Cut and Projection - Part II

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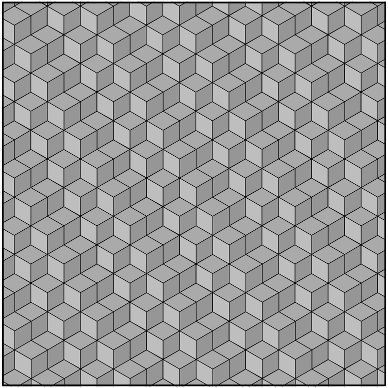
CNRS & Univ. Paris 13

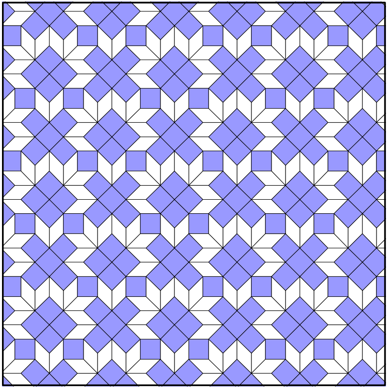
CIMPA School
Tilings and Tessellations
August 24 - September 4, 2015

Outline

Outline

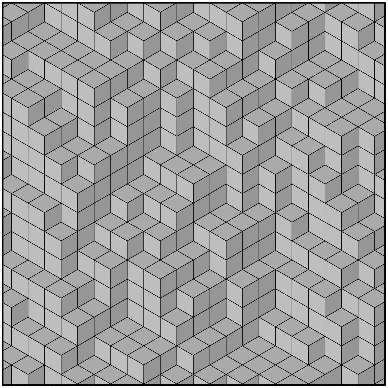
General $n \to d$ tilings

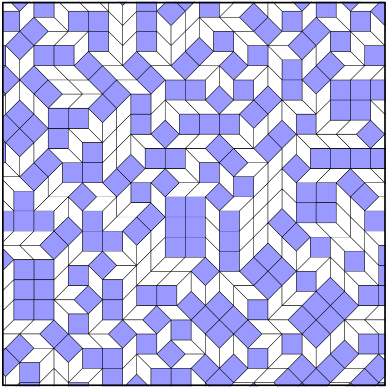




Planar tilings are well ordered. . .

General $n \to d$ tilings



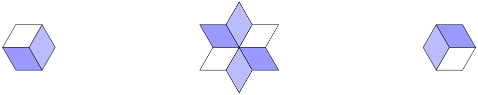


Planar tilings are well ordered. . . but they can easily be messed up!

Local rules

Definition (Local rules)

A planar tiling of slope E has *diameter* r and *thickness* t *local rules* if any tiling with a smaller or equal r-atlas lifts into $E + [0, t]^n$.



Main Open Question

Which planar tilings do admit local rules?

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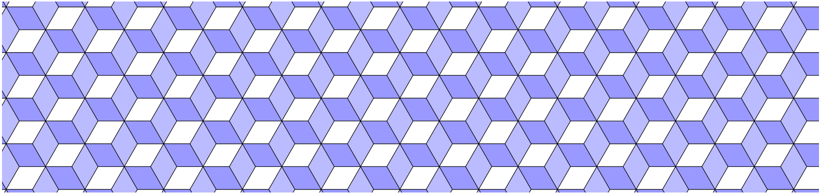

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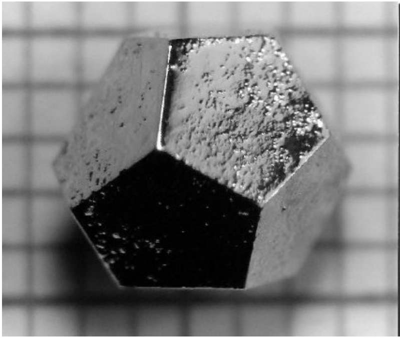
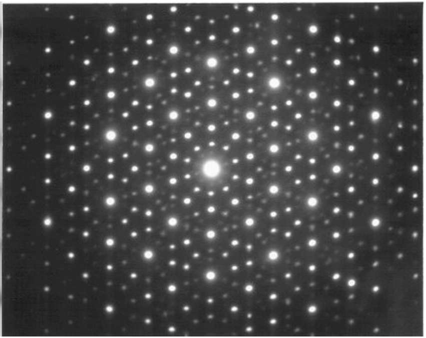
Link with quasicrystals

Planar $n \to d$ tilings aim to model the *structure* of *quasicrystals*.





Local rules aim to model their *stability* (*i.e.*, energetic interactions).

