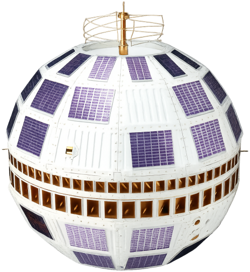
Telstar Balls Gone Wild

Thomas Fernique

Telstar balls

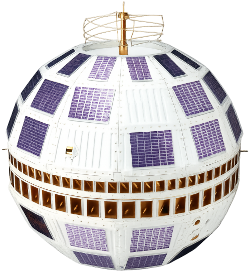
Introduced for the 1968 European Football Championship.
Official match ball of the 1970 FIFA World Cup in Mexico.

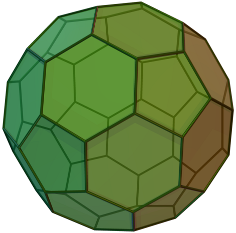
Telstar balls



Named after the 1962 Telstar communications satellite.

Telstar balls





Structure of the truncated icosahedron.

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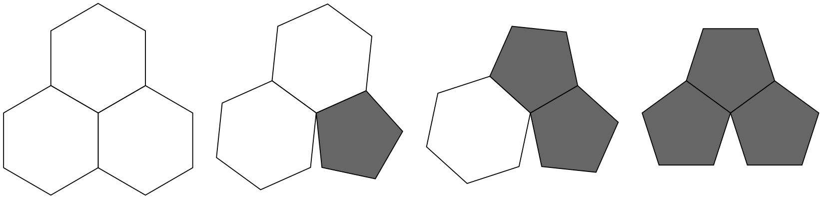
More or less sharp vertices appear, depending on the angle deficit:



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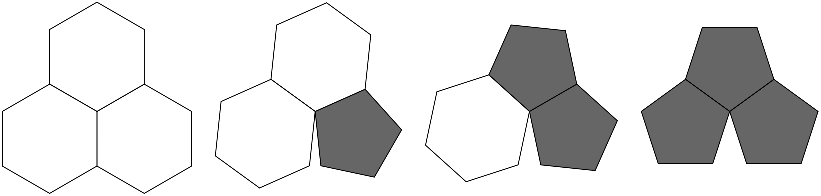


The shape could be strange. . . but can it even be closed into a ball?

Outline

Outline

Curves ahead

32 panels, 810 saddle stitches, 2^{+}h. of work. Why that much!?

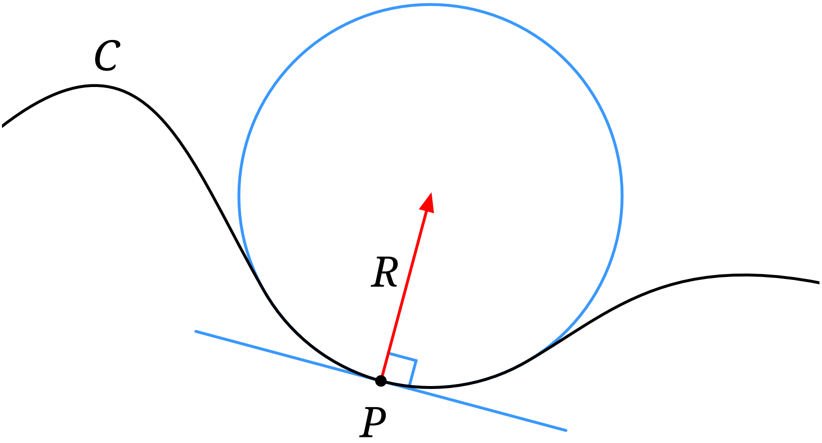
Curves ahead

The existence of spherical cows is controversial.

