Cut and Projection - Part I

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CIMPA School
Tilings and Tessellations
August 24 - September 4, 2015

Outline

1. Planar Tilings
2. Multigrid dualization
3. Grassmann coordinates
4. Patterns

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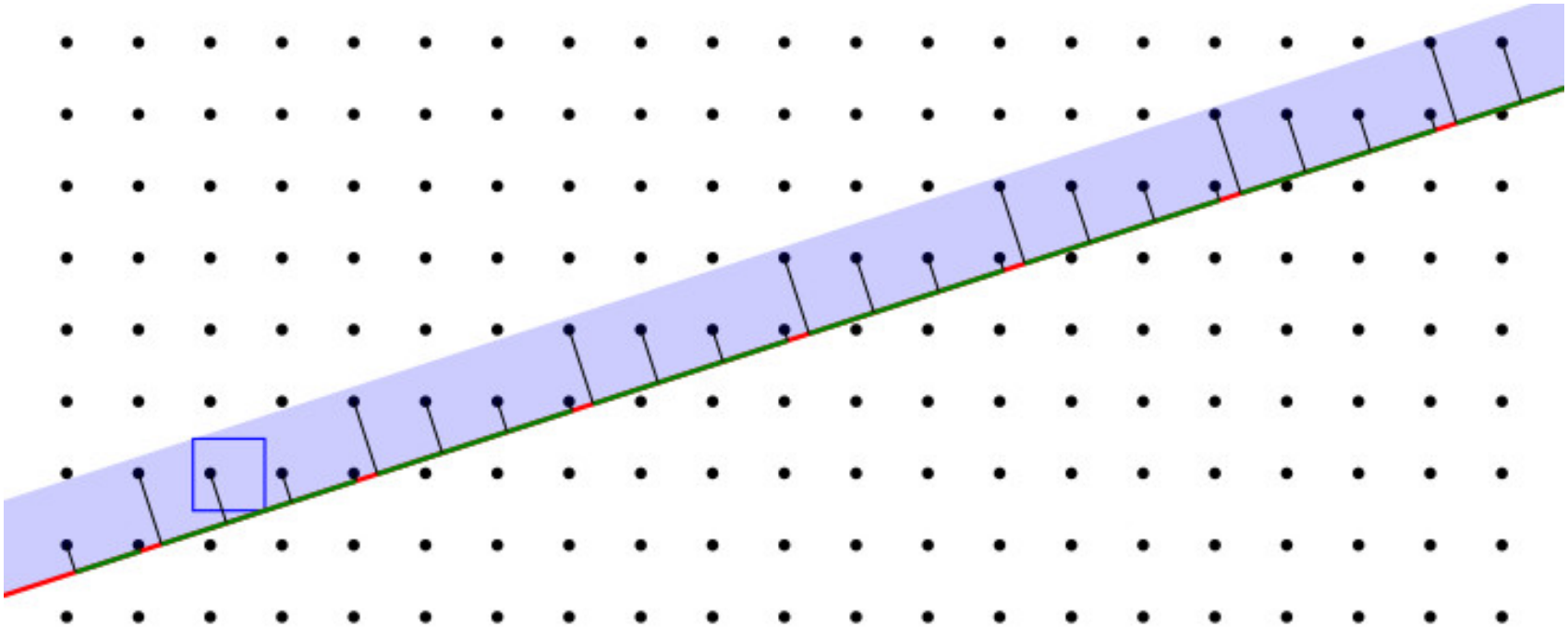
Cut the plane and project onto a line



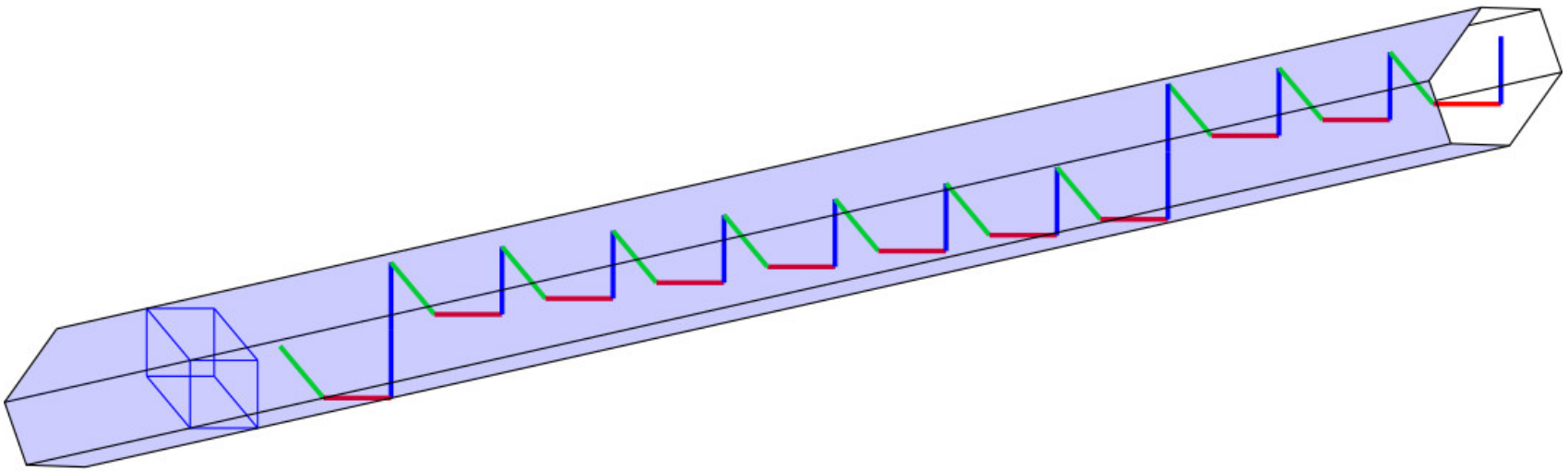
Cut $\mathbb{R}^n$ and project onto $\mathbb{R}^d$

Definition (Planar tiling)

Let E be a d-dim. affine space in $\mathbb{R}^n$ such that $E \cap \mathbb{Z}^n = \emptyset$.
Select the d-dim. faces with vertices in $\mathbb{Z}^n$ lying in $E + [0,1]^n$.
Project them onto E to get a so-called *planar $n \to d$ tiling*.

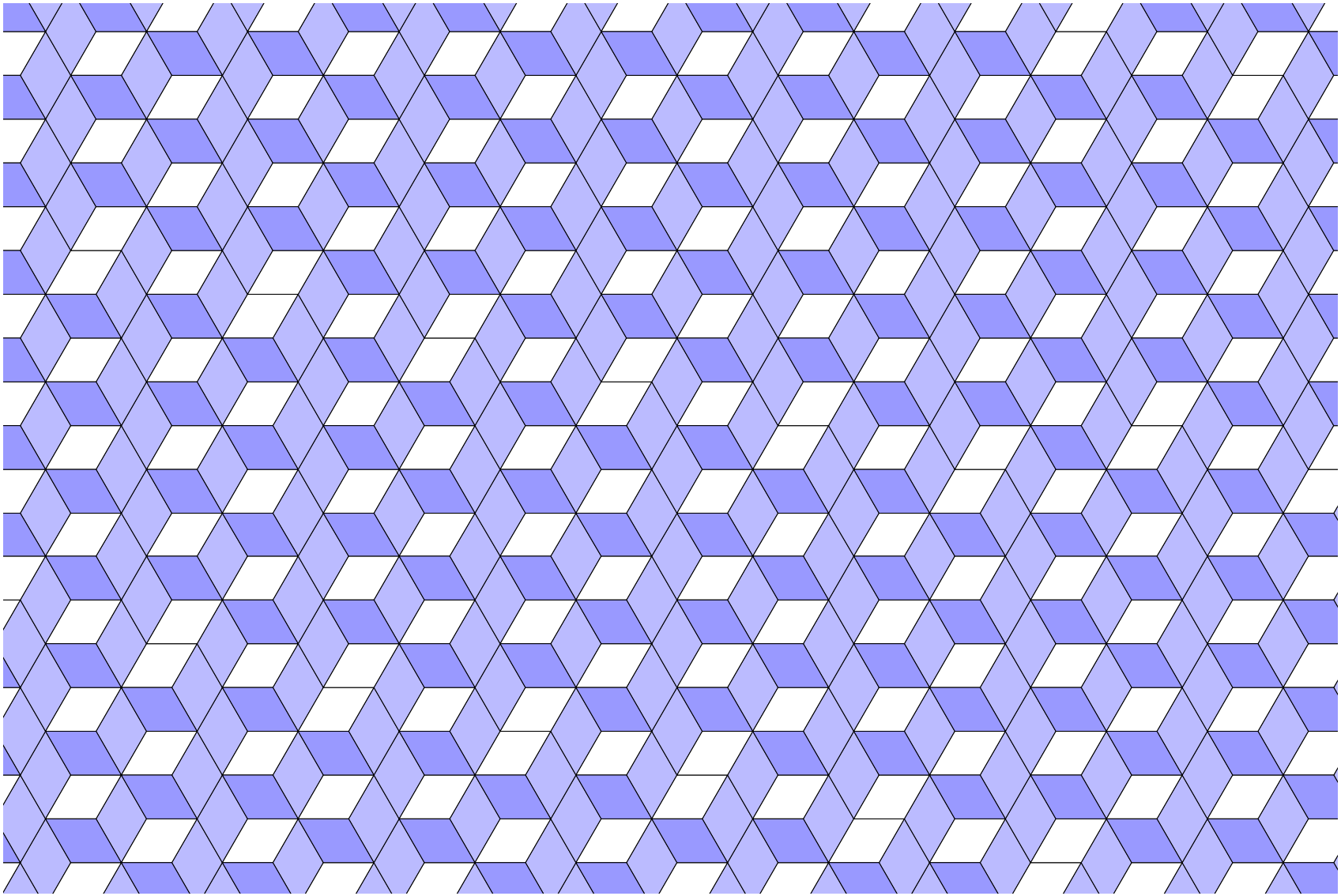
Q. Are such tilings periodic or not?