Outline

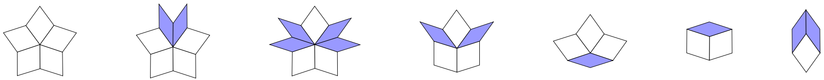
Penrose tilings

Definition (Penrose tiling)

A *Penrose tiling* is a planar $5 \to 2$ tiling with slope

$$\frac{1}{5}(1,1,1,1,1) + \mathbb{R}\left(\cos\frac{2k\pi}{5}\right)_{0\leq k\leq 4} + \mathbb{R}\left(\sin\frac{2k\pi}{5}\right)_{0\leq k\leq 4}.$$

It is the dualization of the multigrid with vectors $\mathrm{e}^{\frac{2ik\pi}{5}}$ and shifts $\frac{1}{5}$.



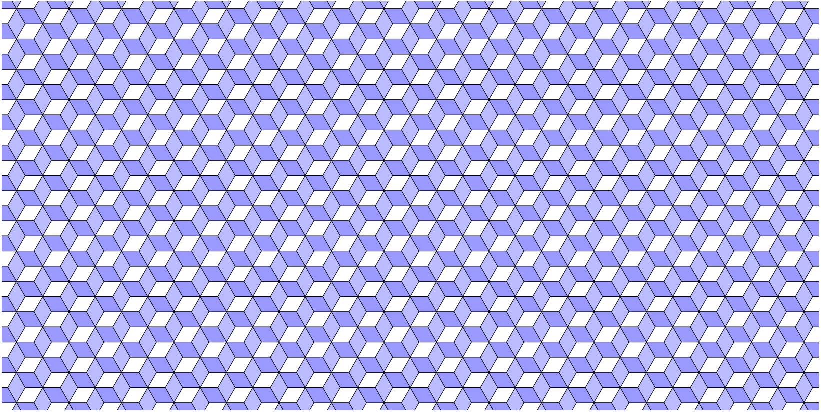
Theorem (de Bruijn, 1981)

Penrose tilings have local rules of diameter 0 *and thickness* 1*.*

n-fold tilings

Definition (n-fold tiling)

A *n-fold tiling* is a planar $n \to 2$ tiling which has the same finite patterns as its image under a rotation by $2\pi/n$.



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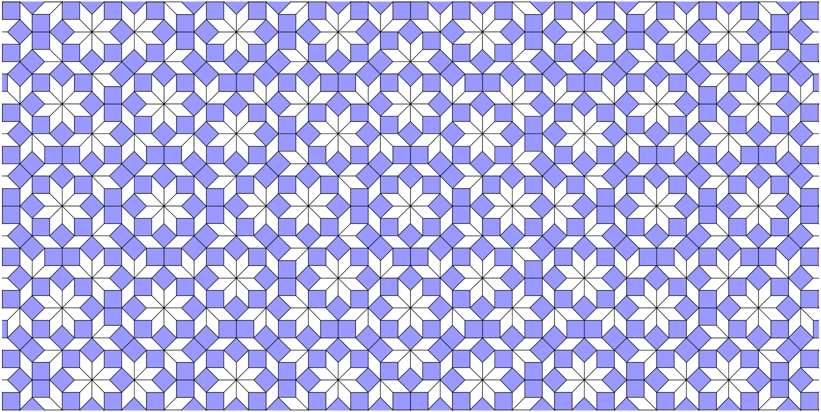

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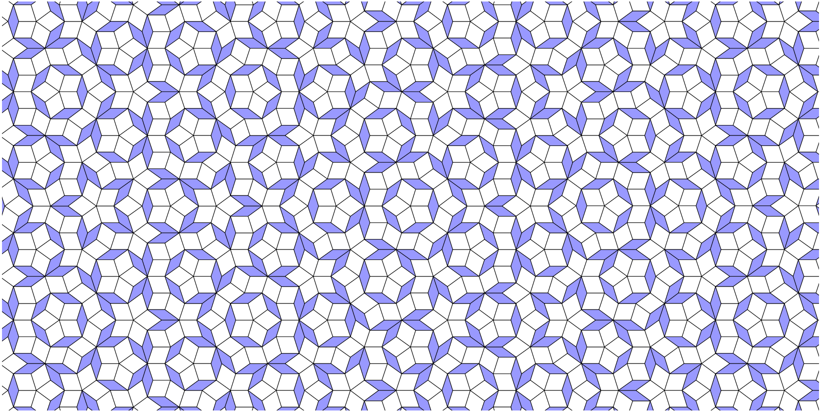

