchberger’s algorithm, find a Gröbner basis for this ideal.

(b) Let \( G = \{g_1, \ldots, g_k\} \) be a Gröbner basis for \( I \). Then, we know that the set of solutions of the above system is equal to the set of solutions of the system \( g_1 = \cdots = g_k = 0 \). Using this fact find all solutions of the above system.

(c) Find a remainder of the polynomial \( y^2 + z^2 - 8 \) on division by \( \{x^2 + y^2 + z^2 - 12, xyz - 8, x^2 - y^2 - z^2 + 4\} \).

(d) Does the polynomial \( y^2 + z^2 - 8 \) belong to \( I \)?

**Question 2.** In this exercise we train how to compute Gröbner bases over integers. In doing so, let \( R = \mathbb{Z}[x, y] \) and \( y \prec_{\text{lex}} x \). Let \( f_1 = 3x^2y + 7y \) and \( f_2 = 4xy^2 - 5x \). So, we follow Buchberger’s algorithm. We set \( G = \{f_1, f_2\} \) and consider the set of critical pairs \( P = \{(f_1, f_2)\} \). We choose and remove the only pair from this set and therefore \( P = \{\} \). We recall that Spolynomial of \( f_1 \) and \( f_2 \) over integers is defined to be

\[
\text{Spoly}(f_1, f_2) = \frac{lcm(LT(f_1), LT(f_2))}{LT(f_1)}f_1 - \frac{lcm(LT(f_1), LT(f_2))}{LT(f_2)}f_2.
\]

So we shall first compute the lcm of \( LT(f_1) = 3x^2y \) and \( LT(f_2) = 4xy^2 \) which is equal to \( 12x^2y^2 \). This follows that \( \text{Spoly}(f_1, f_2) = 4yf_1 - 3xf_2 = 15x^2 + 28y^2 \). We note that this polynomial is not reducible by \( f_1 \) and \( f_2 \). Thus, we add the new polynomial \( f_3 = 15x^2 + 28y^2 \) to \( G \).

(a) Complete this process to compute a Gröbner basis for \( \langle f_1, f_2 \rangle \).

(b) Does the polynomial \( -28y^3 - 15x^2y \) belong to this ideal?