Cut the space and project onto a line

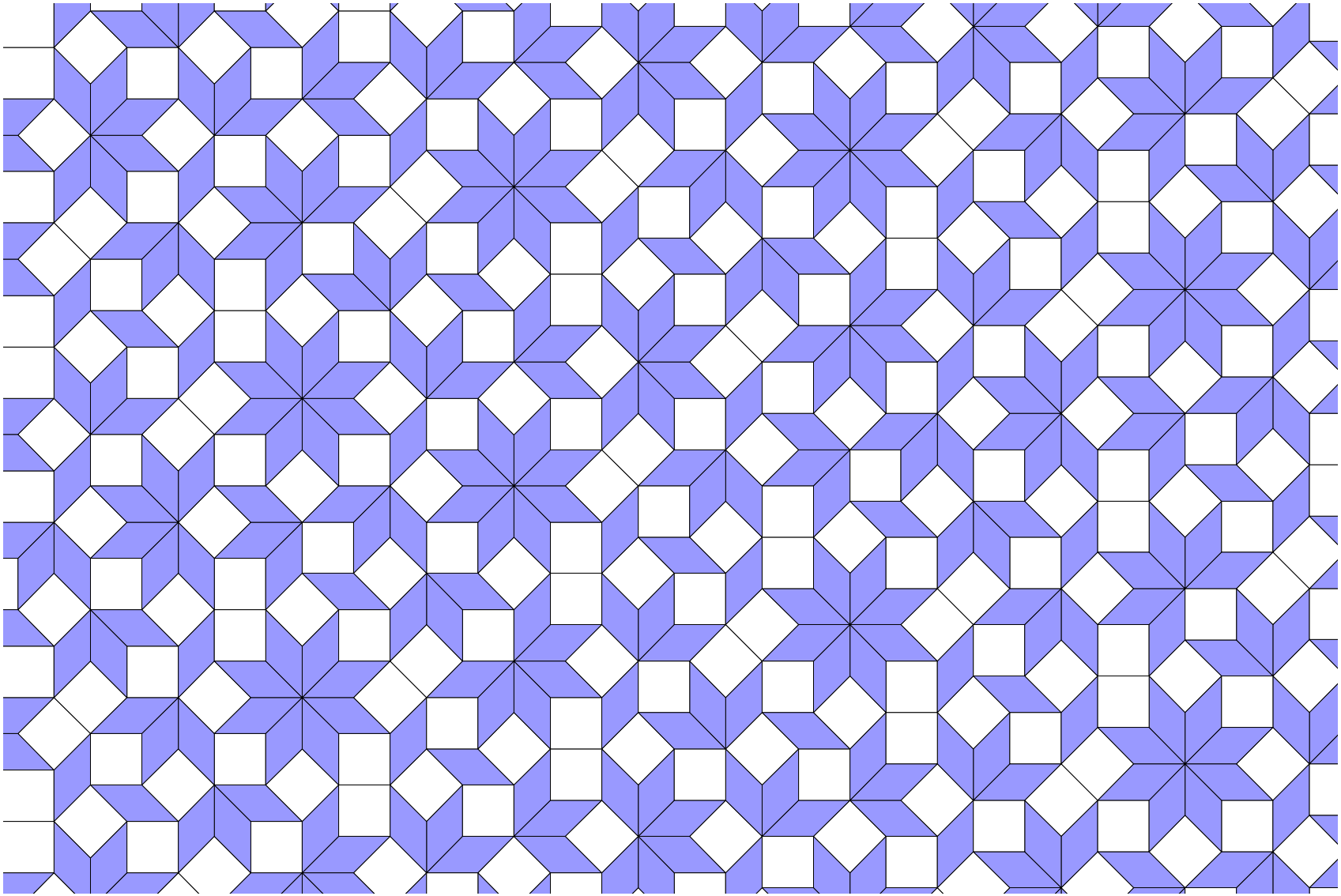


Billiard words are planar $3 \to 1$ tilings, but not the *Tribonacci word*.

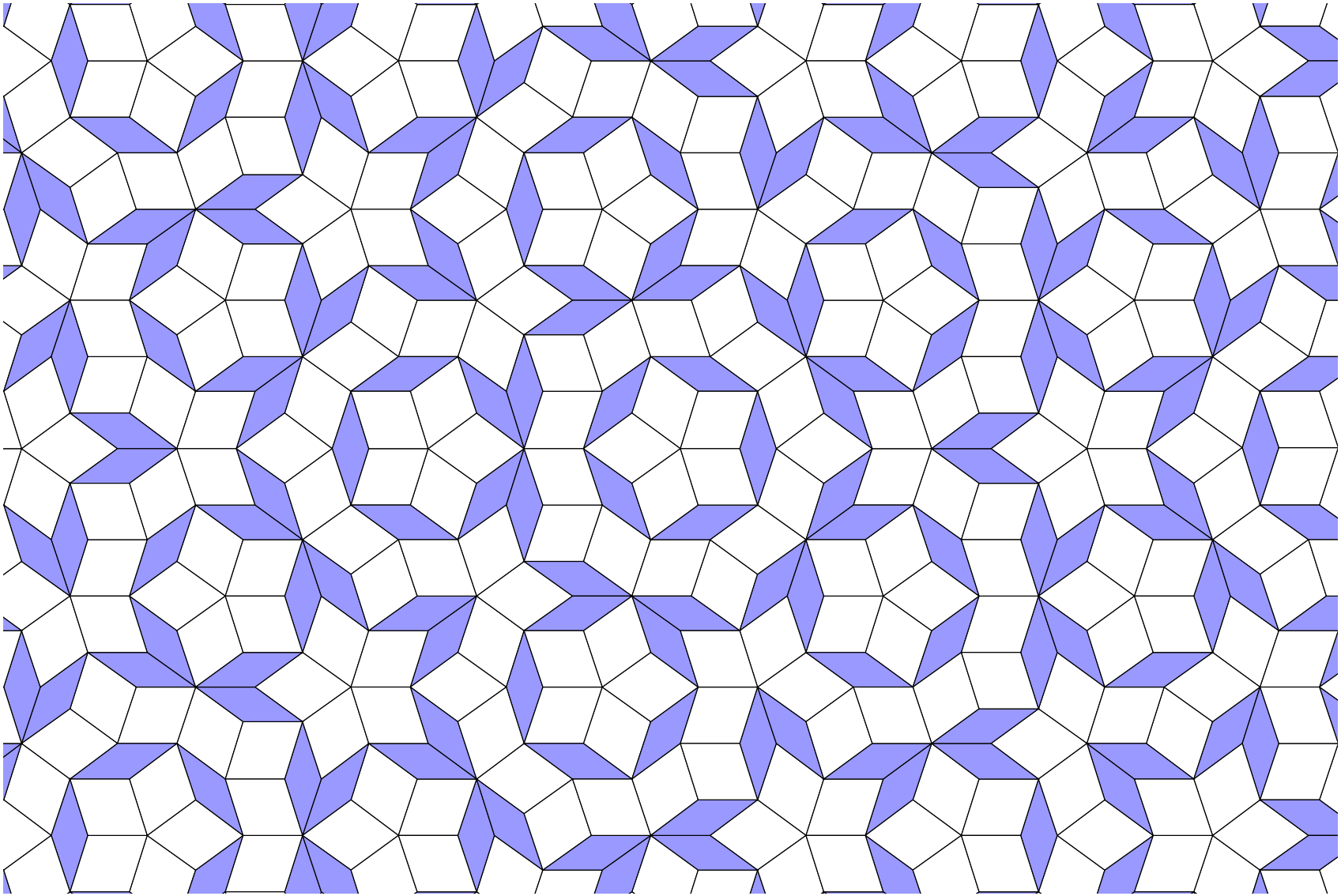
Cut the space and project onto a plane



Cut an higher dim. space and project onto a plane



And get a Penrose tiling (De Bruijn, 1981)



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Multigrid

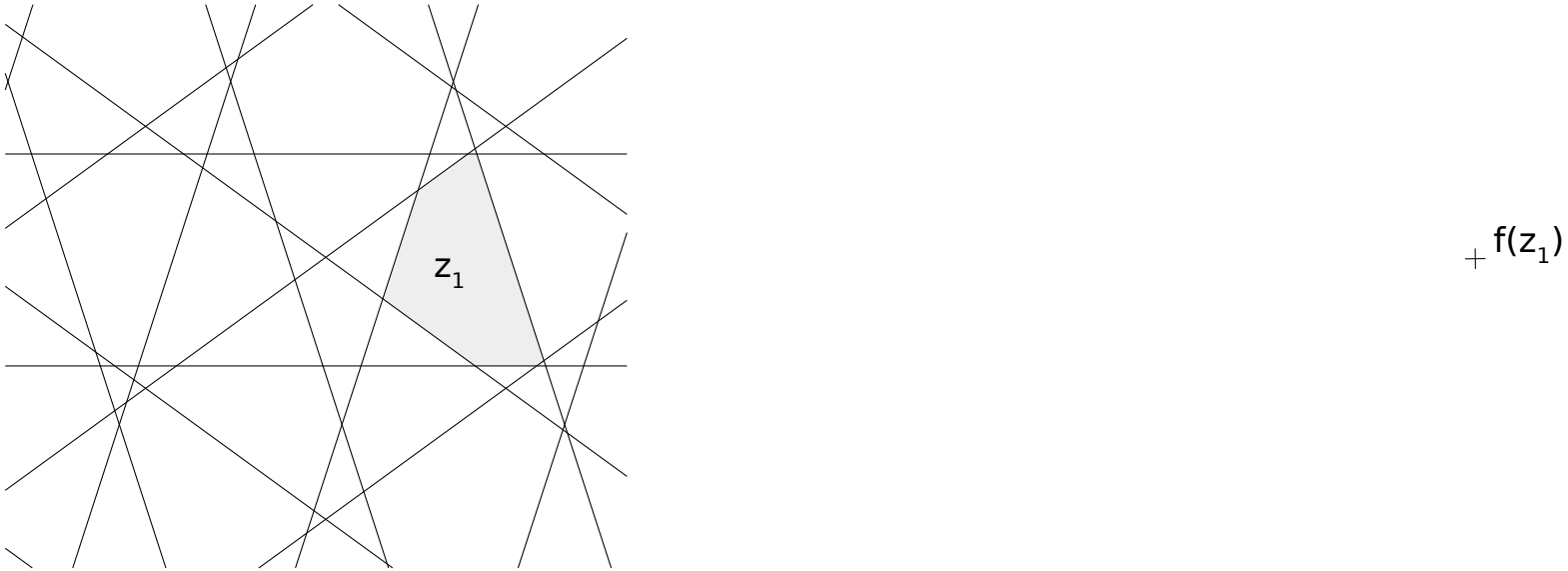
Definition (Multigrid)

The *multigrid* with shifts $s_1, \ldots, s_n$ in $\mathbb{R}$ and *grid vectors* $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^d$ is the set of n families of equally spaced parallel hyperplanes

$$H_i := \{\vec{x} \in \mathbb{R}^d \mid \langle \vec{x} | \vec{v}_i \rangle + s_i \in \mathbb{Z}\},$$

where at most d hyperplanes are assumed to intersect in a point.