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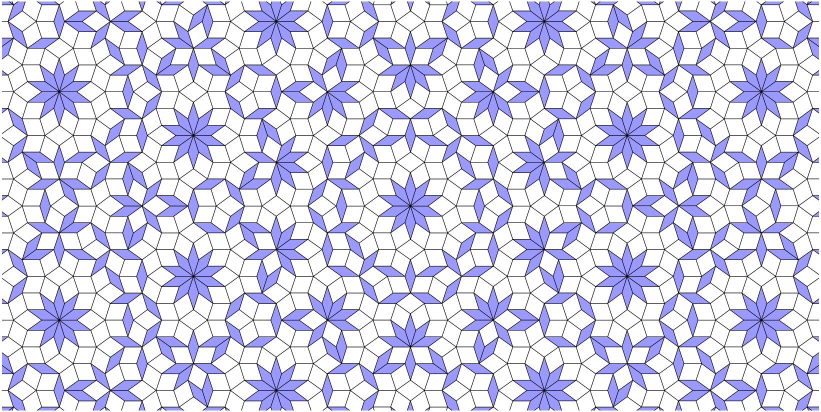

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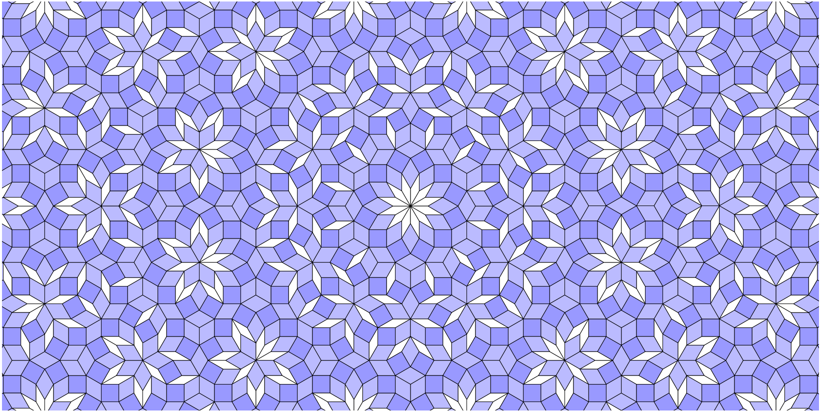

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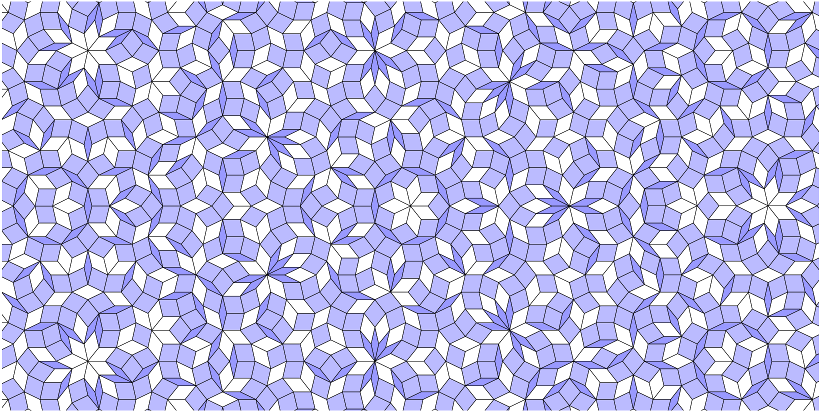

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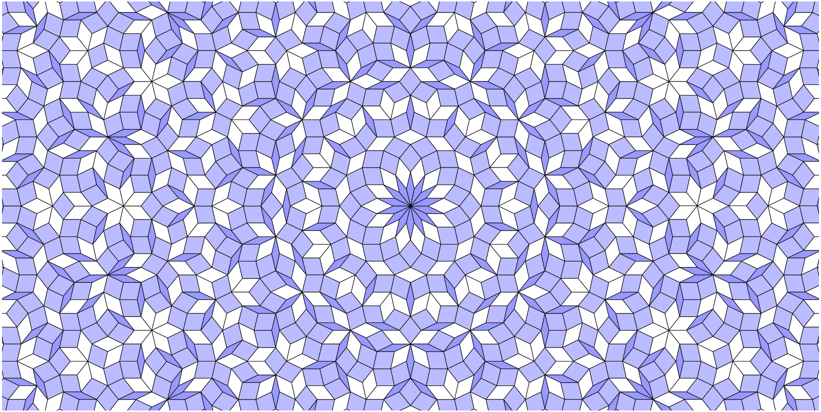

