Brun expansions of stepped surfaces

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Main result:

Action of *dual maps of free group morphisms* over stepped planes and surfaces (extends substitutions on words).

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Applications :

Brun expansions of stepped planes and surfaces.

Recognition of stepped planes among stepped surfaces.

Stp. planes & stp. surfaces 0000	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces 0000



- 2 Dual maps of free group morphisms
- 3 Brun expansions of stepped planes
- 4 Brun expansions of stepped surfaces

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces

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1 Stepped planes and stepped surfaces

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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Stepped plane			

 $(\vec{e}_1, \ldots, \vec{e}_d)$ basis of \mathbb{R}^d . $\vec{x} \in \mathbb{Z}^d$, $i \in \{1, \ldots, d\} \rightsquigarrow face (\vec{x}, i^*)$:



Definition

Stepped plane of normal vector $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \{0\}$:

$$\mathcal{P}_{\vec{\alpha}} = \{ (\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle \leq 0 < \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle \}.$$

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Stp. planes & stp. surfaces ○●○○	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Stepped plane			



A stepped plane.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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Stepped surface			

Let π be the orthogonal projection along $\vec{u} = \vec{e}_1 + \ldots + \vec{e}_d$.



By extension:

Definition [Jamet]

Stepped surfaces : any set of faces homeomorphic to \vec{u}^{\perp} by π .

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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Stepped surface			



A stepped surface.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms ●○○○	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Definition			

Morphism of the free group over $\{1, \ldots, d\}$ (here, d = 3):

$$\sigma : \begin{cases} 1 & \mapsto & 3 \\ 2 & \mapsto & 3^{-1}1 \\ 3 & \mapsto & 3^{-1}2 \end{cases}$$

For example: $\sigma(1^{-1}312) = \sigma(1)^{-1}\sigma(3)\sigma(1)\sigma(2) = 3^{-2}21$. Incidence matrix: $(M_{\sigma})_{ij} = |\sigma(i)|_j - |\sigma(i)|_{j^{-1}}$. Here:

$$M_{\sigma}=\left(egin{array}{ccc} 0 & 1 & 0 \ 0 & 0 & 1 \ 1 & -1 & -1 \end{array}
ight).$$

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms ○●○○	Brun expansions of stp. planes 0000	Brun expansions of stp. surfaces
Definition			

 σ unimodular f. g. morph. \rightsquigarrow dual map $E_1^*(\sigma)$ (Arnoux-Ito, Ei). $E_1^*(\sigma)$: linear map over weighted sums of faces.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms ○●○○	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Definition			

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For σ previously defined:

$$E_1^*(\sigma) : \begin{cases} (\vec{0}, 1^*) & \mapsto & (\vec{e}_1, 2^*) \\ (\vec{0}, 2^*) & \mapsto & (\vec{e}_1, 3^*) \\ (\vec{0}, 3^*) & \mapsto & (\vec{0}, 1^*) - (\vec{e}_1, 2^*) - (\vec{e}_1, 3^*). \end{cases}$$

and, for $\lambda \in \mathbb{Z}$, $\vec{x} \in \mathbb{Z}^d$:

$$E_1^*(\sigma)(\lambda.(\vec{x},i^*)) = M_{\sigma}^{-1}\vec{x} + \lambda.E_1^*(\sigma)(\vec{0},i^*).$$

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms ○○●○	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Properties			

Theorem (B. F. 2007)

For σ unimodular free group morphism and $\vec{\alpha} \in \mathbb{R}^d_+ \backslash \vec{0}$:

$$M_{\sigma}^{ op}ec{lpha} \in \mathbb{R}^d_+ \; \Rightarrow \; E_1^*(\sigma)(\mathcal{P}_{ec{lpha}}) = \mathcal{P}_{M_{\sigma}^{ op}ec{lpha}}.$$



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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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Properties			

Theorem (B. F. 2007)

For σ unimodular free group morphism: if the image by $E_1^*(\sigma)$ of a stepped surface has faces with weights in $\{0,1\}$, then it is a stepped surface. This holds, in particular, when $M_{\sigma} \geq 0$.



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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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Prup expansion of a vector			

Brun map T, defined for
$$\vec{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d \setminus \{0\}$$
:

$$T(\alpha_1,\ldots,\alpha_d) = \left(\frac{\alpha_1}{\alpha_i},\ldots,\frac{\alpha_{i-1}}{\alpha_i},\frac{1}{\alpha_i}-a,\frac{\alpha_{i+1}}{\alpha_i},\ldots,\frac{\alpha_d}{\alpha_i}\right),$$

where $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_{\infty}\}$ and $a = \lfloor 1/\alpha_i \rfloor$. Matrix viewpoint:

$$(1, \vec{\alpha})^{\top} \propto B_{\mathbf{a},i}(1, T(\vec{\alpha}))^{\top}$$
 with $B_{\mathbf{a},i} = \begin{pmatrix} \mathbf{a} & 1 & \\ & \mathbf{I}_{i-1} & & \\ 1 & & 0 & \\ & & & \mathbf{I}_{d-i} \end{pmatrix}$

