Tileability

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Tiling

Definition

A *tiling* is a way of covering a *given region* using a given set of *tiles* completely and without any overlap.

FIGURE: Regions

FIGURE: Tiles

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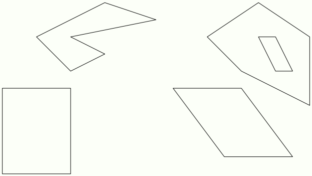

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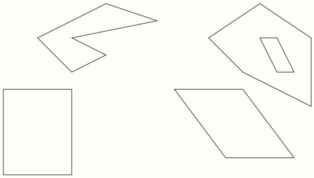

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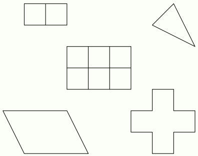

FIGURE: Tiles

Statement of the problem

More precisely here we consider the following problem :

Problem

Is a finite figure drawn on a plane grid (*i.e.*, a set of cells of $\mathbb{Z} \times \mathbb{Z}$ tilable with a given set of tiles ?

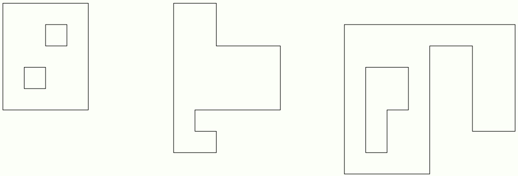

FIGURE: Set of cells of $\mathbb{Z}^2$

Tiles

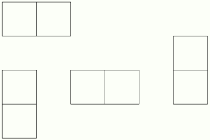

FIGURE: Dominoes

FIGURE: Bars

FIGURE: Rectangles

FIGURE: Polyominoes

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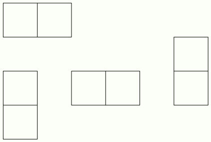

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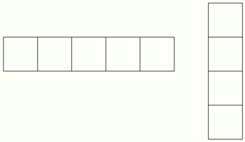

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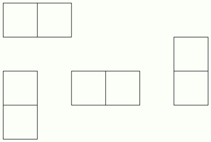

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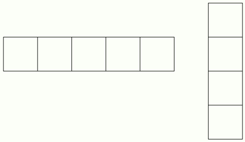

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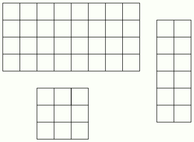

FIGURE: Rectangles

FIGURE: Polyominoes

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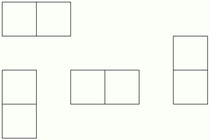

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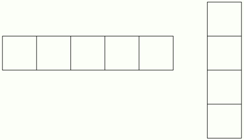

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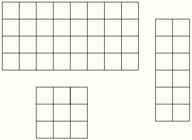

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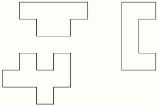
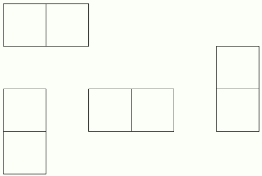


FIGURE: Polyominoes

Outlines of the lecture

- Tiling with dominoes (see also Lecture on Random Tilings)
- Tiling with bars
- Tiling with rectangles
- Tiling with a polyomino

I. Tiling with dominoes



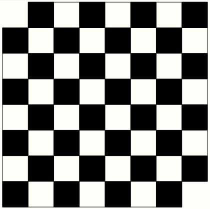
Problem

Can we tile a region of $\mathbb{Z}^2$ with horizontal or vertical dominoes, that is 1×2 and 2×1 bars ?

A first example

Problem

Remove two opposite corners of an 8×8 chessboard.
Is it possible to tile the resulting figure with dominoes ?



- The cells of chessboard are colored black and white alternatingly. This coloring is crucial in answering the question.
- Regardless of where it is placed, a domino will cover one black and one white square of the board.
- Therefore, 31 dominoes will cover 31 black squares and 31 white squares. However, the board has 32 black squares and 30 white squares in all, so *a tiling does not exist*.