Dualization



- The grid hyperplanes divide the space into cells;
- To each cell z_i corresponds a vertex $f(z_i)$ of the tiling;
- If z_i and z_j are adjacent along $a + \vec{v}_k^{\perp}$, then $f(z_j) - f(z_i) = \vec{v}_k$

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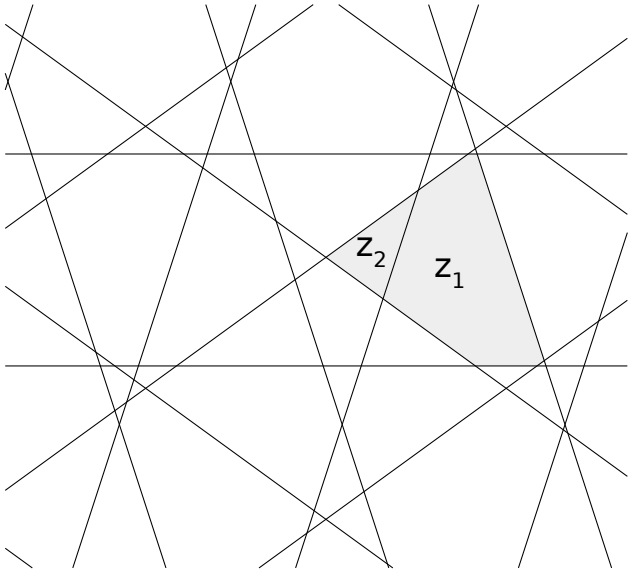

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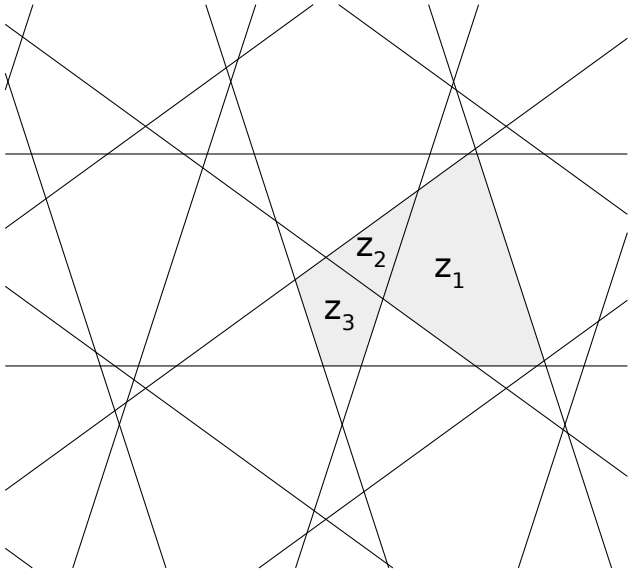

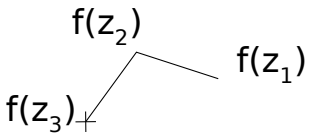

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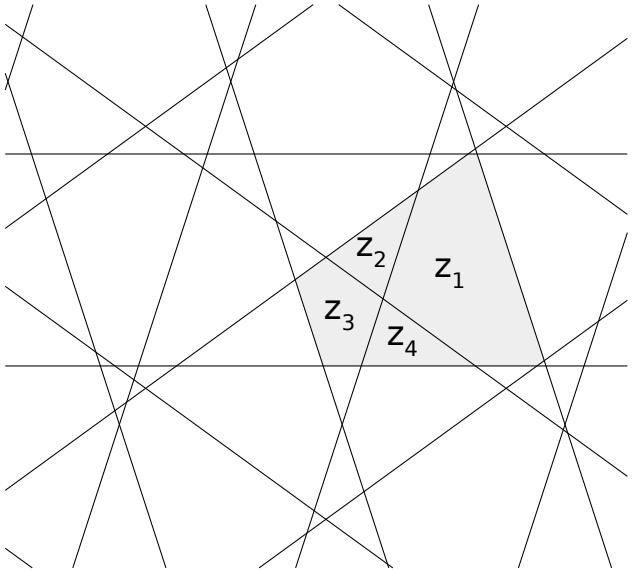

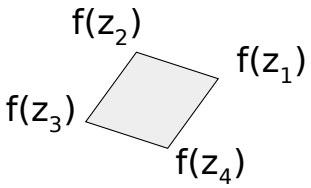

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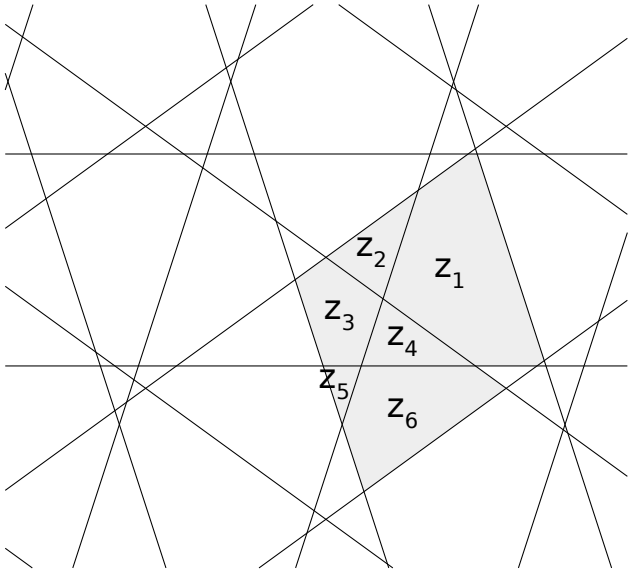

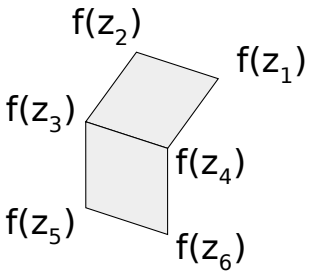

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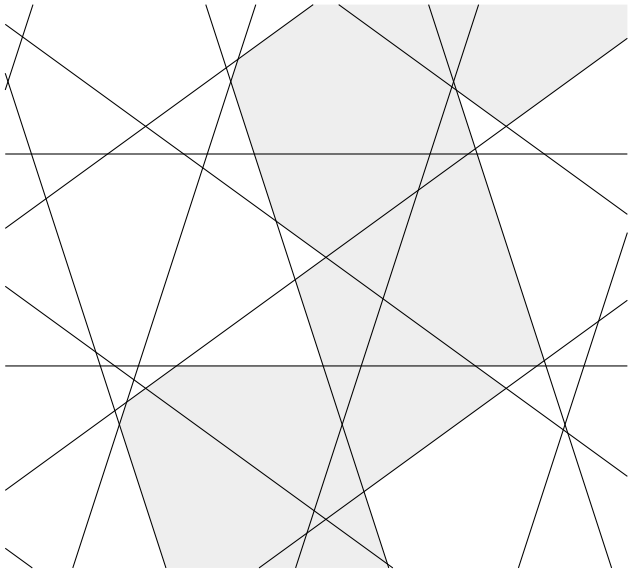

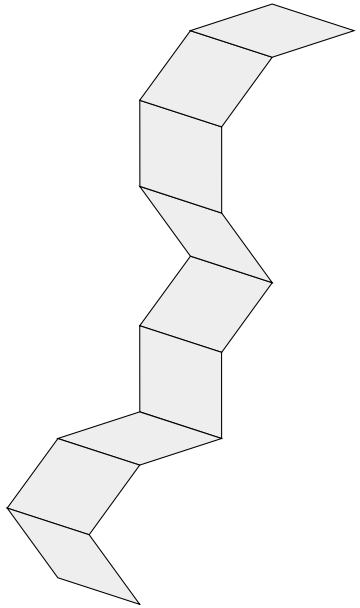

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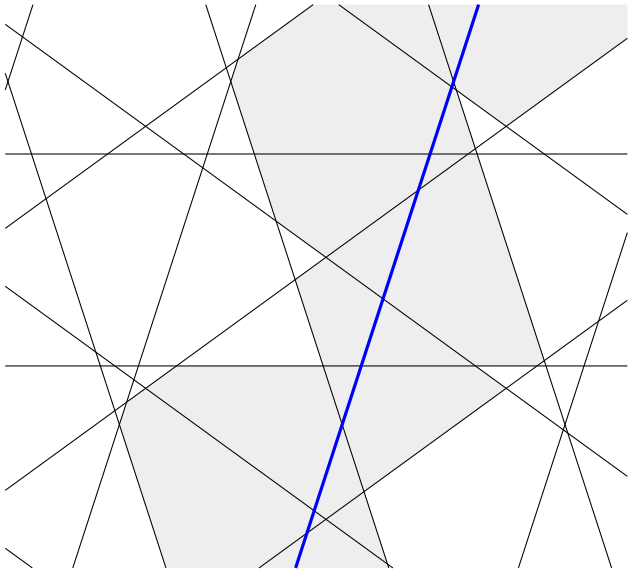

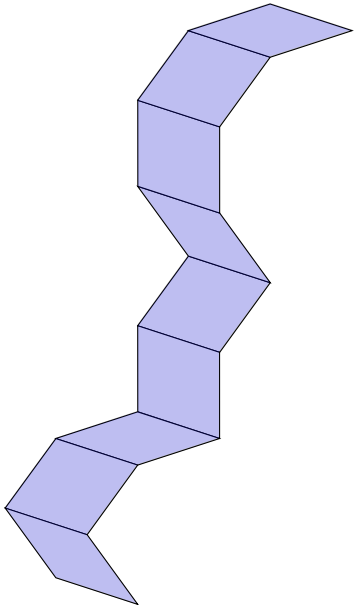

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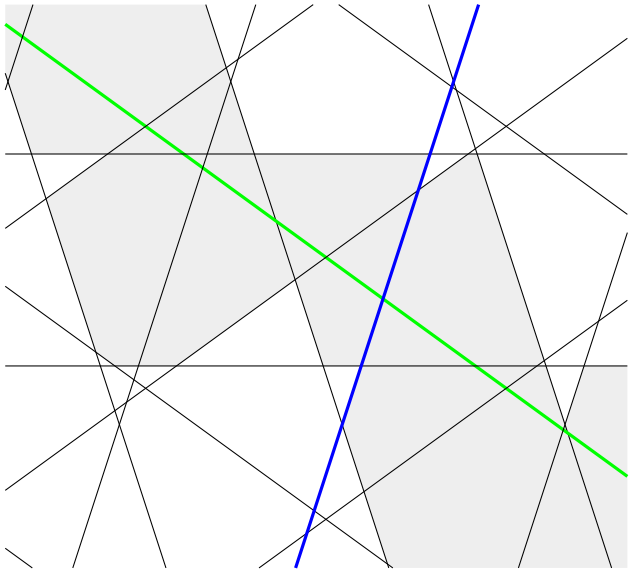