Curves ahead

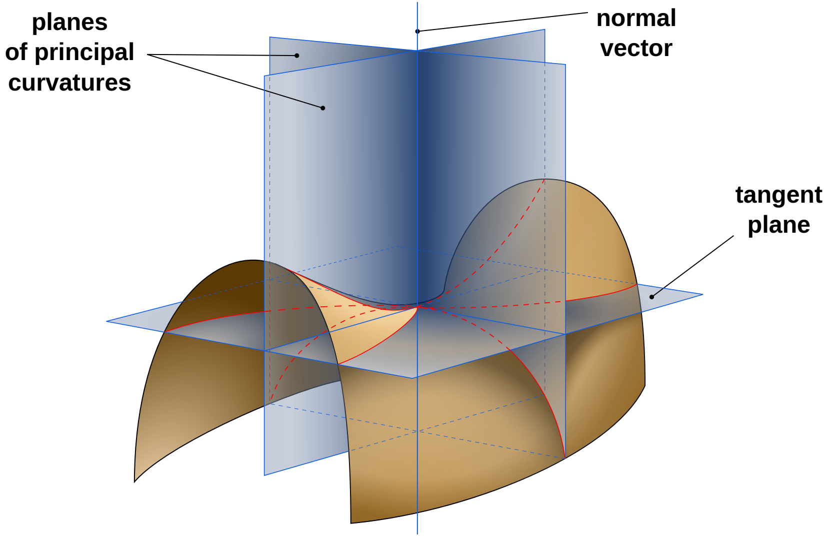
Anyway, cows are too large: *curvature* mismatch!

Curves ahead



Curvature of $\mathcal{C}$ in P: $1/R$ (radius of the osculating circle).

Curves ahead



Principal curvatures: min. and max. in normal planes.

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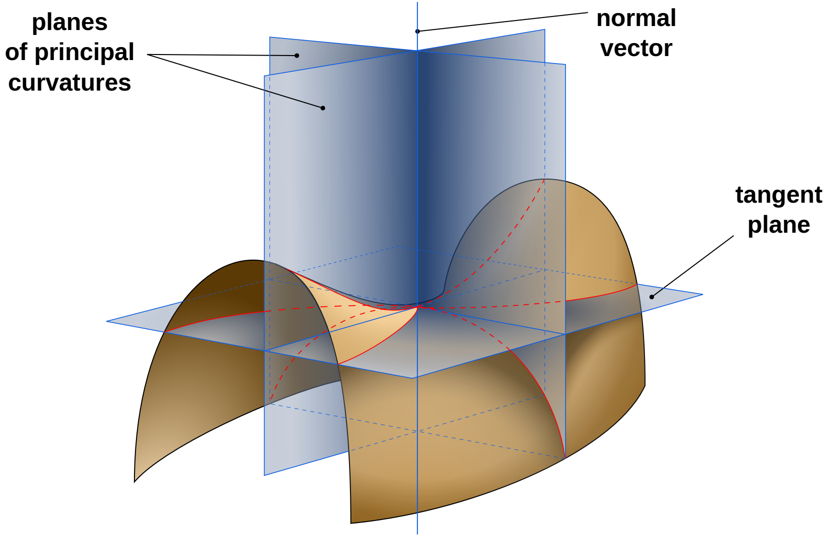

Gaussian curvature: product of the principal curvatures.