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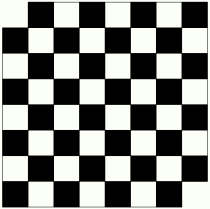

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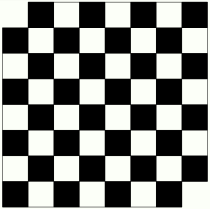

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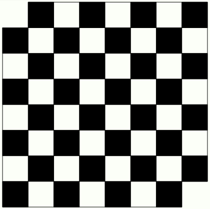

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A second example

Problem

Remove one black square and one white square from the chessboard. Is it possible to tile the resulting board with dominoes ?

The answer is yes, regardless of which squares we remove.

- Consider any closed path that covers all the cells of the chessboard.
- Start traversing the path from the point immediately after the black hole of the chessboard. Cover the two first celld of the path with a domino ; they are white and black, respectively. Then cover the two next ones with a domino ; they are also white and black, respectively. Continue in this way, until the path reaches the second hole of the chessboard. This second hole is white, so there is no gap between the last domino placed and this hole.
- Skip this hole, and continue covering the path with succes- sive dominoes. When the path returns to the first hole, there is again no gap between the last domino placed and the hole.
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A third example

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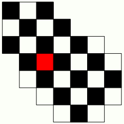
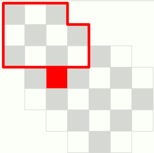
Is it possible to tile the figure below with dominoes ?





- Consider the six black squares in the red perimeter. They are adjacent to a total of five white squares.
- Six different tiles would to cover the six black squares, and each one of these tiles would have to cover one of the five adjacent white squares. This makes a tiling *impossible*.

Hall's theorem (1935)

There exists a dominoes tiling of a figure if and only

- there are the same number of white and black cells in the figure
- and each subset of k white (resp. black) cells has at least k black (resp. white) neighbors.

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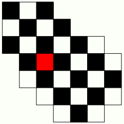

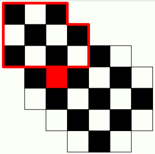

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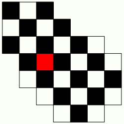

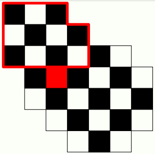

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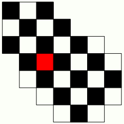

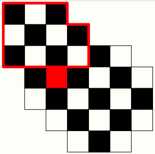

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Bipartite graph and perfect matching

A *bipartite graph* is a graph whose vertices can be partitioned into two disjoint subsets X and Y such that every edge connects a vertex in X to one in Y.

A *perfect matching* is a set of edges without common vertices that cover all vertices.

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There exists a perfect matching in a bipartite graph whose set of vertices is partitioned into $X \cup Y$ if and only

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Tiling with dominoes and perfect matchings

The problem of *tiling a region by dominoes* can be transformed into the search of a **perfect matching in a non-directed graph bipartite graph** in the following way.

- The vertices of the graph are the cells of the region to be tiled.
- There is a non-directed edge between each pair of (horizontally or vertically) adjacent cells.
- The cells are colored black and white alternatingly and partitioned following their color.

▲ Then there exists a tiling of the region with dominoes if and only if there exist a *perfect matching* in the corresponding bipartite graph. The edges of the matching are in bijection with the dominoes of the tiling.

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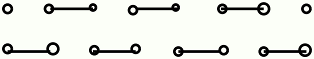
How to find a perfect matching ? A tiling by dominoes ?

Given a matching M, an *M-augmenting path* is is a path in which the edges belong alternatively to the matching and not to the matching that starts from and ends on free (unmatched) vertices.



An augmentating path can be used to make the matching bigger.

Berge theorem (57)

Let M be a matching in a graph. Then either M is a maximum cardinality matching or there exists an M-augmenting path.

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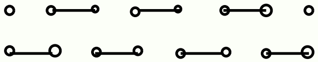

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In a bipartite graph, the **augmenting path algorithm** builds a maximal matching

- beginning from the empty matching,
- by finding an augmenting path from each $x \in X$ to Y
- and adding it to the matching if it exists.

As each path can be found in $\mathcal{O}(|E|)$ time with with for example a breadth-first search or a depth-first search, the augmenting path algorithm finds a perfect matching or prove that there is not such matching, in quadratic time $\mathcal{O}(|V| \cdot |E|)$.

Proposition

A figure of surface n can be tiled by 1×2 and 2×1 tiles, or proved not to be tileable, in quadratic time in n.

