# Exercises

Symbolic dynamics 2

Isfahan, Aug. 28 and 31, 2015

## 1. 2D even subshift

A connected component of 1s in a configuration $x \in A^{\mathbb{Z}^2}$ is a maximal set of cells with state 1, such that any two cells can be connected by paths inside this set (considering the neighborhood with 4 neighboring cells).

1. Let $X' \subset A^{\mathbb{Z}^2}$ be the set of configurations over alphabet $A = \{0, 1\}$ such that all connected components of 1s are finite and have even cardinality.
   Is it a subshift?
   *Hint: Try to define it with finite patterns; can you forbid a pattern whose all symbols are* $1s$*?*

2. Let $X \subset A^{\mathbb{Z}^2}$ be the set of configurations over alphabet $A = \{0, 1\}$ such that all finite connected components of 1s have even cardinality. Is it a subshift?
   *Hint: It is the set of configurations without any connected component of odd cardinality.*

3. Is $X$ a SFT?
   *Hint: What can you say about two configurations with long line of* $1s$*, one of odd length and one of even length?*

4. Show that a connected polyomino is tileable by shapes $\begin{smallmatrix}□\\□\end{smallmatrix}$ and $\begin{smallmatrix} & □\\ □ & \end{smallmatrix}$ together with their rotations, if and only if it has even cardinality.
   *Hint: Induction from the top left.*

5. Is $X$ sofic?
   *Hint: Decorate the* $1s$*.*

## 2. Projective subdynamics

The projective subdynamics of some 2D subshift $\mathbf{X}$ is the 1D subshift $\tau(\mathbf{X})$ of all columns $(x_{i,j})_{j\in\mathbb{Z}}$, for $i \in \mathbb{Z}$ (or equivalently central columns $(x_{0,j})_{j\in\mathbb{Z}}$) appearing in configurations $x \in \mathbf{X}$.

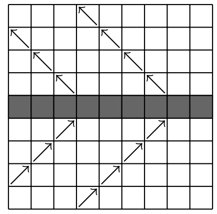
1. Let $\tilde{\mathbf{X}}$ be a 1D SFT. Build a 2D SFT $\mathbf{X}$ such that $\tilde{\mathbf{X}} = \tau(\mathbf{X})$.

2. Let $\tilde{\mathbf{X}}$ be a 1D sofic subshift. Build a 2D sofic subshift $\mathbf{X}$ such that $\tilde{\mathbf{X}} = \tau(\mathbf{X})$.

Let $\mathbf{Y}$ be the 2D SFT of configurations over alphabet $B = \{□, ■, ↗, ↖\}$ such that arrows continue each other, in the good direction, unless blocked by a mirror state ■ (always placed in a line), in which case the arrow changes its direction. Formally, we forbid the following patterns (together with their vertical symetric):

■□, ■↗, ■↖ (mirror states are in lines), $\begin{smallmatrix} & □\\ ↗ & \end{smallmatrix}$, $\begin{smallmatrix} & ↖\\ ↗ & \end{smallmatrix}$ (arrows never end unless finding the mirror...), $\begin{smallmatrix} □ & \\ & ■\\ ↗ & \end{smallmatrix}$, $\begin{smallmatrix} ↗ & \\ & ■\\ ↗ & \end{smallmatrix}$, $\begin{smallmatrix} ■ & \\ & ■\\ ↗ & \end{smallmatrix}$ (... in which case it is turned).

If this definition is too complicated, just consider that configurations look like this:



Moreover, let $X'$ be the 1D subshift of configurations over alphabet $B$ in which $\blacksquare$ appears at most once.

3. Show that $\tau(\mathbf{Y}) \cap X'$ is not a 1D sofic subshift (*i.e.*, the set of labels of biinfinite paths of a labeled graph).
   *Hint: Consider a configuration involving signals, and use the pigeon-hole principle.*

4. Admit (or prove, if you have the time) that the intersection of two sofic subshifts is sofic.
   *Hint: Graph product.*

5. What can one deduce about $\tau(\mathbf{Y})$?

6. Let $\tilde{\mathbf{Y}} = \left\{ z \in \{١,٢,٣\}^{\mathbb{Z}} \middle| \forall i,j \in \mathbb{Z}, i \neq j \text{ and } z_i = ٣ \Rightarrow z_j = z_{2i-j} \neq ٣ \right\}$, the "palindrome" 1D subshift.
   Describe a 2D sofic subshift $\mathbf{Y}$ whose projective subdynamics is $\tau(\mathbf{Y}) = \tilde{\mathbf{Y}}$.

## 3. † Soficity and extension number

We keep $\tilde{\mathbf{Y}}$ as defined above, and consider the 2D subshift $\tilde{\mathbf{Y}}^{\mathbb{Z}} = \left\{ x \in \{١,٢,٣\}^{\mathbb{Z}^2} \middle| \forall i \in \mathbb{Z}, (x_{i,j})_{j\in\mathbb{Z}} \in \tilde{\mathbf{Y}} \right\}$.

1. Let $n > 0$. What is the number of allowed patterns in $\tilde{\mathbf{Y}}^{\mathbb{Z}}$ over the square $[\![0,n]\!] \times [\![0,n]\!]$ that do not involve $٣$?

2. Assume that there is a nearest-neighbor SFT $\mathbf{X}'$ over some (finite) alphabet $A$ that letter-to-letter projects onto $\tilde{\mathbf{Y}}^{\mathbb{Z}}$ (which means every pattern of $\mathbf{X}'$ projects into a pattern of $\tilde{\mathbf{Y}}^{\mathbb{Z}}$ with the same size, and every pattern $\tilde{\mathbf{Y}}^{\mathbb{Z}}$ can be obtained that way). What is the number of allowed patterns in $\mathbf{X}'$ over the unfilled square $[\![0,n]\!] \times [\![0,n]\!] \setminus ]\!]0,n[\![ \times ]\!]0,n[\![$?

3. What does the pigeon-hole principle then tell, for a large $n$?

4. What property share two patterns of $\tilde{\mathbf{Y}}^{\mathbb{Z}}$ over $[\![0,n]\!] \times [\![0,n]\!]$ that are projected from two patterns of $\mathbf{X}'$ that have the same border (*i.e.*, symbols in $[\![0,n]\!] \times [\![0,n]\!] \setminus ]\!]0,n[\![ \times ]\!]0,n[\![$)?
   *Hint: what about their possible extensions as configurations?*

5. For any two of these patterns, give an extension of one that is not an extension of the other one?
   *Hint: use the definition of $\tilde{\mathbf{Y}}$!*

6. Is $\tilde{\mathbf{Y}}^{\mathbb{Z}}$ sofic?

7. Is $\tilde{\mathbf{Y}}^{\mathbb{Z}}$ effective?

## 5. Periodic configurations in SFT

Let $Y \in A^{\mathbb{Z}^d}$ be a SFT such that there exists a configuration $y \in Y$ with $d-1$ independent periodicity vectors.

1. Show that $Y$ admits a configuration with $d$ independent periodicity vectors.

2. Show that such a configuration has a periodicity vector colinear to each vector of the canonical base.