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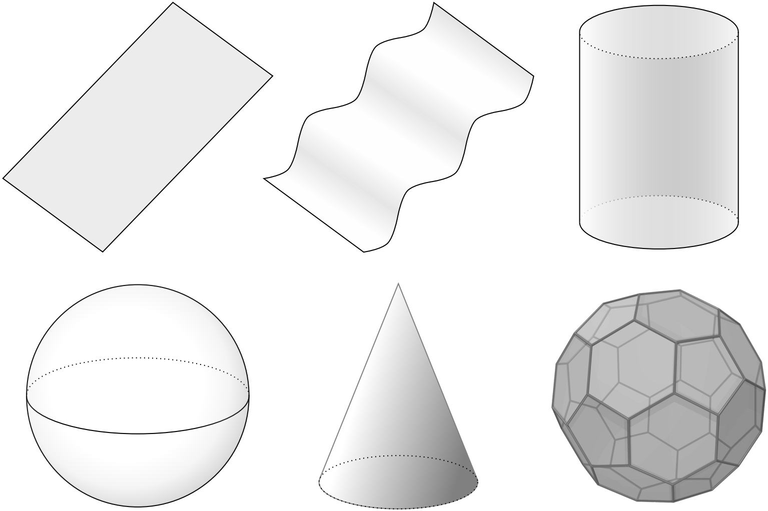
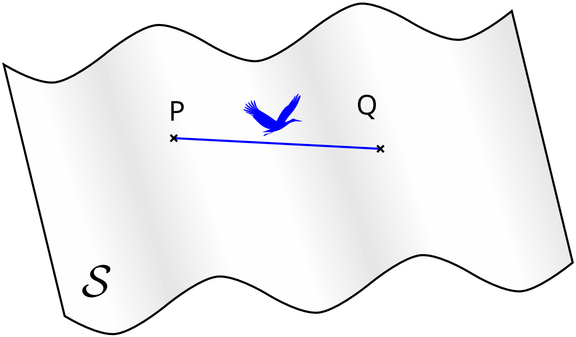


What is the Gaussian curvatures of these surfaces?

In or out?

Distance between two points on a surface $\mathcal{S} \subset \mathbb{R}^3$?

- The length of a segment (usual Euclidean distance in $\mathbb{R}^3$).



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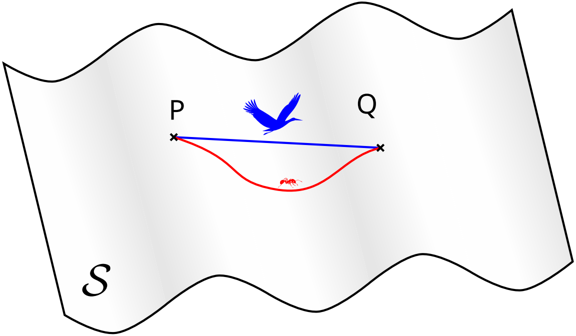

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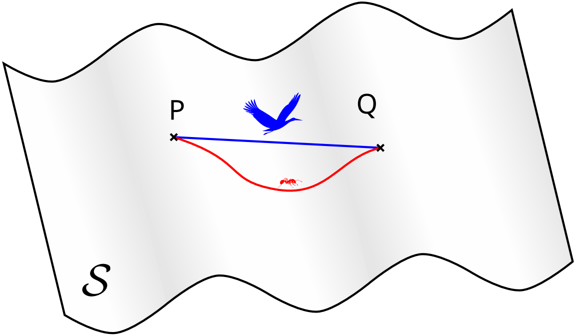

The first metric is extrinsic, the second one intrinsic.
Only the first one depends on the embedding of $\mathcal{S}$.

Theorema egregium

Theorem (Gauss, 1828)

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From a geographer viewpoint: no isometric map projection.
You have to choose what to save:

- angles (conformal), e.g. Mercator (center $\rightsquigarrow$ cylinder)
- areas (equivalent), e.g. Lambert (Earth's axis $\rightarrow$ cylinder)
- geodesics (orthodromic), e.g. gnomonic (center $\rightarrow$ plane)
- ...

Back to the Telstar

From a ball stitcher viewpoint: nonzero curvature $\Rightarrow$ stretch marks.

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- In a vertex of a polyhedron: 2π minus sum of incident angles.

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For the Telstar:

- every vertex sees two hexagons and one pentagon $\rightsquigarrow$ 12°;
- the 60 vertices contribute a total curvature of $720^\circ = 4\pi$.

