Isfahan University of Technology
Department of Mathematical Sciences
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School on Tilings and Tessellations
Exercise Session
A. Hashemi

Exercises

In this exrercise, we focus on the calulation of Gröbner bases over fields and also over integers.

Question 1. Let us consider the following system:

$$\begin{cases} x^2+y^2+z^2 & = & 12 \\ xyz & = & 8 \\ x^2-y^2-z^2 & = & -4. \end{cases}$$

The aim of this exercise is to solve this system. Suppose that $R = K[x,y,z]$ where K is a field and $z \prec_{lex} y \prec_{lex} x$. Let $I = \langle x^2+y^2+z^2-12, xyz-8, x^2-y^2-z^2+4\rangle \subset R$.

(a) Using Buchbergr's algorithm, find a Gröbner basis for this ideal.

(b) Let $G = \{g_1,\dots,g_k\}$ be a Gröbner basis for I. Then, we know that the set of solutions of the above system is equal to the set of solutions of the system $g_1 = \cdots = g_k = 0$. Using this fact find all solutions of the above system.

(c) Find a remainder of the polynomial y^2+z^2-8 on division by $\{x^2+y^2+z^2-12, xyz-8, x^2-y^2-z^2+4\}$.

(d) Does the polynomial y^2+z^2-8 belong to I?

Question 2. In this exercise we train how to compute Gröbner bases over integers. In doing so, let $R = \mathbb{Z}[x,y]$ and $y \prec_{lex} x$. Let $f_1 = 3x^2y+7y$ and $f_2 = 4xy^2-5x$. So, we follow Buchberger's algorithm. We set $G = \{f_1, f_2\}$ and consider the set of critical pairs $P = \{\{f_1, f_2\}\}$. We choose and remove the only pair from this set and therefore $P = \{\}$. We recall that Spolynomial of f_1 and f_2 over intgers is defined to be

$$\text{Spoly}(f_1,f_2) = \frac{lcm(\text{LT}(f_1),\text{LT}(f_2))}{LT(f_1)} f_1 - \frac{lcm(\text{LT}(f_1),\text{LT}(f_2))}{LT(f_2)} f_2.$$

So we shall first compute the lcm of $\text{LT}(f_1) = 3x^2y$ and $\text{LT}(f_2) = 4xy^2$ whcih is equal to $12x^2y^2$. This follows that $\text{Spoly}(f_1,f_2) = 4yf_1 - 3xf_2 = 15x^2+28y^2$. We note that this polynomial is not reducible by f_1 and f_2. Thus, we add the new polynomial $f_3 = 15x^2+28y^2$ to G.

(a) Complete this process to compute a Gröbner basis for $\langle f_1, f_2\rangle$,

(b) Does the polynomial $-28y^3-15x^2y$ belong to this ideal?