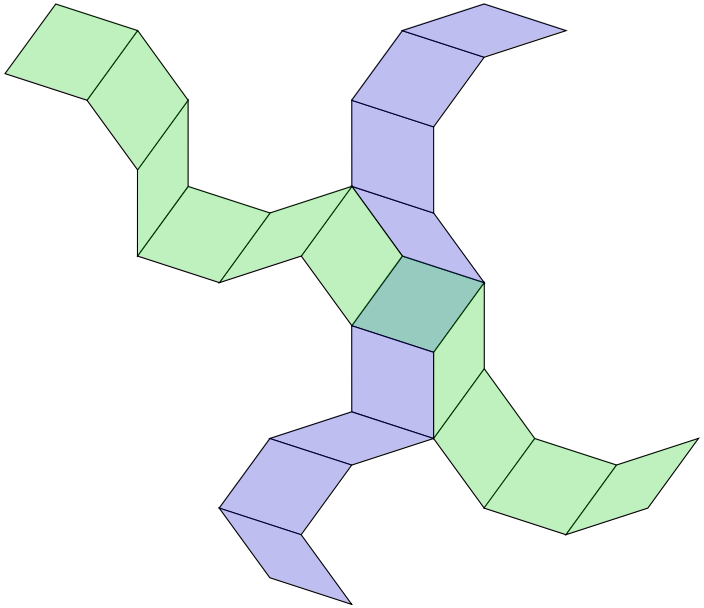

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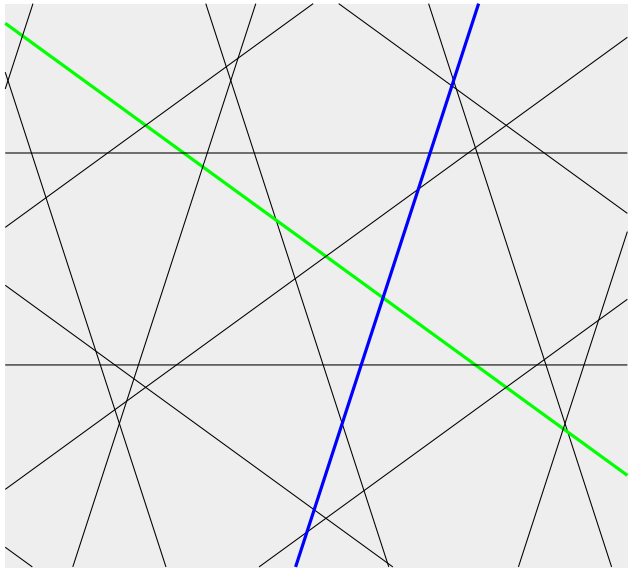

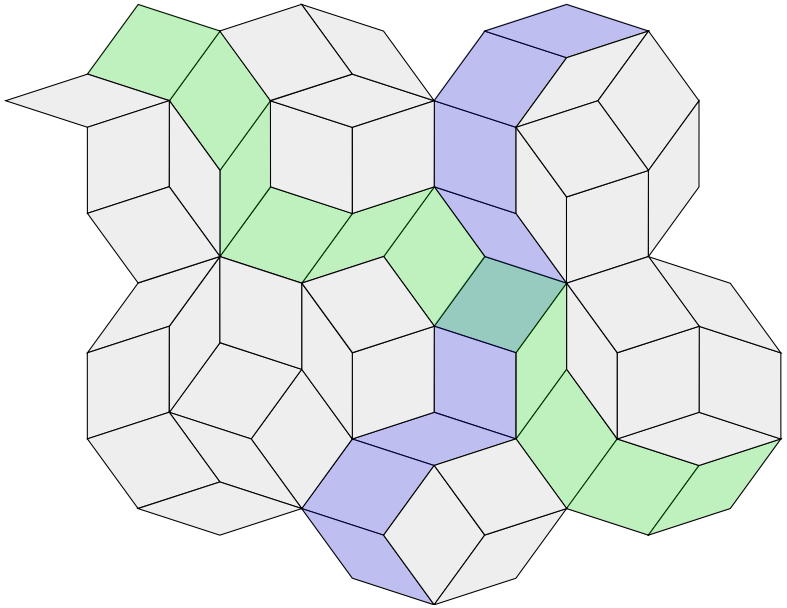

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Equivalence

Theorem (Gähler-Rhyner 1986)

Any multigrid dualization is a planar tiling, and conversely.

The grids are the intersection of the slope with the hyperplanes

$$G_i = \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x} | \vec{e}_i \rangle \in \mathbb{Z}\}$$

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Another way to define vectorial spaces

Definition (Grassmann coordinates)

The *Grassmann coordinates* of a vector space $\mathbb{R}\vec{u}_1 + \ldots + \mathbb{R}\vec{u}_d$ are the $d \times d$ minors of the matrix whose columns are the $\vec{u}_i$'s.

Q. How many Grassmann coordinates does have a subspace of $\mathbb{R}^n$?

Q. What are the Grassmann coordinates of a hyperplane?

Plücker relations

Theorem

A vector space is characterized by its Grassmann coordinates.

Q. What is the dimension of the set of d-dim. vector spaces of $\mathbb{R}^n$?

Plücker relations

Theorem

A vector space is characterized by its Grassmann coordinates.

Q. What is the dimension of the set of d-dim. vector spaces of $\mathbb{R}^n$?

Theorem

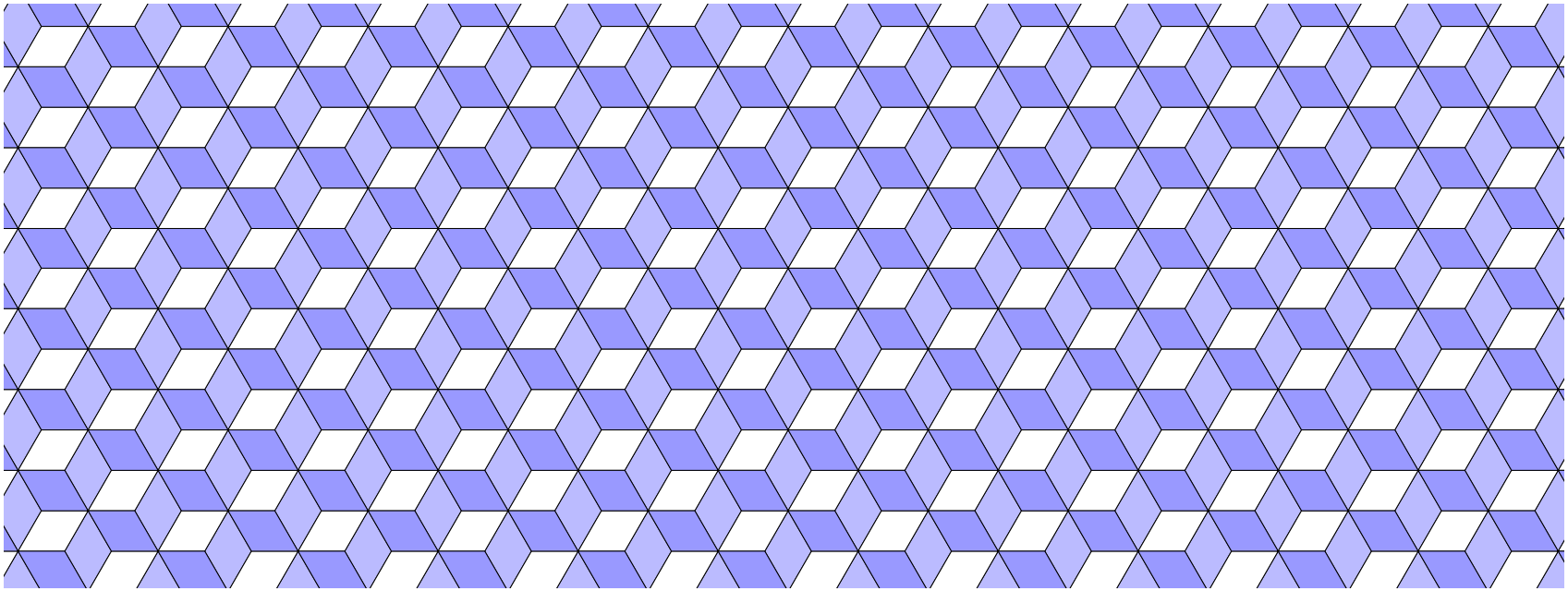
A non-zero real tuple $(G_{i_1,\dots,i_d})$ are the Grassmann coordinates iff, for any $1 \leq k \leq n$ and any two d-tuples of indices they satisfy

$$G_{i_1,\dots,i_d} G_{j_1,\dots,j_d} = \sum_{l=1}^{d} \underbrace{G_{i_1,\dots,i_d} G_{j_1,\dots,j_d}}_{swap\ i_k\ and\ j_l}.$$

Link with planar tilings

Proposition

The tile generated by $\vec{v}_{i_1}, \dots \vec{v}_{i_d}$ has frequency $|G_{i_1,\dots,i_d}|$.



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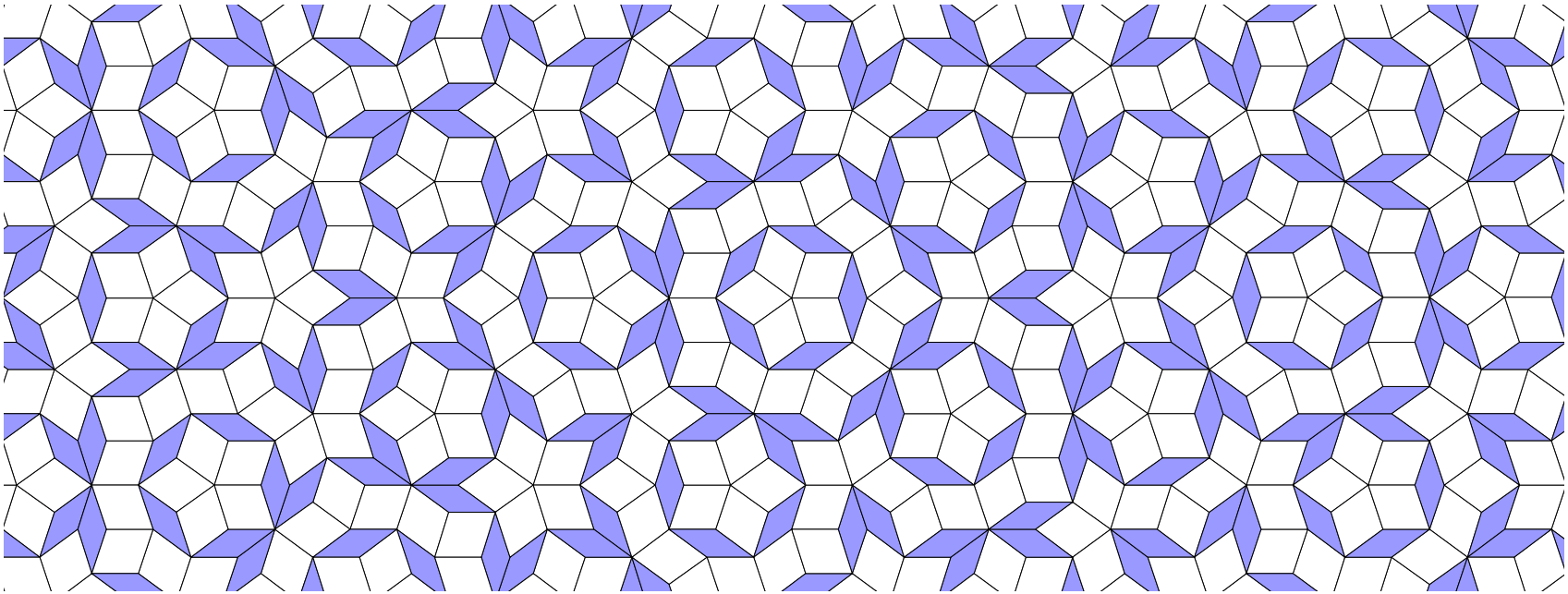