An improvement over this is the Hopcroft–Karp algorithm, which runs in $\mathcal{O}(n^{3/2})$

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Enumeration of tilings by dominoes

- A lot of things are known about tiling by dominoes
- A theorem independently proved to Kasteleyn and Temperley and Fisher (see Lecture on Random tilings) give the number of tilings of a figure by dominoes.
- The statement of this result in the particular case of tilings of rectangles is the following

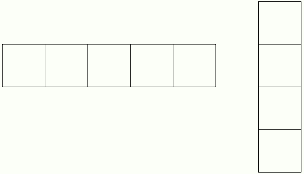
Theorem (Kasteleyn, Temperley-Fisher 1961)

The number of tilings of a $2m \times 2n$ rectangle with $2mn$ dominoes is equal to

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

Proof is based on the fact that the number of perfect matchings of a graph can be calculated via the determinant of a matrix closely related to the graph.

II. Tiling with bars

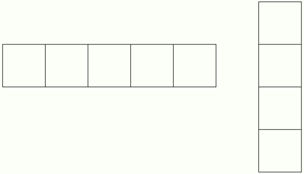


Problem

Can we tile a region of $\mathbb{Z}^2$ with horizontal or vertical bars, at least one of them having a length greater than two ?

If the region is a rectangle ? A polygon ? A general region with possible holes ?

II. Tiling with bars



Problem

Can we tile a region of $\mathbb{Z}^2$ with horizontal or vertical bars, at least one of them having a length greater than two ?
If the region is a rectangle ? A polygon ? A general region with possible holes ?

Tiling a rectangle with a bar

Proposition (Klarner 1966)

An $m \times n$ rectangle can be tiled with $1 \times b$ rectangles if and only if b divides m or n.

• If b divides m or n an $m \times n$ rectangle can be cut into $1 \times b$ rectangles in a natural way.

• Conversely suppose an $m \times n$ rectangle has been tiled with $1 \times b$ rectangles and that m is not divisible by b :

$$m = qb + r, \text{ with } 0 < r < b.$$

- Then number from 1 to m the m rows of length n.
- Let $f_1, f_2, \ldots f_b$ denote distinct colors and color the c-th row f_i if $c \sim i \mod b$.

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Tiling a rectangle with a bar

We shall use a coloring argument (counting colored cells in two different ways) to prove that n is necessarily divisible by b.

- Let c_i denote the total number of cells in the rectangles colored f_i, then

$$c_i = \begin{cases} qn + n, & \text{for } 1 \leq i \leq r, \\ qn, & \text{for } r < i \leq b. \end{cases}$$

- In the tiling, each $1 \times b$ rectangle is monocolor if it is horizontal and contains exactly one cell of each color is it vertical.
  - • Let x denote the number of vertical $1 \times b$ rectangles
  - • Let y_i denote the number of horizontal $1 \times b$ rectangles with every cell colored f_i for $1 \leq i \leq b$.

  Then $c_i = x + by_i$, and $c_i = x + by_i \sim x + by_j = c_j \mod b$.
- In particular $c_1 = qn + n \sim qn = c_{r+1} \mod b$, so $n \sim 0 \mod b$.

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Tiling a polygon

Theorem (Kenyon-Kenyon 1992)

A polygon of surface n can be tiled by $1 \times \ell$ and $k \times 1$ tiles, or proved not to be tileable, in time linear in n.

- This algorithm generalizes a domino tiling algorithm due to Thurston based on ideas of Conway and Lagarias which rely on geometric group theory and the notion of height function.
- The notion of height function will presented in detail and used in both Lectures on *Random tilings* and *Flip dynamics*
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A decision problem

- A **decision problem** is a question in some formal system with a yes-or-no answer, depending on the values of some input parameters.
- For example, the problem ”given a set of tiles T and a plane region R, does T tile R ?” is a decision problem. The answer can be either ’yes’ or ’no’, and depends upon T and R.

An NP-complete problem

- A decision problem is **NP-complete** when it is both in NP and NP-hard.
- Intuitively, **NP** is the set of all decision problems for which the instances where the answer is ”yes” have efficiently verifiable proofs of the fact that the answer is indeed ”yes”.
  More precisely, these proofs have to be verifiable in polynomial time by a deterministic Turing machine (see Lecture on *Cellular automata*).
- **NP-hard** is a class of problems that are, informally, ”at least as hard as the hardest problems in NP”.
  More precisely, a problem H is NP-hard when every problem L in NP can be reduced in polynomial time to H.
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Tiling an arbitrary region.