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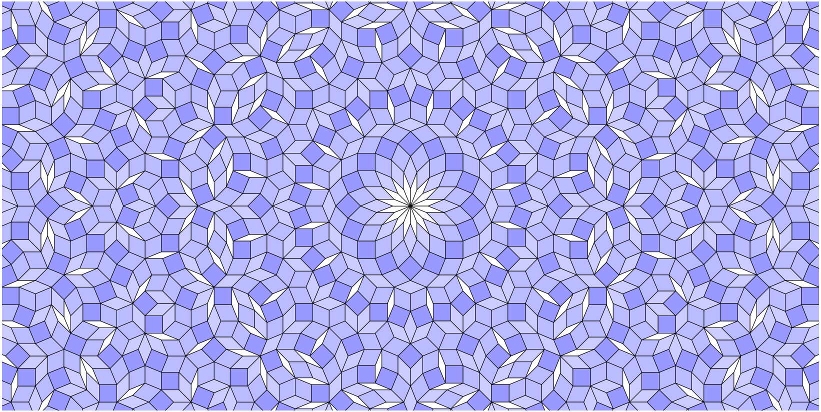
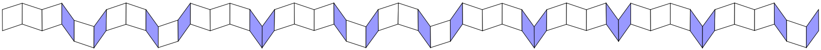


Local rules for n-fold tilings

Theorem (Socolar 1990)

An n-fold tiling has local rules when n is not a multiple of 4.

Local rules actually enforce an *alternation condition*:



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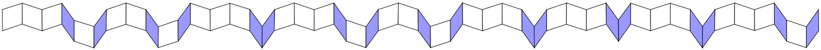

When n is a multiple of 4, there are square tiles. . .

Subperiods

Definition (Subperiod)

A planar $n \to d$ tiling has a *subperiod* if one gets a periodic tiling by an orthogonal projection onto $d+1$ well-chosen basis vectors.

For example, a Penrose tiling has 10 subperiods (video).

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This translates in linear rational dependencies between Grassmann coordinates over $d+1$ indices. For Penrose:

$$G_{12} = G_{23} = G_{34} = G_{45} = G_{51}, \qquad G_{13} = G_{35} = G_{52} = G_{24} = G_{41}.$$

Planarity issues

Proposition

The subperiods of a planar tiling can be enforced by local rules.

But these local rules may not suffice to enforce planarity. . .

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But these local rules may not suffice to enforce planarity...

Theorem (Bédaride-Fernique 2015)

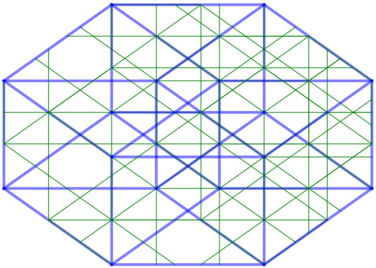
A planar $4 \to 2$ tiling has local rules iff its slope is characterized by its subperiods. In particular the slope is quadratic (or rational).

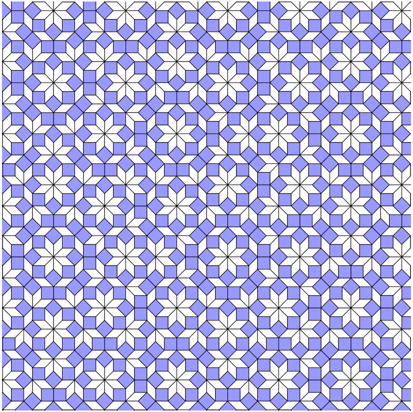
Outline

4p-fold tilings

Theorem (Bédaride-Fernique 2015)

The 4p-fold tilings do not have local rules.





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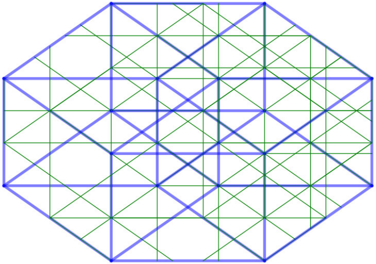

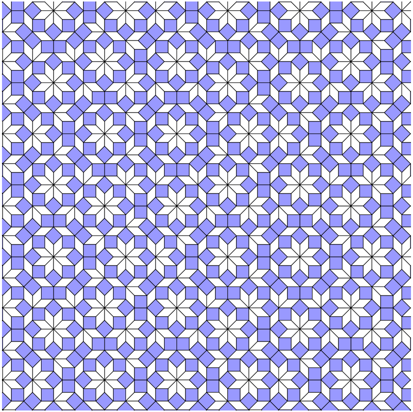

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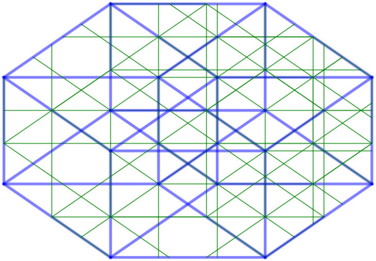

