Substitutions: Answers to Exercises
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Exercice 3. Complexity of the language of the subshift of the Fibonacci substitution.

We prove that $p(n) = n + 1$ for every integer n. Remark that a 0 is at the beginning of an image of a letter, since every image of a letter begins with a 0. The alphabet is of size 2, thus a word can only be extended by two letters on the right or on the left.

We need to introduce some notations : For a word v of length n, we denote the number of extensions on words of length $n + 1$ by $r(v)$ on the right, and by $l(v)$ on the left. Finally $b(v)$ denote the number of extensions in a word of length $n + 2$ with one letter on each side. For example we have $r(0) = 2, l(1) = 1, b(0) = 3$.

— First we consider a word v with two extensions on the right. (i.e $r(v) = 2$) This word must finish by 0. We look at the images of letters inside the fixed points. If the last 0 is an image of a letter, then it implies that its extension by the 1 is an image of a letter which is impossible by definition of σ. Thus the last 0 must be separated and we obtain that there exists only one $s \in \{\varepsilon, 1\}$ and $w \in \mathcal{L}$ such that $v = s\sigma(w)0$ and w also has two extensions on the right.

— Now consider word v with two extensions on the left. The word must begins by a 0. As previously we obtain that $v = \sigma(w)p$ with $p \in \{\varepsilon, 1\}$.

— Now consider word v with two extensions on the left **and** on the right. The two previous arguments show that $v = \sigma(w)0$. For $n = 1$ 0 is such a word, and the next one is $010 = \sigma(0)0$. We consider the map $\varphi(v) = \sigma(v)0$, and these words are of the form $\varphi^n(0)$ with $n \in \mathbb{N}$.

— We prove now that $p(n + 2) - p(n + 1) - [p(n + 1) - p(n)]$ counts the numbers of words of length n with two extensions on the left **and** on the right with multiplicity. The multiplicity of such a word v s equal by definition to $b(v) - r(v) - l(v) + 1$.

$$p(n+1) - p(n) = \sum_{v \in \mathcal{L}_n} (r(v) - 1), \quad p(n+2) - p(n+1) = \sum_{w \in \mathcal{L}_{n+1}} (r(w) - 1)$$

$$\begin{aligned} p(n+2) - p(n+1) - [p(n+1) - p(n)] &= \sum_{w=va} (r(av) - 1) - \sum_{v \in \mathcal{L}_n} (r(v) - 1) \\ &= \sum_{v \in \mathcal{L}_n} [\sum_{av} (r(av) - 1) - r(v) + 1] \\ &= \sum_{v \in \mathcal{L}_n} [b(v) - l(v) - r(v) + 1] \end{aligned}$$

— The word 0 can be extended in $001, 100, 101$ thus its index is equal to 0. Remark that every word in the sequence $(\varphi^n(0))_{n \in \mathbb{N}}$ has the same multiplicity as 0. We deduce by the previous point that for every integer $p(n + 2) - p(n + 1) - [p(n + 1) - p(n)] = 0$. Thus the sequence $p(n + 1) - p(n)$ is constant and is equal to 1. We deduce $p(n) = n + 1$.

Exercice 1 First we consider the Fibonacci substitution : Remark that now we can deduce that there exists only one word of each length with two extensions on one side. We deduce that the Rauzy graphs have the following form : only vertex has two outgoing edges. For the other vertices there is one ingoing edge and one outgoing edge.

For the Thue Morse substitution things are different as can be seen on the first Rauzy graphs.

Exercice 5.

For Tribonacci we can use the same method : The fixed point begins with

$$01.02.01.0.01.02.01.01.02.01.0\ldots$$

- Let v be a word such that $r(v) > 1$. Then the last letter is 0. This 0 can not be an image of a letter. Thus $v = s\sigma(w)0$ with $|s| < 2$.
- Let v be a word such that $l(v) > 1$. Then it also begins by 0 and we obtain $v = \sigma(w)p$ with $|p| < 2$.
- Let v be a word with $r(v) > 1, l(v) > 1$. We deduce of the two previous points that $v = \sigma(w)0$. Thus every word with at least two extensions on each side belongs to $(\varphi^n(0))_{n\in\mathbb{N}}$.
- The extansions of 0 are $001, 002, 101, 102, 201$. Thus we have $b(0) = 5, r(0) = l(0) = 3$. Its multiplicity is equal to $5 - 6 + 1 = 0$.
- We deduce that $p(n+1) - p(n)$ is constant, and we remark that $p(1) = 3, p(2) = 5$. We deduce $p(n) = 2n + 1$.