Problem

Can we tile a region of $\mathbb{Z}^2$ with horizontal or vertical bars, at least one of them having a length greater than two ?

In the general case, the result is completly different.

Theorem (Beauquier-Nivat-Rémila-Robson 1995)

Deciding whether a figure can be tiled with $1 \times \ell$ and $k \times 1$ bars is an NP-complete problem as soon as k or $\ell > 2$.

- The result is based on a reduction of this problem to a classical NP-complete problem.
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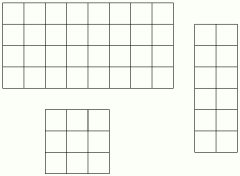
III. Tiling with rectangles



A. Tiling a rectangle with another rectangle

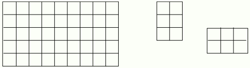
Problem

When can an $m \times n$ rectangle be tiled with $a \times b$ rectangles (in any orientation) ?

Tiling a rectangle with another rectangle

First example

Can a 5×9 rectangle be tiled with 2×3 rectangles ?



- This is impossible.
- Each 2×3 rectangle contains 6 cells,
- while the number of cells in a 5×9 rectangle is 45, which is not a multiple of 6.

A necessary condition

For a tiling to be possible, the number of cells of the large rectangle must be divisible by the number of cells of the small rectangle.

▲ Is this condition enough ?

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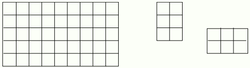

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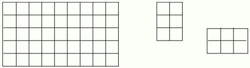

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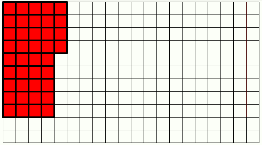
Tiling a rectangle with another rectangle

Second example

Can a 11×20 rectangle be tiled with 4×5 rectangles ?



- The number of tiles needed is 11
- But if the 11 cells of the first column can be covered with 4×5 tiles, then 11 can be written as a sum of 4s and 5s, which is impossible.

A necessary condition

For a tiling to be possible, the lengths of the sides of the large rectangle must be decomposable as sums of the lengths of the sides of the small rectangle.

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Tiling a rectangle with another rectangle

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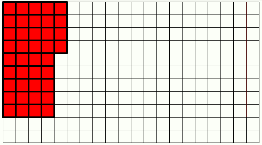

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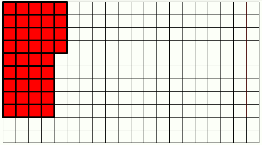

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Tiling a rectangle with another rectangle

Third example

Can a 10×15 rectangle be tiled with 1×6 rectangles ?

- 150 is a multiple of 6
- Both 10 and 15 can be written as a sum of 1s and 6s.
- However, this tiling problem is still impossible !
- Neither 10 nor 15 are divisible by 6 ! (Previous proposition)

Theorem (de Bruijn-Klarner 1969)

An $m \times n$ rectangle can be tiled with $a \times b$ rectangles if and only if :

- The first row and first column can be covered (*i.e.*, m and n can be expressed in the form $ax + by$ with $x, y \geq 0$).
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Tiling a rectangle with another rectangle (Proof)

- If a divides m and b divides n, an $m \times n$ rectangle can be cut into $a \times b$ in a natural way.
- If ab divides m and $n = ax + by$, with $x, y \geq 0$, the rectangle can be cut into x strips $a \times m$ and y strips $b \times m$, and these strips can be cut into $a \times b$ rectangles because both a and b divide m.

Conversely, recall that

Proposition

An $m \times n$ rectangle can be tiled with $1 \times b$ rectangles if and only if b divides m or n.

- If an $m \times n$ rectangle R has been tiled with $a \times b$ rectangles each side of R can be expressed in the form $ax + by$, with $x, y \geq O$, because each side of R is the union of length a or b segments.
- Since each $a \times b$ rectangle can be cut into $1 \times b$ rectangles, b divides m or n by the previous proposition ;
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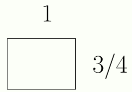
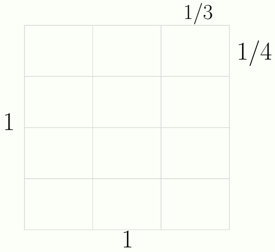
B. Tiling a square with similar rectangles

Problem

Can a square be tiled with finitely many rectangles similar to a $1 \times x$ rectangle (in any orientation) ?
In other words, can a square be tiled with finitely many rectangles, all of the form $a \times ax$ (where a may vary) ?