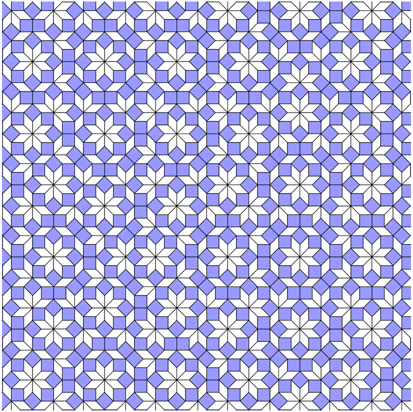

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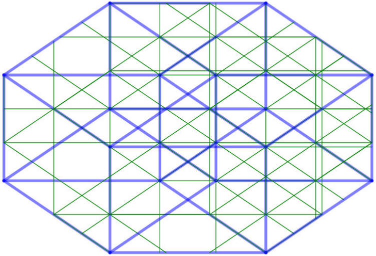

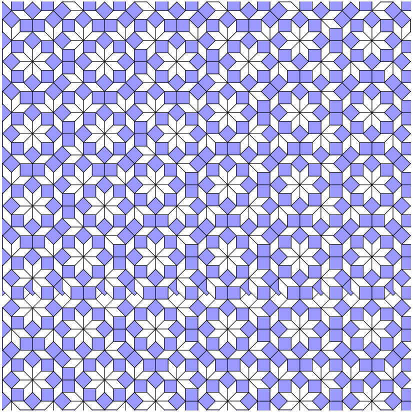

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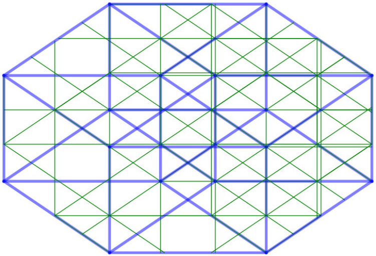
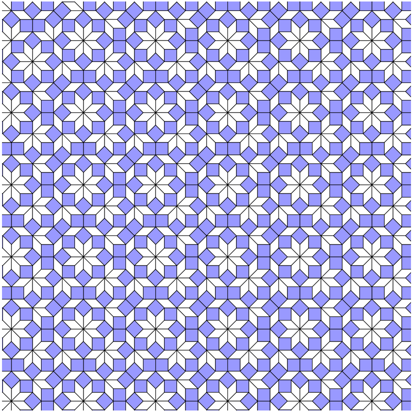




Local rules of thickness 1

Full subperiods: any projection on $d+1$ basis vector is periodic.

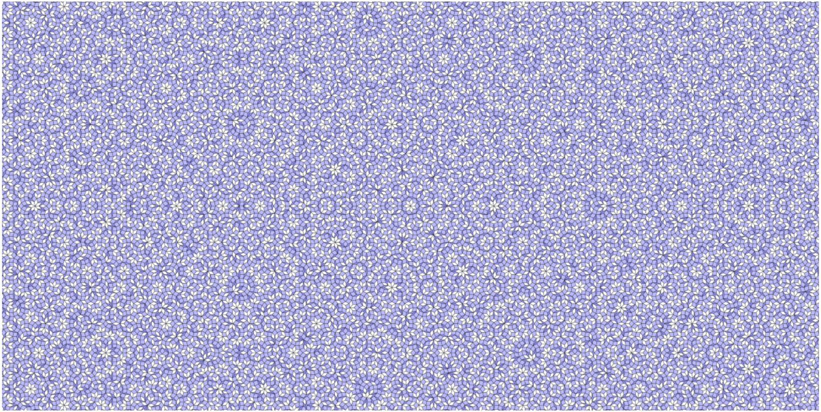
Theorem (Levitov 1988)

A planar tiling with thickness 1 *local rules has full subperiods.*

For n-fold tilings, this yields $n \in \{4, 6, 8, 10, 12\}$.