Outline

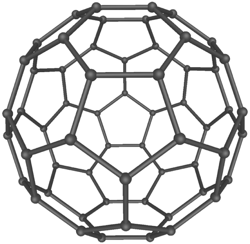
Vertices

Edges

Faces

Fullerenes

Telstar with carbon atoms at vertices and covalent bonds as edges:



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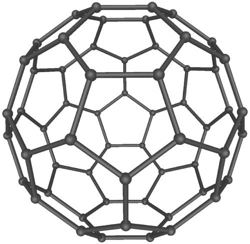

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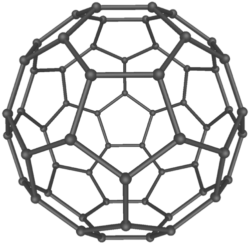

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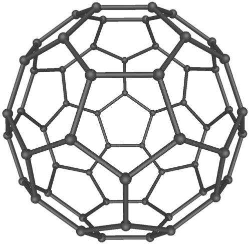

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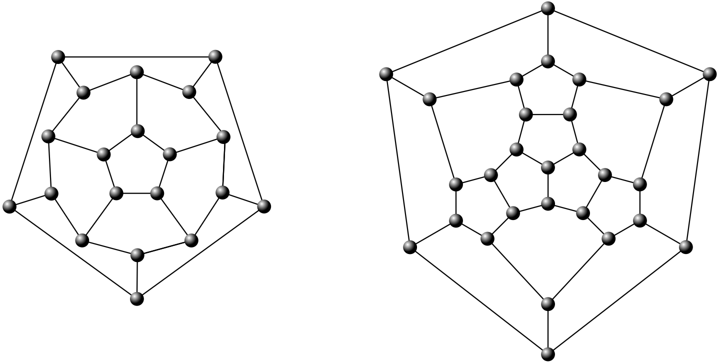
Exercise: prove there are 12 pentagons; what about the hexagons?

Enumeration

BuckyGen algorithm (Brinkmann-Goedgebeur-McKay, 2012): Zipper/annulus expansions generate every fullerene from 2 seeds.



The two seeds.

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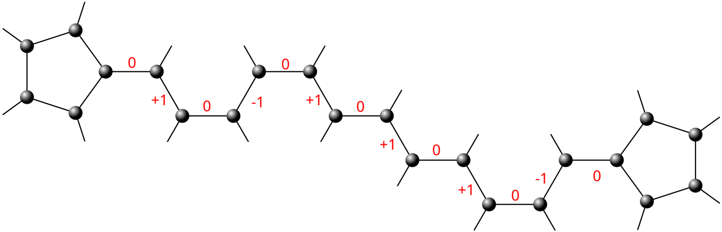

Follow a path of $k \geq 3$ edges between two pentagons.
Starting from 0, add ± 1 depending whether you turn right or left.
Assume you end in 0 and see only $-1, 0, +1$.

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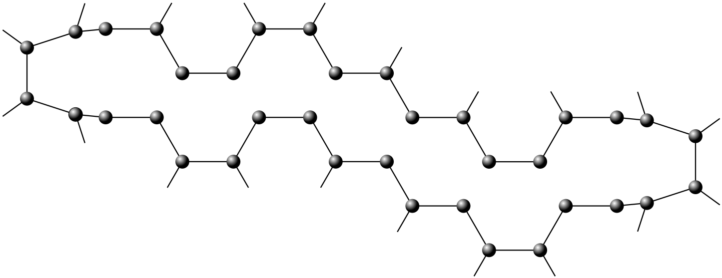

Break the pentagons to "unzip" the fullerene along this path.

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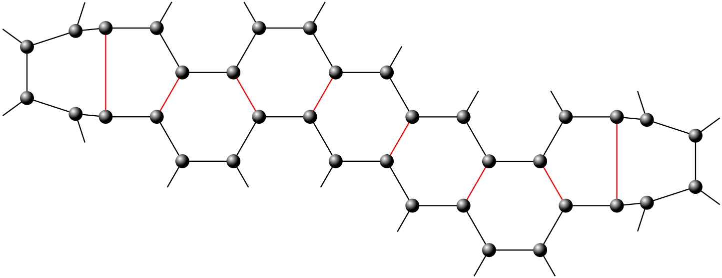

Zip it back up with two pentagons and hexagons (one for each 0).