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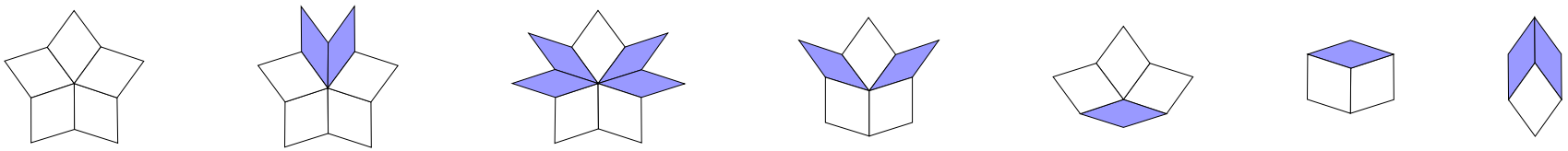
Pattern

Definition

A *pattern* of a tiling is a finite subset of the tiles of this tiling.



A *r-map* is a pattern formed by the tiles intersecting a closed r-ball.

The *r-atlas* of a tiling is th set of its r-maps.

Window

Definition

The *window* of a planar $n \to d$ tiling of slope E is the orthogonal projection of $E + [0,1]^n$ onto $E^\perp$.

Q. What is the window of a $2 \to 1$ planar tiling?

Window

Definition

The *window* of a planar $n \to d$ tiling of slope E is the orthogonal projection of $E + [0,1]^n$ onto $E^\perp$.

Q. What is the window of a $3 \to 1$ planar tiling?

Window

Definition

The *window* of a planar $n \to d$ tiling of slope E is the orthogonal projection of $E + [0,1]^n$ onto $E^\perp$.

Q. What is the window of a $4 \to 2$ planar tiling?

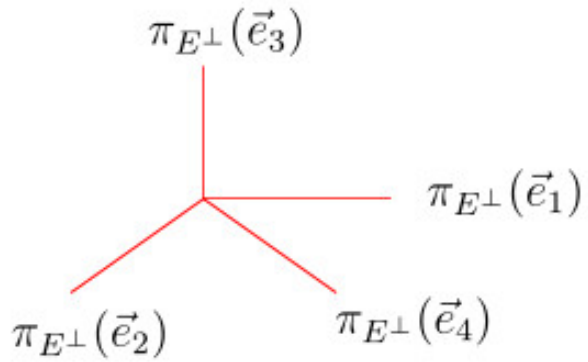
Window

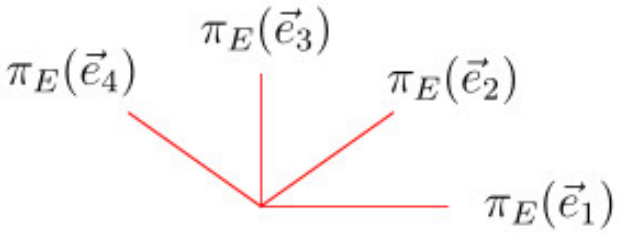
Definition

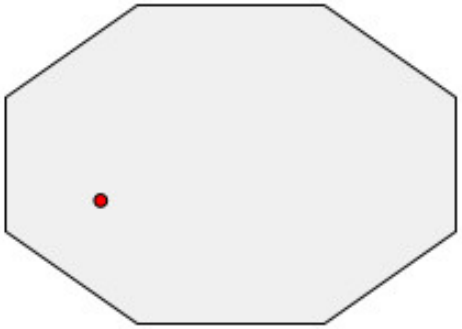
The *window* of a planar $n \to d$ tiling of slope E is the orthogonal projection of $E + [0,1]^n$ onto $E^\perp$.

Q. What is the window of a $5 \to 2$ planar tiling?

Tilings seen from the window







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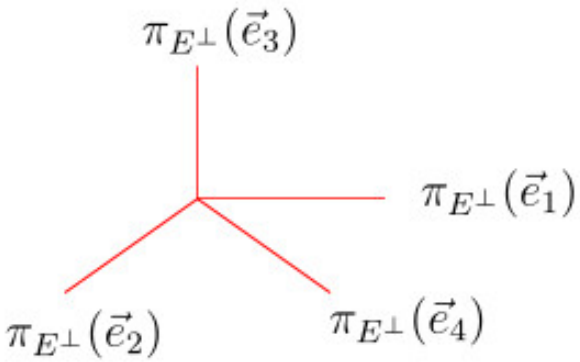

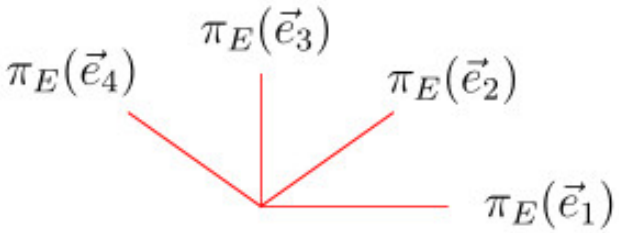

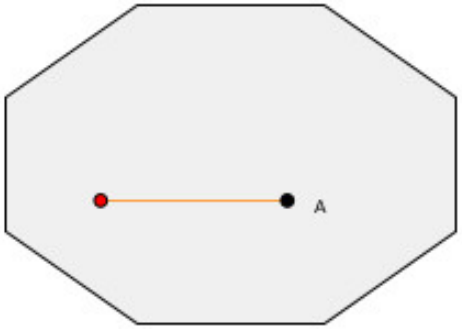

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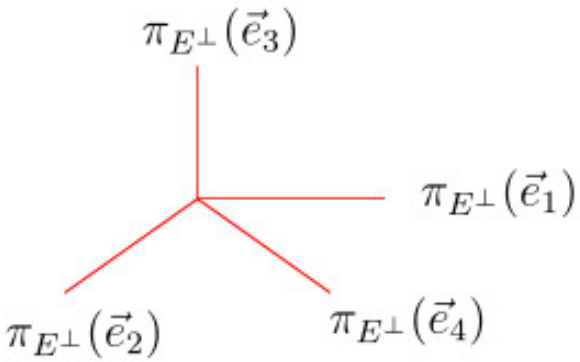

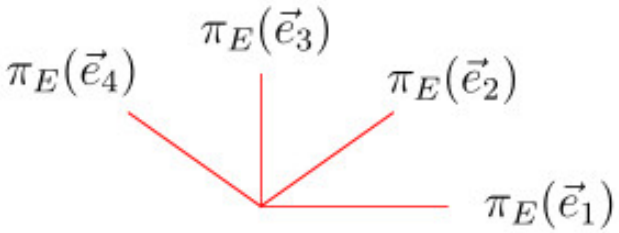

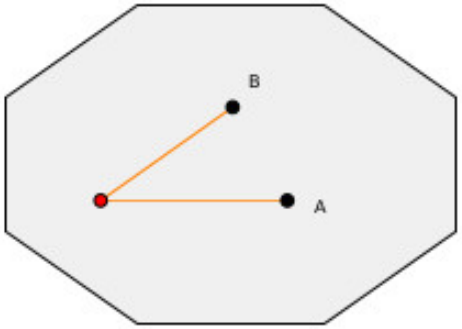

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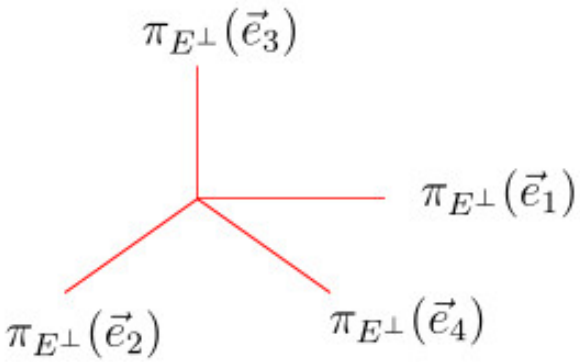