We begin with a first simple example ;
Let $x = \frac{3}{4}$.





In this case, we only need one size of rectangle, because we can tile a

- a 1×1 square with **12** $\frac{1}{3} \times \frac{1}{4}$ rectangles ($a = \frac{1}{3}$)
- or equivalently a 3×3 square with **12** $1 \times \frac{3}{4}$ rectanglses.

Note that $x = \frac{3}{4}$ satisfies the equation $4x - 3 = 0$.

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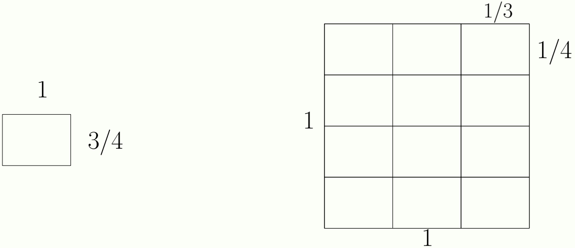

In this case, we only need one size of rectangle, because we can tile a

- a 1×1 square with **12** $\frac{1}{3} \times \frac{1}{4}$ rectangles ($a = \frac{1}{3}$)
- or equivalently a 3×3 square with **12** $1 \times \frac{3}{4}$ rectanglses.

Note that $x = \frac{3}{4}$ satisfies the equation $4x - 3 = 0$.

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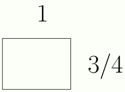

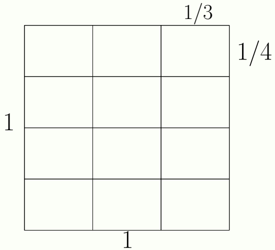

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A similar construction will work for any positive rational number

$$x = \frac{p}{q}.$$

When $x = \frac{p}{q}$ is a positive rational number, a tiling of the unit square is obtained

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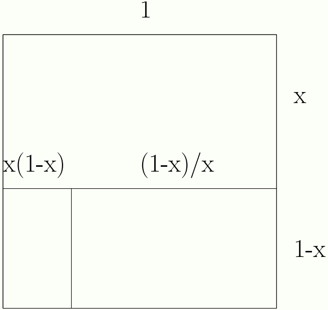
Tiling a square with similar rectangles- Example

Let us find a tiling using three similar rectangles of different sizes.



- Say that the largest rectangle has dimensions $1 \times x$.
- Then the second largest one has dimensions $(1-x) \times \frac{(1-x)}{x}$.
- The last one has dimension $(1-x) \times x(1-x)$.
- Therefore $x(1-x) + \frac{(1-x)}{x} = 1$, x satisfies the equation

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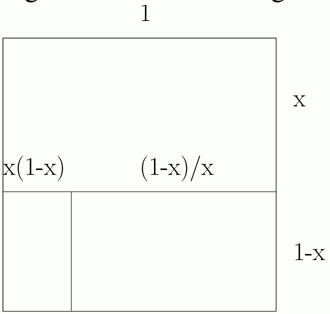

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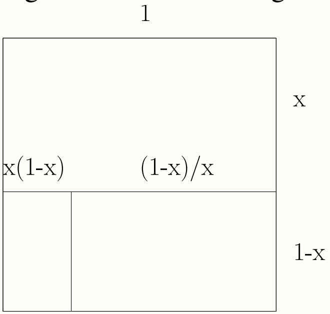

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- One value of x which satisfies this equation is
  $x = 0.5698402910\ldots$
  For this value of x, the tiling problem can be solved as above.
- The two other solutions are approximately $0.215 + 1.307i$ and $0.215 - 1.307i$.
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Tiling a square with similar rectangles

Theorem (Freiling-Rinne, Laczkovich-Szekeres 1995)

Let $x > 0$. The three following statements are equivalent

1. It is possible to tile a square with rectangles similar (up to a rotation) to the $1 \times x$ rectangle.
2. There exist rational positive numbers $c_1, \ldots, c_n$ such that
$$c_1 x + \cfrac{1}{c_2 x + \cfrac{1}{\ddots + \cfrac{1}{c_n x}}} = 1.$$
3. The real x is algebraic and every (complex) conjugate of x has positive real part.