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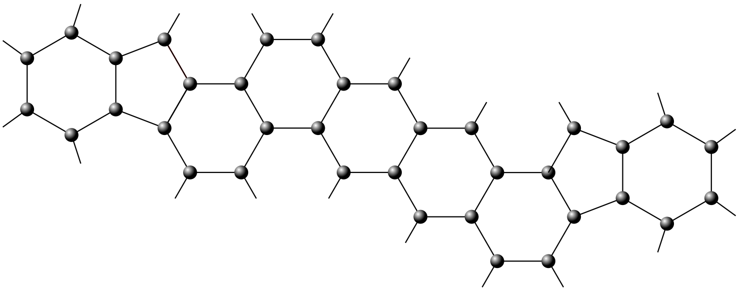

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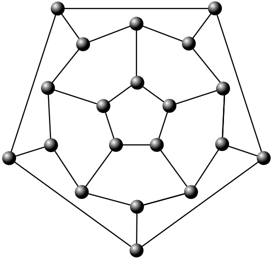

The annulus expansion.

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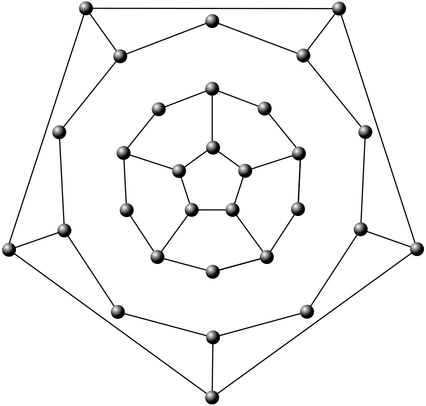

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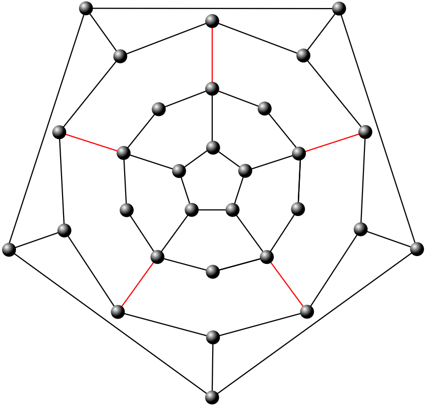

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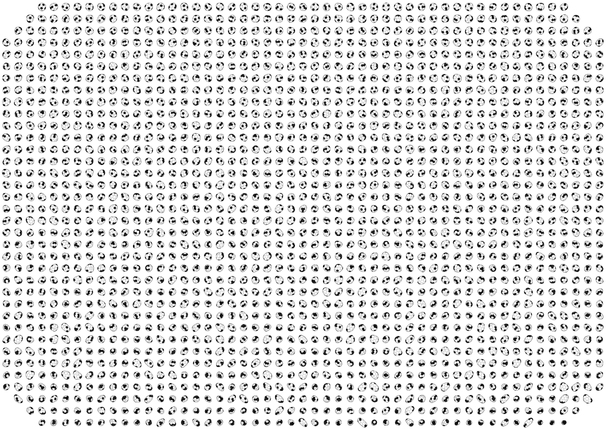

$\rightsquigarrow$ 1812 isomers of the Telstar (of which 1720 are chiral) in 42ms.

Counting

Theorem (Engel-Smillie, 2025)

The generative function of the number of fullerenes is an "explicit" linear combination of specific modular forms (Eisenstein series):