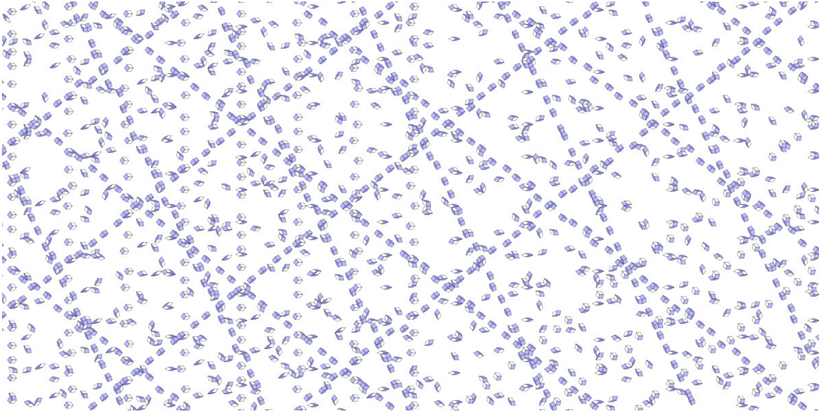
These are the only symmetries yet observed in real quasicrystals. . .

Proof sketch



Consider a planar tiling which does not have full subperiods.

Proof sketch



Shifting the slope creates *flips*. We shift without creating patterns.

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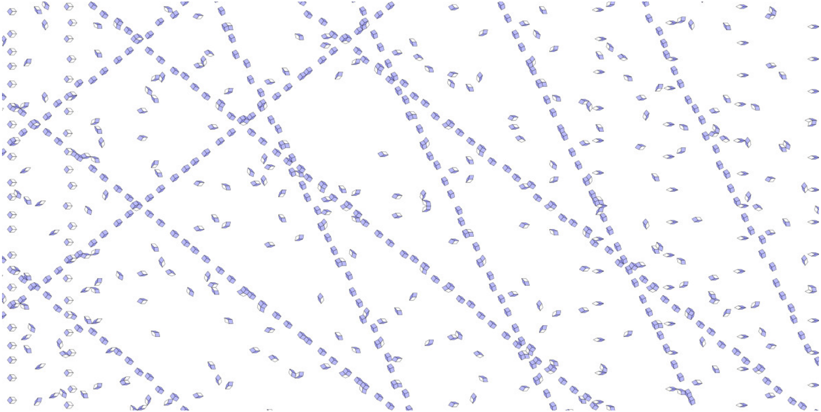
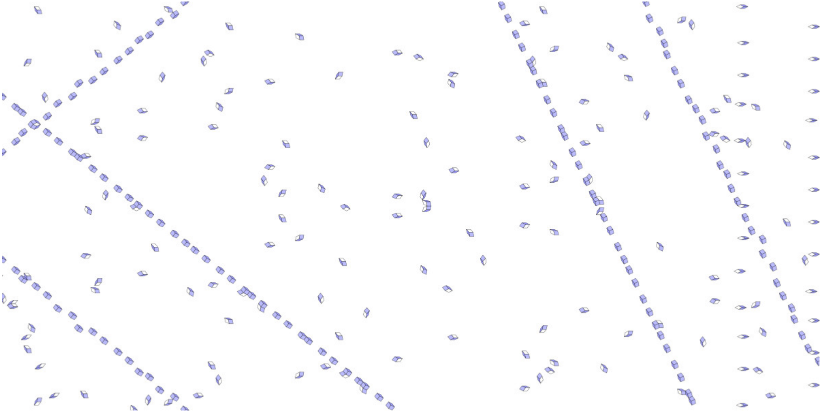


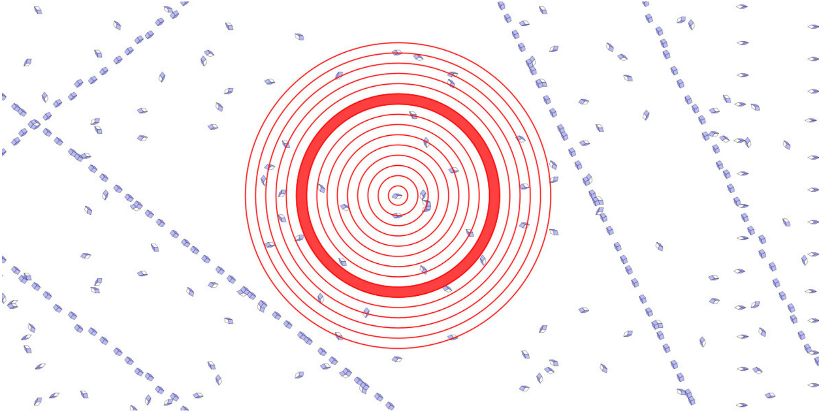
There are lines of flips (corresp. to subperiods) and isolated flips.