Here the key ingredient is the following theorem

Theorem (Wall 1945)

Let $P(z) = z^n + p_{n-1}z^{n-1} + \cdots + p_0$.
Let $Q(z) = p_{n-1}z^{n-1} + p_{n-3}z^{n-3} + \cdots$ be the alternant of $P(z)$.
All roots of $P(z)$ have positive real part if and only if

$$\frac{Q(z)}{P(z) - Q(z)} = \frac{-1}{c_1 z + \frac{1}{c_2 z + \frac{1}{\ddots + \frac{1}{c_n z}}}}$$

where each $c_i > 0$.

- Let $x > 0$ be an algebraic number having conjugate roots with positive real parts.
- Let P be the minimal polynomial of degree n for x over the rationals. By Wall's Theorem, we have

$$\frac{Q(z)}{P(z) - Q(z)} = \frac{-1}{c_1 z + \frac{1}{c_2 z + \frac{1}{\ddots + \frac{1}{c_n z}}}}$$

where each $c_i > 0$.

- Since $P(x) = 0$, $P(x) - Q(x) = -Q(x)$ so $\frac{-Q(x)}{P(x)-Q(x)} = 1$ and

$$1 = \frac{1}{c_1 x + \frac{1}{c_2 x + \frac{1}{\ddots + \frac{1}{c_n x}}}}$$

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Tiling a square with similar rectangles : Statement 2 $\to$ Statement 1

- Take a unit square.
- Cut off a rectangle of ratio (that is, the length of the horizontal side divided by the length of the vertical one) $c_1 x$ from the square by a vertical cut. The remaining part is a rectangle of ratio

$$1 - c_1 x = \cfrac{1}{c_2 x + \cfrac{1}{\ddots + \frac{1}{c_n x}}}.$$

- Now cut off a rectangle of ratio $\frac{1}{c_2 x}$ from the remaining part by a horizontal cut. We get a rectangle of ratio

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- The value $x = \sqrt{2}$ does satisfy a polynomial equation with integer coefficients, namely $x^2 - 2 = 0$.
  However, the other root of the equation is $-\sqrt{2} < 0$.
  Thus a square cannot be tiled with finitely many rectangles similar to a $1 \times \sqrt{2}$ rectangle.
- On the other hand, $x = \sqrt{2} + \frac{17}{12}$ satisfies the quadratic equation

$$144x^2 - 408x + 1 = 0,$$

  whose other root is $-\sqrt{2} + \frac{17}{12} = 0.002453\ldots > 0$.
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C. Tiling a rectangle with squares

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Can a rectangle be tiled with finitely many squares ?

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Tiling a rectangle with squares

- An $m \times n$ rectangle, where m and n are integers, can be tiled by mn unit squares.
- Thus a rectangle with rational side ratio can be tiled by squares.

The following result shows that this condition is sufficient :

Theorem (Dehn 1903)

A rectangle can be tiled by squares (not necessarily equal) if and only if the ratio of two sides of the rectangle is rational.

- The original proof is complicated.
- In 1940, Brooks, Smith, Stone, and Tutte study tilings by squares introducing a nice interpretation in terms of **electrical networks**.
- Roughly speaking, the study of squared rectangles is transformed into the study of certain flows of electricity in networks of unit resistors (*i.e*, with wires of conductance 1) .

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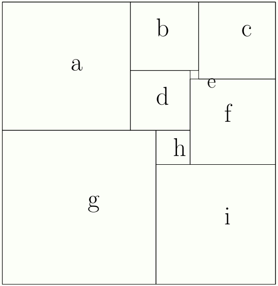
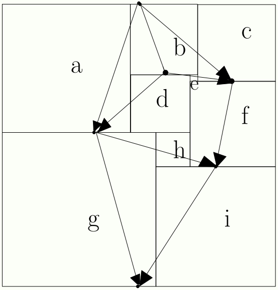
Tiling a rectangle with squares

Define a directed graph such that

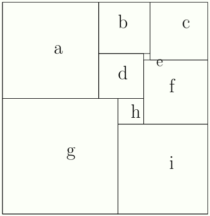
- vertices are the horizontal lines found in the rectangle ;
- for each square there is one edge going from its top horizontal line to its bottom horizontal line ;
- each square corresponds to a wire (in the network) through which the current flowing is equal to the length of the corresponding square.