Exercice 4 For the letters 0 and 1 the frequencies are given by the eigenvector for the Perron value.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \lambda = \frac{1+\sqrt{5}}{2}$$

$$Mv = \lambda v, v = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$

$$\mu([0]) = \frac{\lambda}{1+\lambda} = \frac{1}{\lambda}, \mu([1]) = \frac{1}{\lambda^2}.$$

Then we remark that $[1] = [10]$ and $S^{-1}[1] = [01]$ since 1 can only be prolongated by 0. We obtain

$$\mu([10]) = \mu([01]) = \frac{1}{\lambda^2}$$

Since there are 3 words of length two, we deduce $\mu([00]) = 1 - \frac{2}{\lambda^2}$.

Exercice 6 A periodic point x fulfills $x = x + n\varphi + m$ with n, m integers. There is no periodic point since φ is an irrationnal number.

Consider the translation by φ on $\mathbb{T}^1$. The interval $(0, 1)$ with boundary points identified is a fundamental domain of the circle.

The map S is a bijective exchange of two intervals by translation. The ratio of these lengths is the same as for T. Thus the two maps are conjugated by an inflation map.

We consider one coding with $0, 1$ and one coding with A, B. We pass from one coding to the other by the following morphism $\begin{cases} 0 \mapsto AB \\ 1 \mapsto A \end{cases}$.

Second exercices session

Exercice 1 Remark that the inflations are different from each square. Thus the first think to do is to remark that it is possible to iterate the substitution. Indeed the image of 1 is a square of size 2, and the image of 3 is a rectangle of size 2. Thus their images match on an edge of length 2. The first iterations are

2	4	2	2	4	2
1	3	1	1	3	1
2	4	2	2	4	2
1	3	1	1	3	1
2	4	2	2	4	2
1	3	1	1	3	1

On every line and row you see the subshift of Fibonacci but the alphabets can be different. For example on the first line the alphabet is $\{1, 3\}$.

We deduce the incidence matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The matrix is equal to the tensor product of the Fibonacci matrix by itself.

Exercice 2 Consider a dilatation by λ. The system for the lengths is $\begin{cases} x + y = \lambda x \\ \lambda y = x \end{cases}$ We remark it is again the Fibonacci matrix. The two lengths form the eigenvector for the Perron value.

Exercice 3 The triangle appears with different orientations all multiples of the same number θ. The angle θ is not a rational multiple of π since $\tan\theta = \frac{1}{2}$. Thus the rotation of this angle has infinite order. Thus this map is not a substitution with the definition of the notes, or you need to use an infinite number of prototiles.

To finish we prove by contradiction that θ is not a rational multiple of π. First remark that $\cos\theta = \frac{2}{\sqrt{5}}$. Then we deduce $\cos(2\theta) = 2\cos^2\theta - 1 = \frac{3}{5}$. Now we assume that $\theta \in \pi\mathbb{Q}$, then 2θ is in the same field, and is equal to some $\frac{p}{q}\pi$. Thus $\cos(2q\theta) = (-1)^p$. Now consider T_q the Chebychev polynomial : this is the polynomial such that $T_q(\cos x) = \cos(qx)$. We have $\cos(2q\theta) = T_q(\cos 2\theta) = (-1)^p = T_q(3/5)$. We deduce that $\frac{3}{5}$ is a root of the polynomial $T_q + (-1)^q$. By a classical argument this proves that 5 divides the dominant coefficient of T_q. But this coefficient is equal to some power of 2. We deduce a contradiction.