Proof sketch



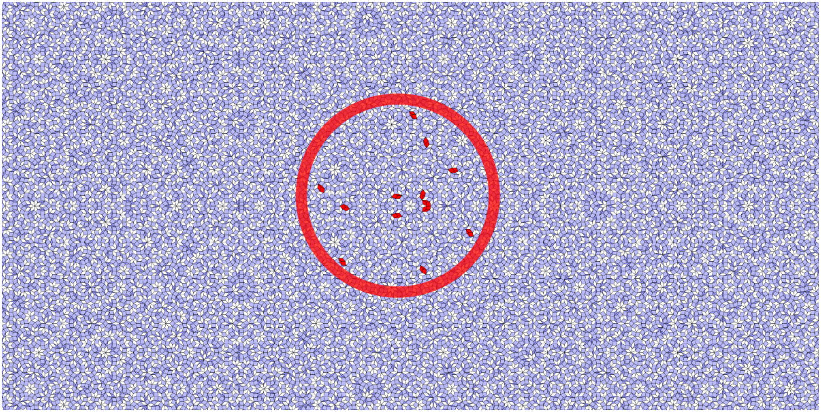
The smaller the shift is, the sparser these flips are.

Proof sketch



Given r, we eventually find a ring of thickness r without any flip.

Proof sketch

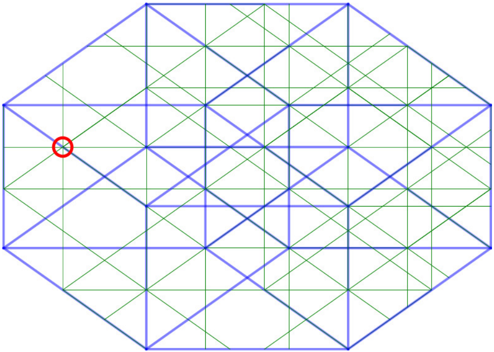


This yields a planar tiling of thickness $t > 1$ with the same r-atlas.

Algebraic obstruction

Theorem (Le 1995)

The slope of a planar tiling with local rules is algebraic.



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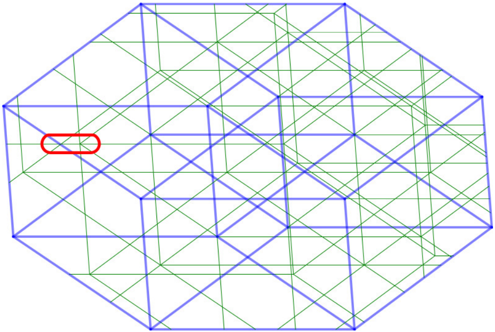

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From colors to computability

Definition (Colored local rules)

A planar tiling has *colored local rules* if it is obtained by removing the colors of a colored planar tilings which has local rules.

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This allows to go far beyond the previous algebraic obstruction:

Theorem (Fernique-Sablik 2012)

A planar tiling has colored local rules iff its slope is computable.

We would like such a characterization for uncolored local rules. . .

Computable slope?

Definition

A number is *computable* if it can be computed to within any desired precision by a finite, terminating algorithm.

Q. Are rational numbers computable? Algebraic numbers? e? π?

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A slope is computable if it has computable Grassmann coordinates.

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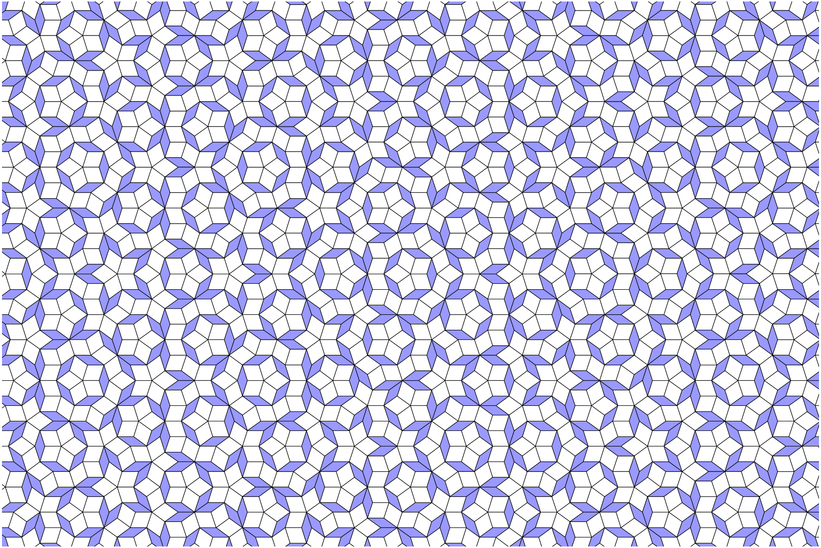

