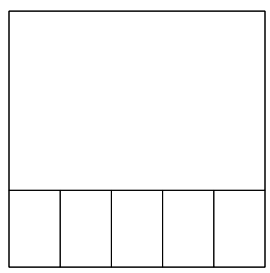
Tileability I

CIMPA Research School - Tilings and Tessellations -
August 24 – September 4, 2015 - Isfahan, IRAN

Tiling a Square with Similar Rectangles

▶ **Exercice 1** (**Determination of the ratio**)

1. In order to tile a square with rectangles similar to $1 \times x$ (up to rotation) as described in the figure below, which ratio x can you choose ?



2. What happens now if you choose $x = \frac{5-\sqrt{5}}{10}$?

Simple Proofs of a Rectangle Tiling Theorem

Theorem 1 *If a finite number of rectangles, every one of which has at least one* integer *side, perfectly tile a big rectangle, then the big rectangle also has at least one integer side.*

Fourteen proofs of theorem 1 were published by Wagon :
Wagon, S. (1987) Fourteen proofs of a result about tiling a rectangle. *The American Mathematical Monthly* 94 (7) : 601-617.

In the following we establish two proofs of this theorem. These proofs generalize to other situations.

In particular Theorem 1 generalizes to the two following statements

Theorem 2 *If a finite number of rectangles, every one of which has at least one* rational *side, perfectly tile a big rectangle, then the big rectangle also has at least one* rational *side.*

Theorem 3 *If a finite number of rectangles, every one of which has at least one* algebraic side*, perfectly tile a big rectangle, then the big rectangle also has at least one* algebraic *side.*

▶ **Exercice 2** (**Rational sides vs integer sides**)
In order to prove the second theorem why is it enough to prove the first one ?

▶ **Exercice 3** (**Chessboard proof of Theorem 1**)
Take the big rectangle and align its bottom left corner with a half-integer chessboard, that is, a chessboard whose squares have side 1/2. Let the bottom left corner be black.

1. Prove that if the upper corner of the big rectangle is at (x, y) the excess black area is

$$|x - r(x)| \cdot |y - r(y)|,$$

   where $r(x)$ denotes the integer nearest to x.
2. Conclude that if the big reactangle can be tiled by a finite number of rectangles, every one of which has at least one integer side, the the big rectangle must have an integer side.

▶ **Exercice 4** (**Proof of Theorems 1, 2 and 3 and walks in a graph**)

In the three theorems, at least one of the big rectangle's sides must have a special property. More precisely depending whether we wish to prove Theorem 1,2 or 3, we will call a number *special* if it is *integer, rational, or algebraic*, respectively. We define a point (x, y) in the plane to be *special* if both coordinates x and y are special.

We associate with each small rectangle four vertices and two edges (see Figure 4). The four vertices are the four corners of the rectangle. The two edges are two parallel sides of the rectangle that are both special in length. (Every small rectangle has two such special edges, by the statement of the theorem.)

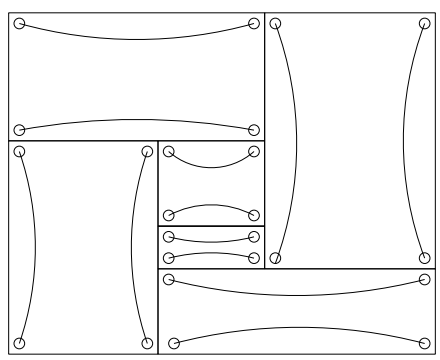


FIGURE 1 – Vertices and edges associated with rectangles

1. Prove that if a vertex is special, then its neighbouring vertex is also special.

   We now define a graph by identifying all the vertices that have identical coordinates. In this graph there may be double edges connecting two vertices. Such double edges are not merged into a single edge
2. Prove that apart from the four vertices at the corners of the big rectangle, which have degree 1, all other vertices have degree two or four.

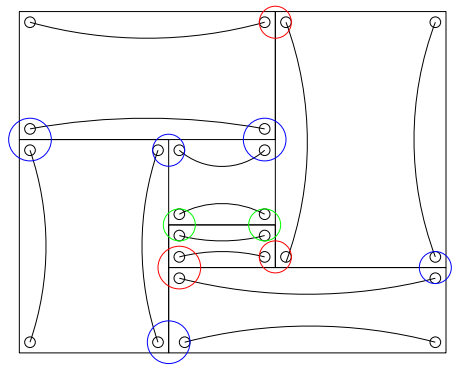


FIGURE 2 – The graph associated with rectangles

3. Aligning the coordinate system such that the bottom left vertex of the big rectangle is at the origin, prove that at least one side that is special in length.

▶ **Exercice 5** (**Proof of Theorems 1, 2 and 3 and higher dimensions**)
Both proofs can be applied to the analogous problem in higher dimensions where each small hyperrectangle has one special dimension.

1. How can you generalize the the chessboard proof ? Then what is the excess black volume of a hyperrectangle that has one corner at the origin and one at x ?
2. How do you define the graph associated with each hyperrectangle in higher dimension ?