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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where $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_{\infty}\}$ and $a = \lfloor 1/\alpha_i \rfloor$. Matrix viewpoint:

$$(1, \vec{\alpha})^{\top} \propto B_{a,i}(1, T(\vec{\alpha}))^{\top}$$
 with $B_{a,i} = \begin{pmatrix} a & 1 \\ I_{i-1} & \\ 1 & 0 \\ & & I_{d-i} \end{pmatrix}$

Brun expansion of $\vec{\alpha}$: sequence $(a_n, i_n)_{n \ge 0}$ of $\mathbb{N}^* \times \{1, \ldots, d\}$:

 $a_n = \lfloor ||T^n(\vec{\alpha})||_{\infty}^{-1} \rfloor$ and $i_n = \min\{j \mid \langle T^n(\vec{\alpha})|\vec{e}_j \rangle = ||T^n(\vec{\alpha})||_{\infty}\}.$

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes	Brun expansions of stp. surfaces
From vectors to stepped plan	es		

Let $\beta_{a,i}$ be an automorphism with incidence matrix $B_{a,i}$ (it exists).

If $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_{\infty}\}$ and $a = \lfloor 1/\alpha_i \rfloor$ are known:

 $E_1^*(\beta_{a,i}^{-1})(\mathcal{P}_{(1,\vec{\alpha})}) = \mathcal{P}_{(1,T(\vec{\alpha}))}.$

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes ○●○○	Brun expansions of stp. surfaces 0000
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Brun expansions could be computed directly over stepped planes by "reading" (a, i). By abuse: Brun expansions of stepped planes.

Note: we do not need to know $\vec{\alpha}$ but just to perform entries comparisons and floor computation.

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
From vectors to stepped plan	es		

Definition

An (i, j)-run of length a is a set of faces of the form:

 $\{(\vec{x} + k\vec{e}_j, i^*) \mid 0 \le k < a\}.$

entries comparisons:

 $\mathcal{P}_{\vec{\alpha}}$ admits an (i, j)-run of length at least 2 iff $\alpha_i > \alpha_j$.

floor computation:

The smallest (i, j)-run of $\mathcal{P}_{\vec{\alpha}}$ has length max $(\lfloor \alpha_i / \alpha_j \rfloor, 1)$.

From vectors to stepped plan	ies		
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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfa



Stepped plane $\mathcal{P}_{(1,\alpha,\beta)}$, with unkown $\alpha,\beta \geq 0$.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces 0000
From vectors to stepped plan	es		



(1,2)-run and (1,3)-run of length at least 2 \rightsquigarrow $(lpha,eta)\in [0,1]^2$

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From vectors to stepped plan	ies		
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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfa



(2,3)-run of length 2 $\rightsquigarrow \alpha > \beta \rightsquigarrow i = 1$.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes ○○○●	Brun expansions of stp. surfac
From vectors to stepped play	nes		



Smallest (1,2)-run of length 2 \rightsquigarrow $a = \lfloor 1/\alpha \rfloor = 2$.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
From vectors to stepped plan	es		

Finally: $E_1^*(\beta_{2,1}^{-1}(\mathcal{P}_{(1,\alpha,\beta)}) = \mathcal{P}_{(1,\mathcal{T}(\alpha,\beta))}$.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes 0000	Brun expansions of stp. surfaces
From stepped planes to stepp	ed surfaces		

Reading over stepped planes ~>> Brun exp. of stepped planes.

By analogy (runs and dual maps are still defined):

Reading over stepped surfaces \rightsquigarrow Brun exp. of stepped surfaces.

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Relation with Brun exp. of vectors? (no more normal vectors)

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes	Brun expansions of stp. surfaces
From stepped planes to stepp	ed surfaces		

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Theorem (B. F. 2007)

Stepped surfaces having the Brun expansion of $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \{0\}$ are:

- the stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$ (finite or infinite expansion);
- some stepped surfaces almost equal to $\mathcal{P}_{(1,\vec{\alpha})}$ (idem);
- some non-plane stepped surfaces (only finite expansion).

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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The stepped plane case (finit	to or infinite expansion)		



(a, i) = (4, 1)

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
The stepped plane case (finit	e or infinite expansion)		



(a, i) = (1, 2)

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces



 $a = \infty$: (rational) stepped plane recognized.

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Stp. planes & stp. surfaces 0000	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes	Brun expansions of stp. surfaces
The stepped quasi-plane case	(finite or infinite expansion)		



(a, i) = (4, 1)

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes	Brun expansions of stp. surfaces $\circ \circ \bullet \circ$
The stepped quasi-plane case	(finite or infinite expansion)		



(a,i)=(1,2)

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes	Brun expansions of stp. surfaces ○○●○
The stepped quasi-plane case	(finite or infinite expansion)		



 $a = \infty$: not a stepped plane... but almost.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes

Brun expansions of stp. surfaces $\circ \circ \circ \circ \bullet$

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The stepped surface case (only finite expansion)



(a, i) = (4, 1)

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces $\circ \circ \circ \bullet$
The stepped surface case (on	ly finite expansion)		



(a,i)=(1,2)

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes

Brun expansions of stp. surfaces $\circ \circ \circ \circ$

The stepped surface case (only finite expansion)



(a, i) undefined: not at all a stepped plane.

Where is "digital plane recognition"?

A stepped surface is a rational stepped plane iff it has a finite Brun expansion, with the last obtained stepped surface being $\mathcal{P}_{(1,\vec{0})}$.

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Can be extended for finite subset of stepped surfaces.