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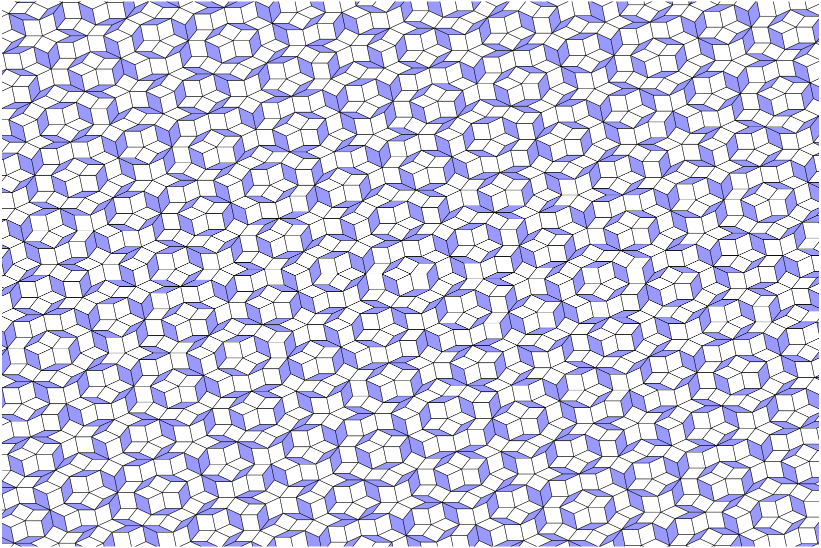

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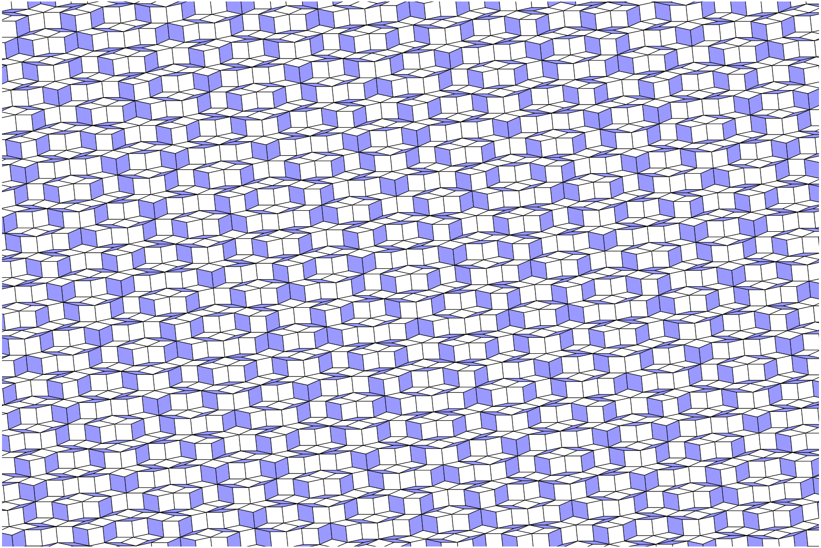

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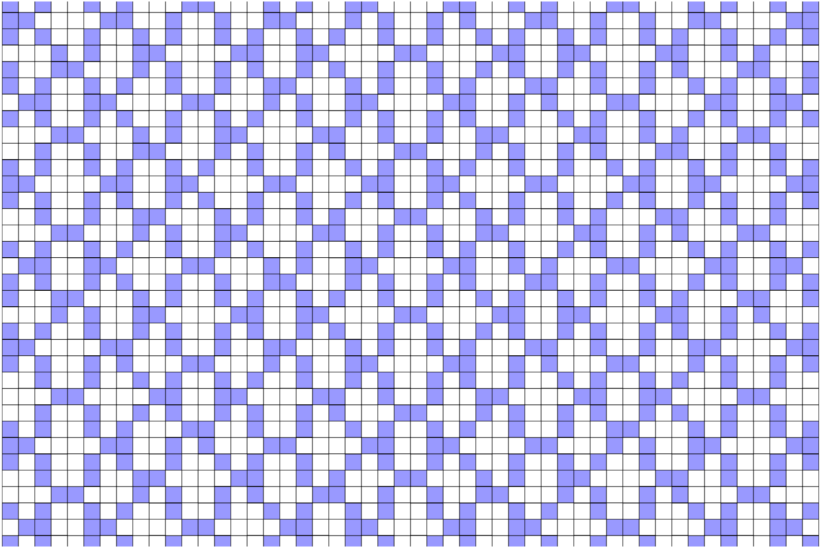

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