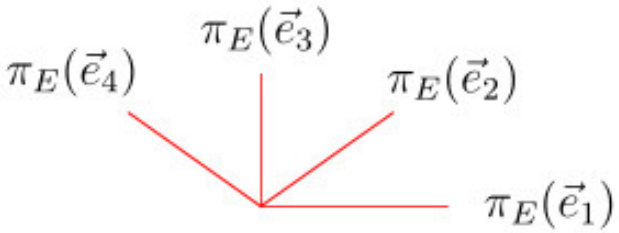

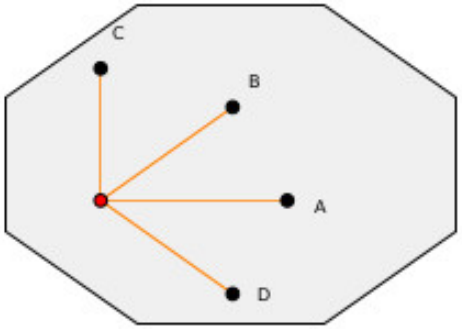

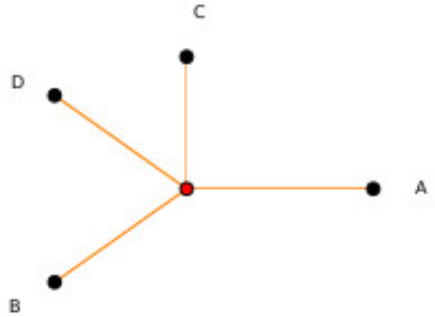

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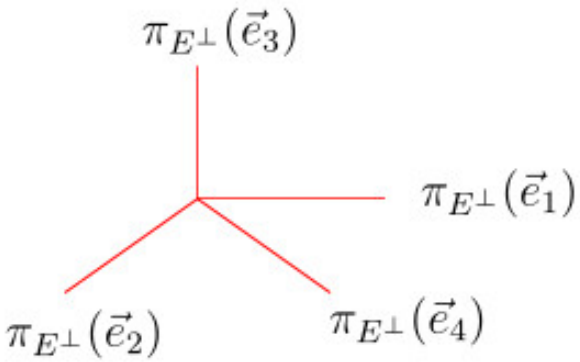
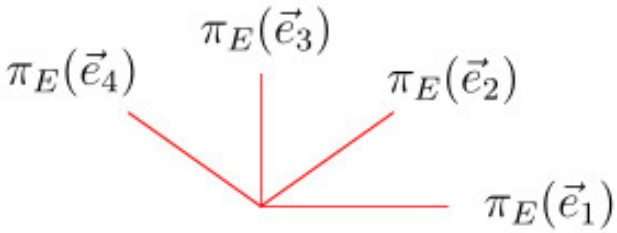
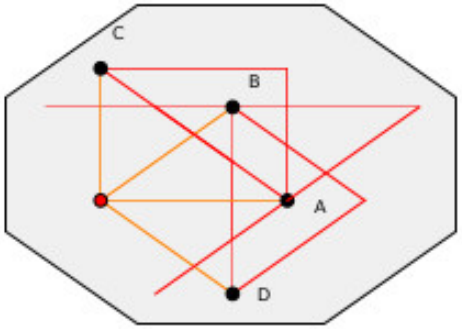
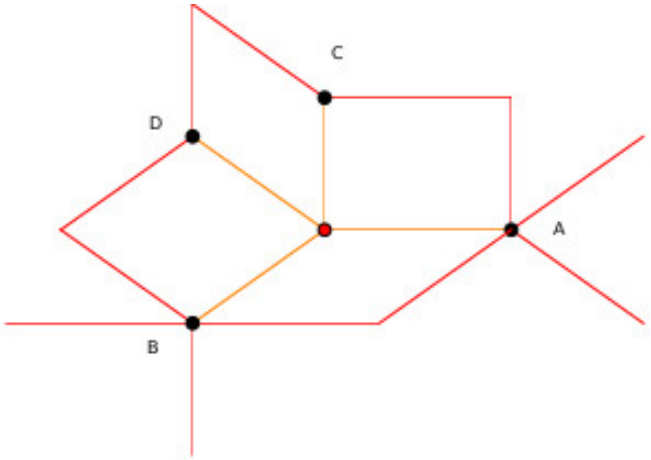
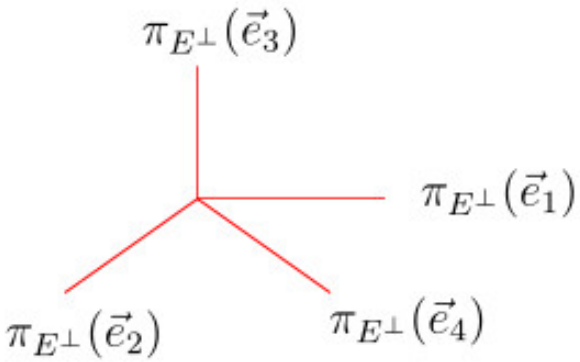


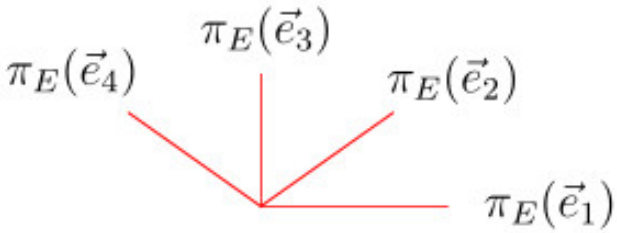


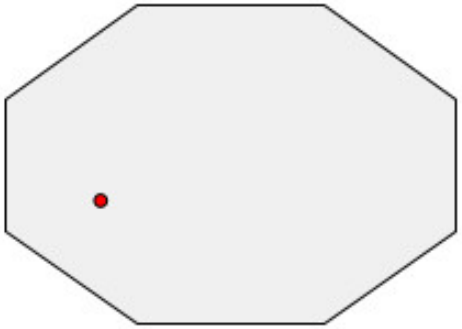


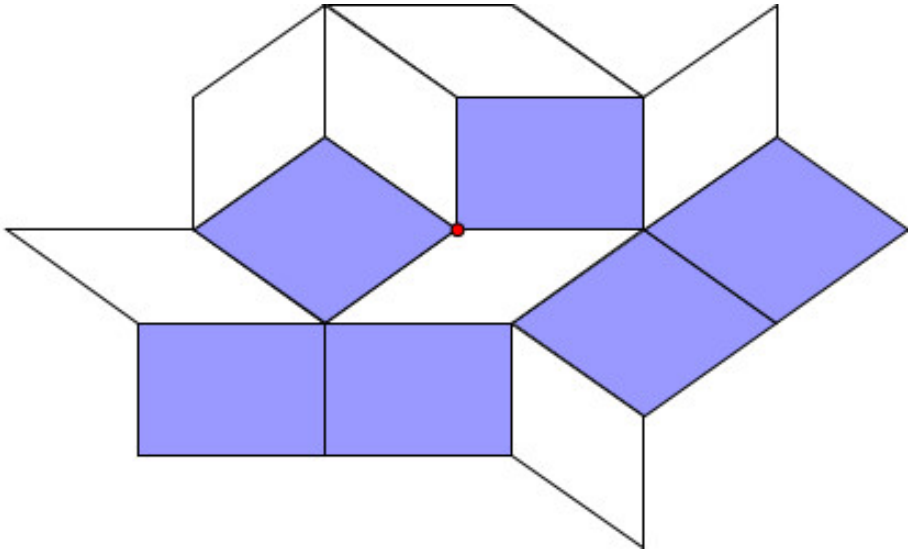


Tilings seen from the window

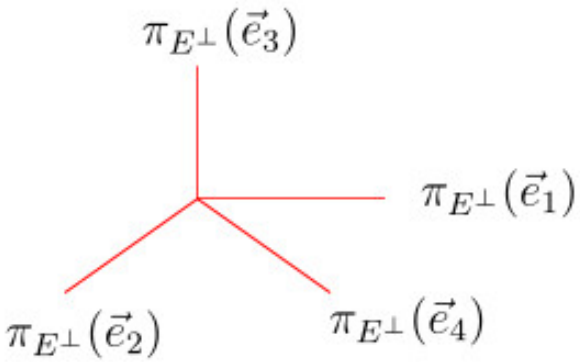


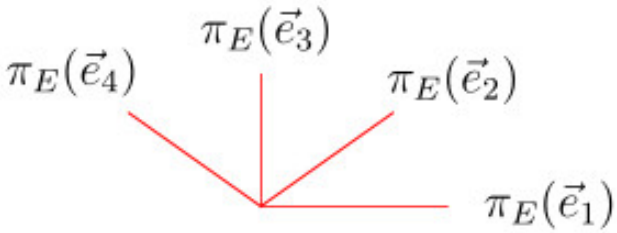


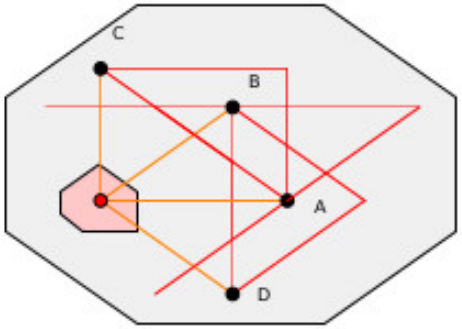


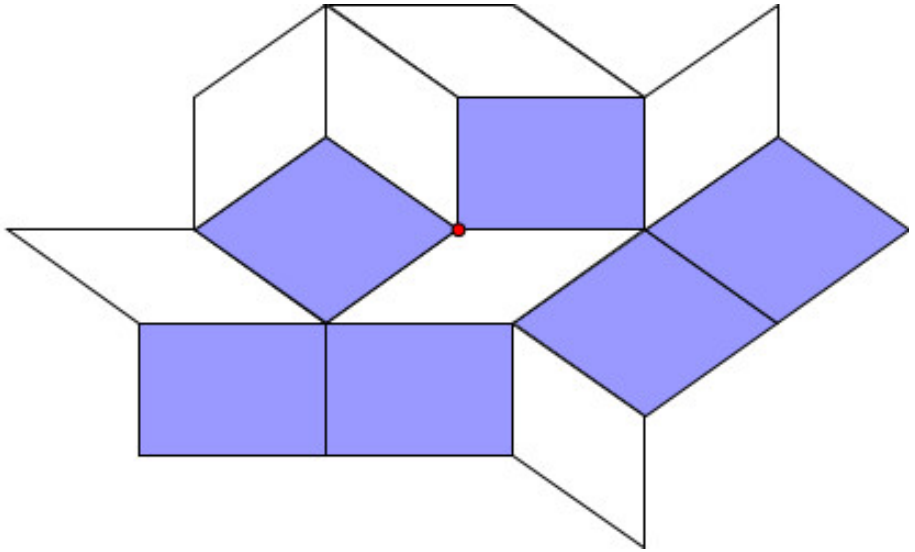


The region of a pattern









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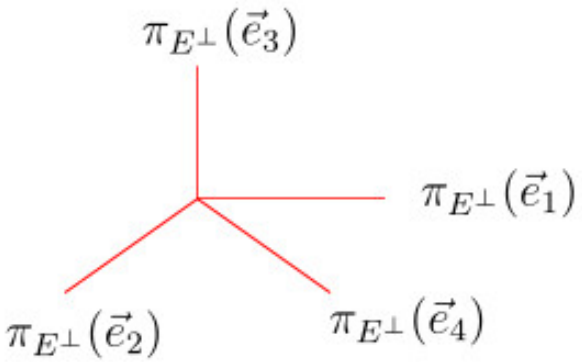

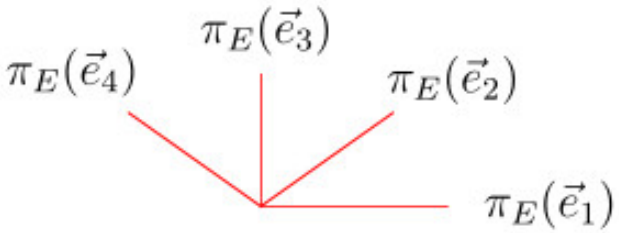

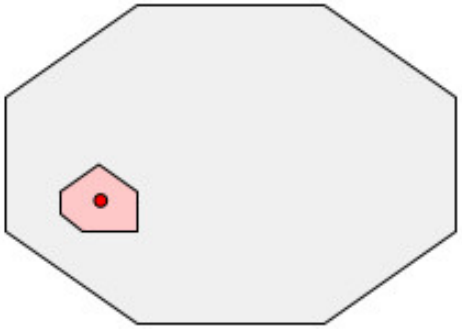

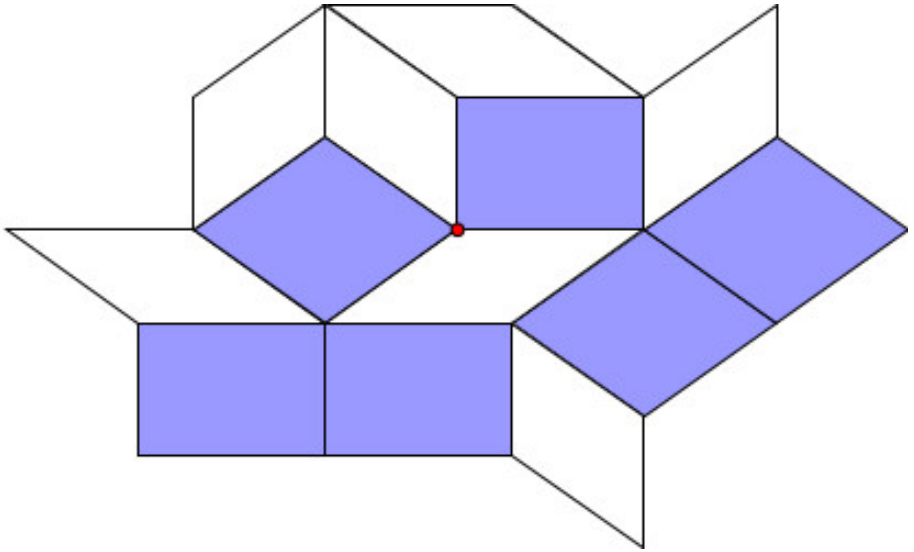

Counting patterns

Complexity: function which counts the size of the r-atlas.