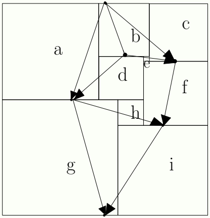




Tiling a rectangle with squares

- To each horizontal line corresponds a ”horizontal equation” for the side lengths of the squares : $a + d = g + h$, $b = d + e$, $\ldots$ and we get one equation saying that the top and bottom sides of the rectangle are equal : $a + b + c = g + i$
  They are equivalent to the equations for conservation of current in this network (Kirshoff’s law).
- we get a ”vertical equation” for each vertical line : $a = b + d$, $d + h = e + f$, $\ldots$ and one equation saying that the left and right sides of the rectangle are equal $a + g = c + f + i$.
  The ”vertical equations” are equivalent to Ohm’s law : $U = RI$ (with $R = 1$ here) or $I = CU$.



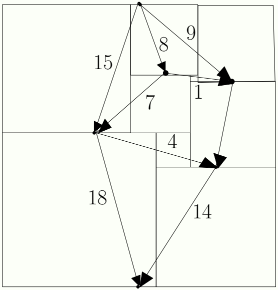


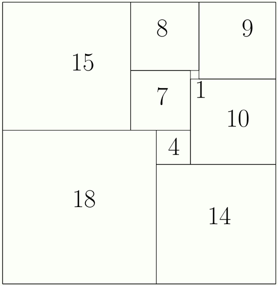
Tiling a rectangle with squares

Theorem (Weyl 1923)

The flow in each wire is determined uniquely, once the potential difference between some two vertices (up to scaling) is known.

- The resulting system of linear equations always has a unique solution up to scaling, for any proposed layout of squares.





- If we fix the flow going out from the top vertex to be the *complexity of the circuit* (here it is also the number of the subtrees of the network),
- then computations involve only integers and rationals numbers
- we obtain

Dehn's Theorem (Rational ratios)

Every rectangle that can be tiled by rectangles has commensurable sides and square tiles.

Characterization of the network

The structure of the networks corresponding to a squared rectangle is characterized as follows

- It is a connected planer graph (can be designed on the plane without crossing edges)
- There are two special vertices (one without ingoing edge, the another without aoutgoing edge)
- All conductances (and resistances) are equal to 1
- The values of the flow in each wire and of the potential difference between each pairs of vertices must satisfies Ohm's law and Kirshoff's law.

▲ Exhaustive generation of rectangles that can be tiled with n squares (n small) building all networks with n wires.

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- To find tilings of **squares instead of rectangles** an additional linear equation is needed, stating that the vertical and horizontal side lengths of the rectangle are equal. In terms of the electrical network, this is equivalent to saying that the network has total resistance 1.
- Using this technique Duijvestijn (1978) with a computer showed that the **smallest possible number of squares** in a perfect tiling of a square is 21.
- See `http://www.squaring.net` for a survey and artwork.
- To described a tiling of a rectangle **by rectangles** with an electrical network, it suffice to take for the conductances the ratios of the rectangle instead of 1.
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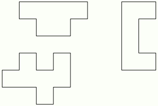
IV. Tiling with a polyomino



Problem

Given a polyomino P, does there exist a rectangle which can be tiled using copies of P ?

Undecidability

- **Roughly speaking**, an *undecidable problem* is a decision problem for which it is known to be impossible to construct a single algorithm that always leads to a correct yes-or-no answer
- that is, any possible program would sometimes give the wrong answer or run forever without giving any answer.
- A **rigorous definition** of the notion of decidabilty/undecidability based on Turing machines will be given during the lecture on *Cellular Automata*.

Tiling a rectangle with a polyomino

Theorem (Berger 1966)

The general problem of whether a given arbitrary polyomino can tile a rectangle is undecidable

- This implies that there is no computable function $f(n)$ which bounds the area of the minimum rectangle that a given n-omino might tile.
- Otherwise we would have a decision procedure :
  "Try all arrangements of the given n-omino in all rectangles of area $< f(n)$."
  While this is no doubt computationally "hard," it is nonetheless much easier than "undecidable."

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Thank you for your attention !