Theorem (Julien 2010)

A generic planar $n \to d$ tiling has complexity $\Theta(r^{d(n-d)})$.

Q. What is the complexity of a Fibonacci word?

Q. What is the complexity of a Penrose tiling?

Quasiperiodicity

Definition (quasiperiodic or repetitive or minimal)

A tiling is *quasiperiodic* if whenever a pattern occurs somewhere, it reoccurs at uniformly bounded distance from any point.

Q. Is it true that periodic tilings are quasiperiodic?

Q. Is it true that non-periodic tilings are quasiperiodic?

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Definition (quasiperiodic or repetitive or minimal)

A tiling is *quasiperiodic* if whenever a pattern occurs somewhere, it reoccurs at uniformly bounded distance from any point.

Q. Is it true that periodic tilings are quasiperiodic?

Q. Is it true that non-periodic tilings are quasiperiodic?

Theorem

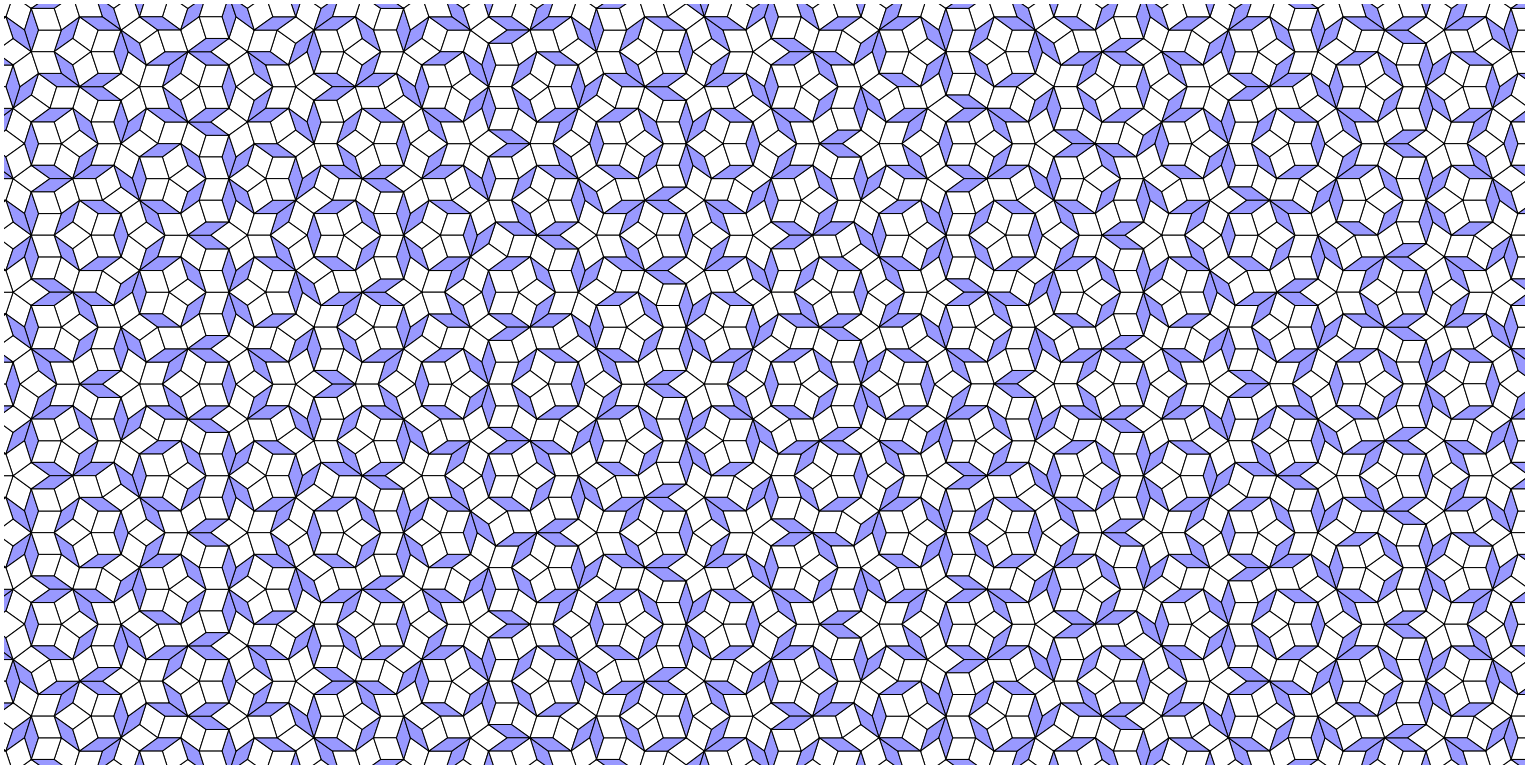
Planar tilings are quasiperiodic.

Patterns even have *frequencies*, related to the area of their regions.

Slope shift

Proposition

If a slope $a + E$ is in the smallest rational space containing $b + E$, then the planar tilings with these slopes have the same patterns.



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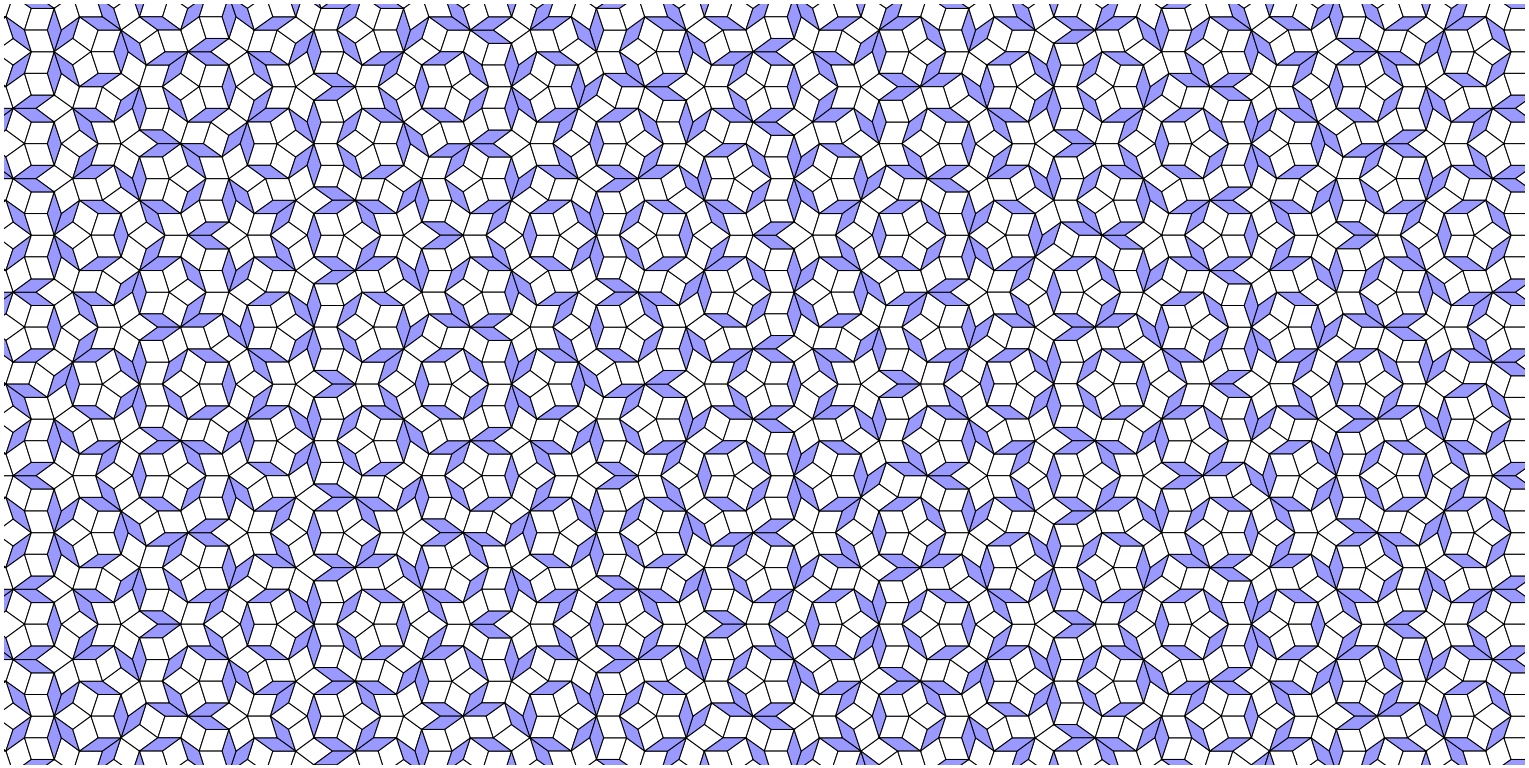

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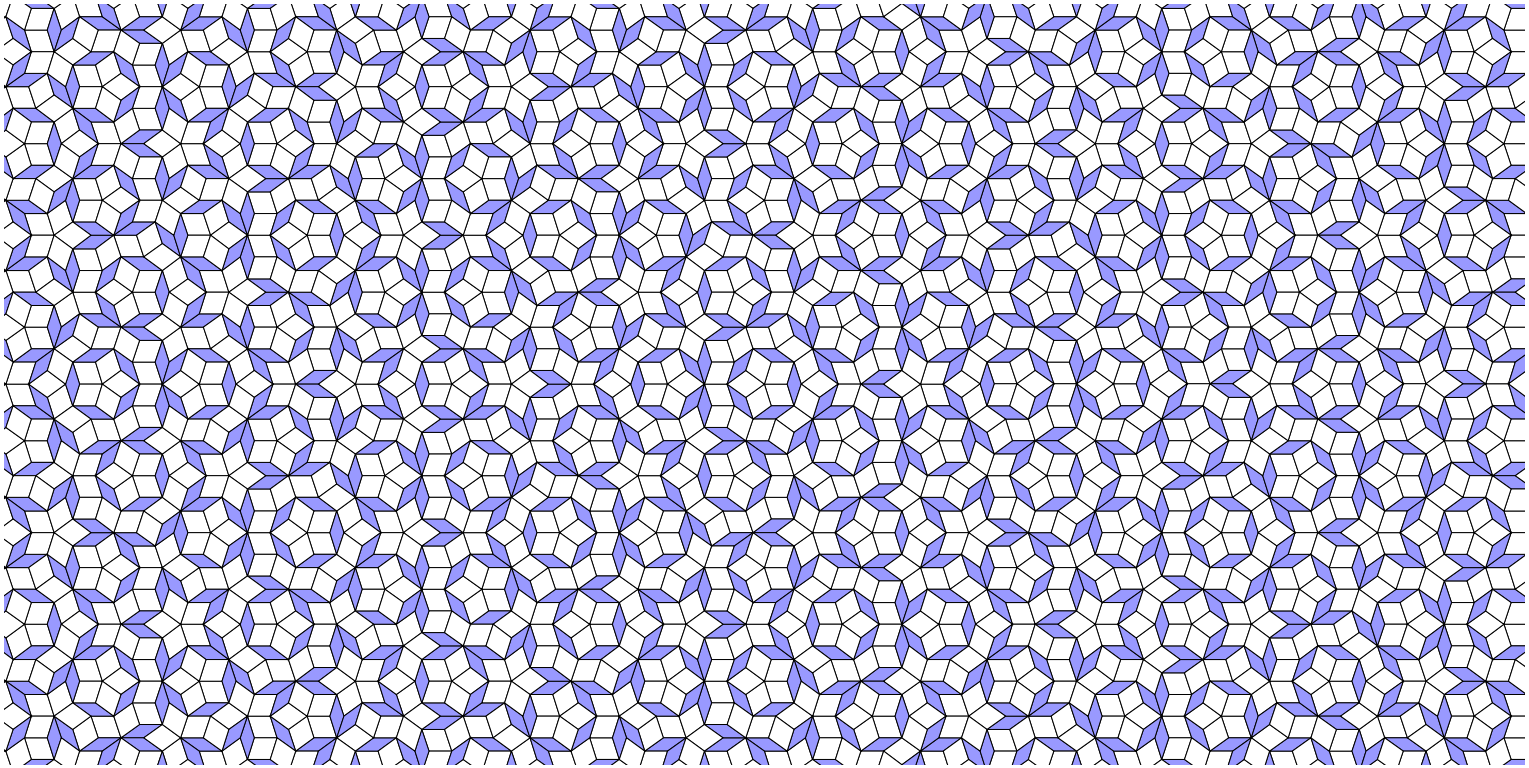

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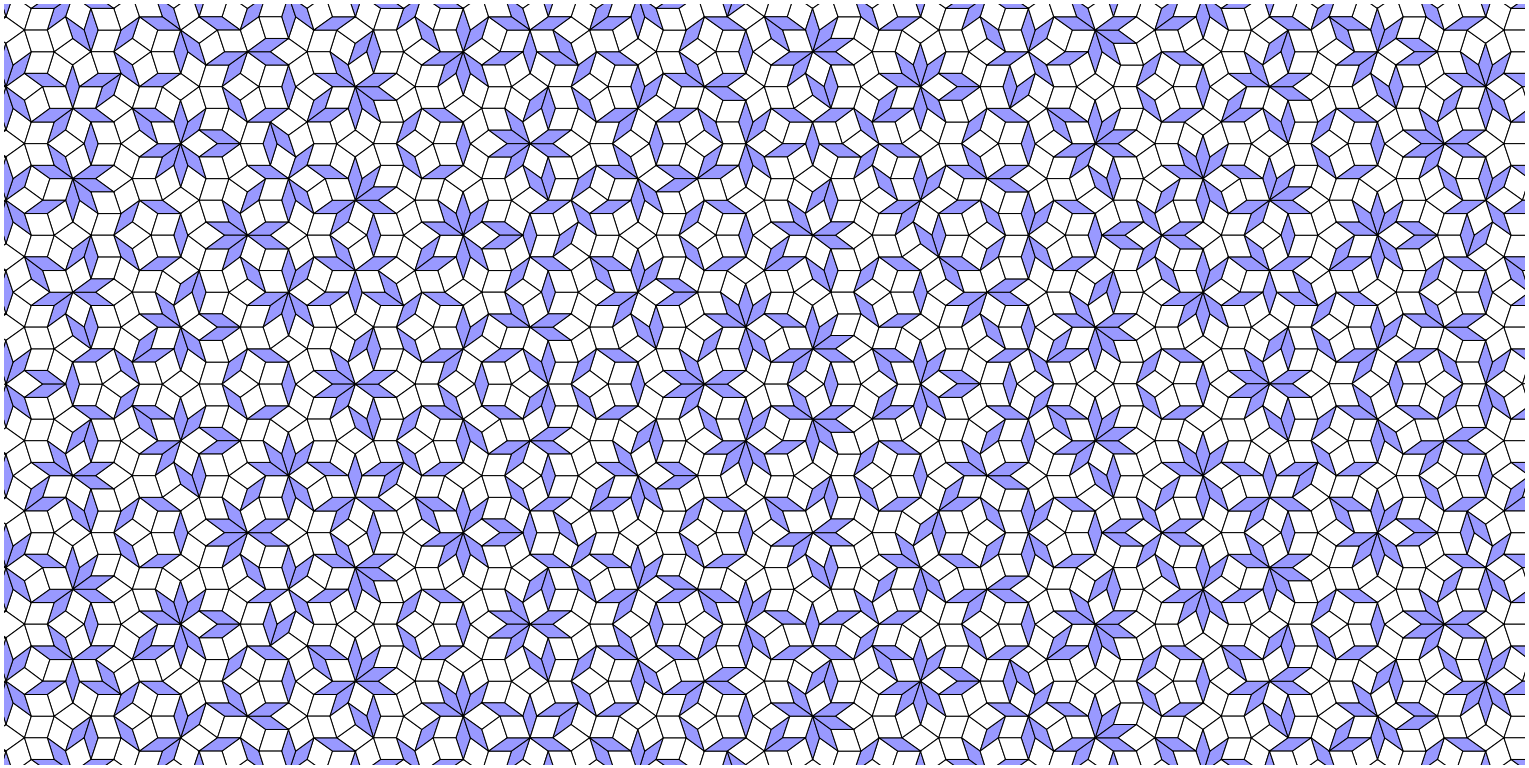

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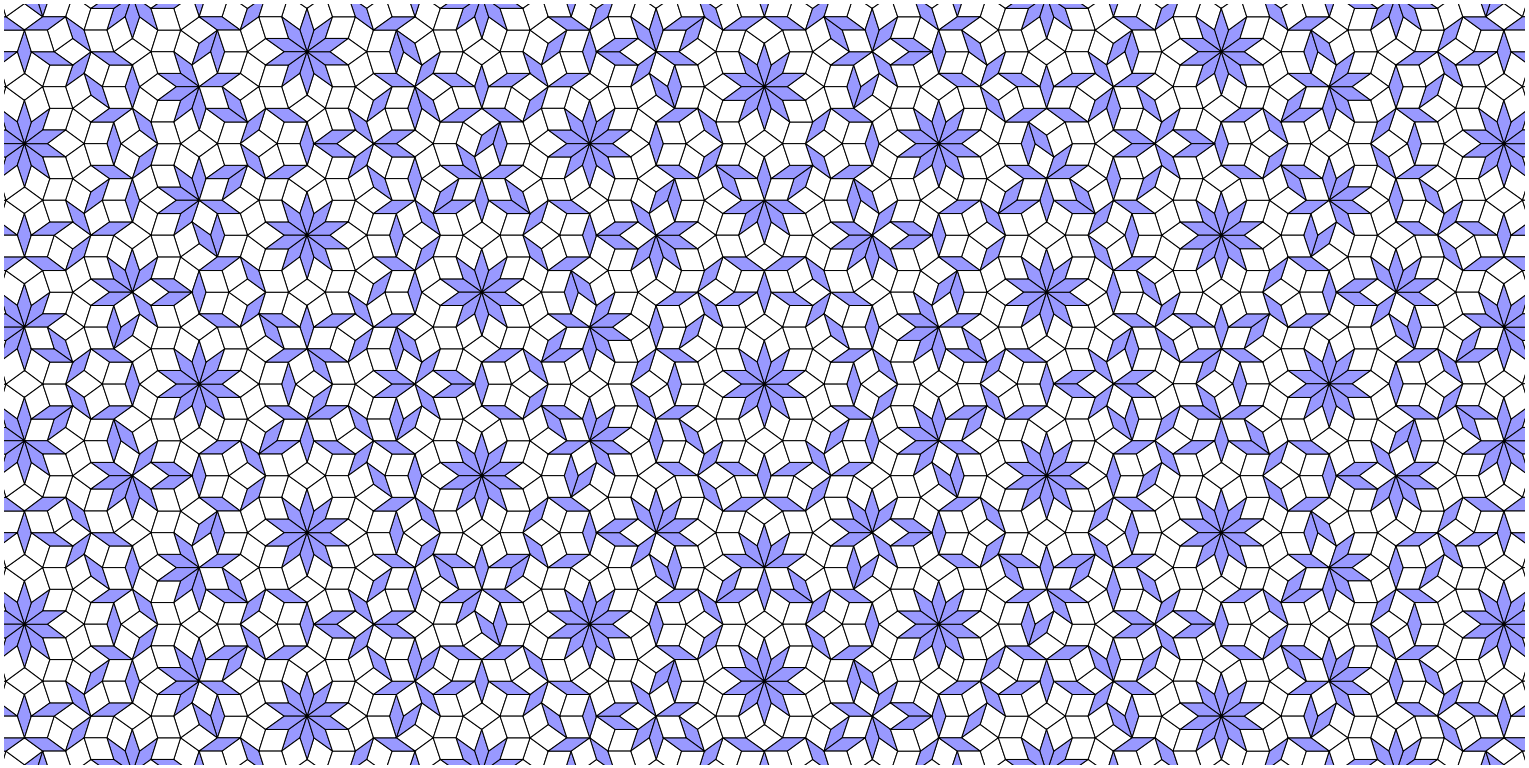

End of the first part

- Now: exercises.
- Evening: cut and project & window with SageMath.
- Tomorrow: which planar tilings do have local rules (SFT)?