$$
\begin{aligned}
&\tfrac{809}{2^{15}3^{13}5^{2}}E_{10}(q) - \tfrac{1}{2^{16}3^{11}5^{2}}E_{9,\mathbf{1},\chi_3}(q) - \tfrac{1}{2^{16}3^{3}5^{2}}E_{9,\chi_3,\mathbf{1}}(q) + \tfrac{515}{2^{13}3^{11}}E_8(q)\\
&- \tfrac{1}{2^{6}3^{9}5}E_8(q^2) - \tfrac{1}{2^{11}3^{4}5}E_8(q^3) + \tfrac{47}{2^{14}3^{12}}E_{7,\mathbf{1},\chi_3}(q) + \tfrac{1}{2^{8}3^{9}5\cdot 7}E_{7,\mathbf{1},\chi_3}(q^2)\\
&- \tfrac{7}{2^{14}3^{3}5}E_{7,\chi_3,\mathbf{1}}(q) - \tfrac{1}{2^{8}3^{3}5\cdot 7}E_{7,\chi_3,\mathbf{1}}(q^2) + \tfrac{2138939}{2^{14}3^{11}5^{2}}E_6(q) + \tfrac{3571}{2^{9}3^{8}5}E_6(q^2)\\
&- \tfrac{29}{2^{11}3^{6}5}E_6(q^3) + \tfrac{1}{2^{6}3^{3}5}E_6(q^6) + \tfrac{274421}{2^{15}3^{11}5^{2}}E_{5,\mathbf{1},\chi_3}(q) - \tfrac{8231}{2^{11}3^{9}5}E_{5,\mathbf{1},\chi_3}(q^2)\\
&- \tfrac{2393}{2^{15}3^{2}5^{2}}E_{5,\chi_3,\mathbf{1}}(q) - \tfrac{203}{2^{11}3^{2}5}E_{5,\chi_3,\mathbf{1}}(q^2) + \tfrac{12074417}{2^{13}3^{12}}E_4(q) + \tfrac{287281}{2^{10}3^{9}5}E_4(q^2)\\
&- \tfrac{2672137}{2^{1}13^{9}5}E_4(q^3) - \tfrac{1}{2^{2}3^{3}}E_4(q^4) - \tfrac{4289}{2^{8}3^{6}}E_4(q^6) + \tfrac{3343037}{2^{14}3^{11}}E_{3,\mathbf{1},\chi_3}(q)\\
&+ \tfrac{227779}{2^{12}3^{8}5}E_{3,\mathbf{1},\chi_3}(q^2) + \tfrac{1}{2^{3}3^{3}}E_{3,\mathbf{1},\chi_3}(q^3) + \tfrac{1}{2^{5}3^{2}}E_{3,\mathbf{1},\chi_3}(q^4) - \tfrac{33103}{2^{14}3^{3}5}E_{3,\chi_3,\mathbf{1}}(q)\\
&- \tfrac{713}{2^{12}3\cdot 5}E_{3,\chi_3,\mathbf{1}}(q^2) + \tfrac{1}{2^{3}3}E_{3,\chi_3,\mathbf{1}}(q^3) - \tfrac{1}{2^{5}3}E_{3,\chi_3,\mathbf{1}}(q^4) + \tfrac{4314057521}{2^{15}3^{11}5^{2}}E_2(q)\\
&- \tfrac{59136661}{2^{14}3^{9}5}E_2(q^2) - \tfrac{32541151}{2^{11}3^{10}5}E_2(q^3) + \tfrac{1}{2^{5}3}E_2(q^4) + \tfrac{2}{3\cdot 5}E_2(q^5) + \tfrac{5023687}{2^{10}3^{8}5}E_2(q^6)\\
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\end{aligned}$$

BuckyGen: 132247999328 fullerenes with 400 vertices (3 y. CPU).
Here: 1203397779055806181762759 fullerenes with 10000 vertices.

Outline

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Edges

Faces

Isometric embedding

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The metric space obtained by gluing these spaces according to the fullerene structure is still geodesic and locally Euclidean... excepts around the vertices. It is also homeomorphic to a sphere.

Isometric embedding

We do not want to play with a carbon cage or a graph!
Faces are *solid* leather panels, not vertices connected by edges!

Exercise: prove that only the Telstar can be made of *rigid* panels.

Each panel defines a *geodesic* and *locally Euclidean* metric space.

The metric space obtained by gluing these spaces according to the fullerene structure is still geodesic and locally Euclidean. . . excepts around the vertices. It is also homeomorphic to a sphere.

We want an isometric embedding of this metric space into $\mathbb{R}^3$.

Alexandrov's theorem on polyhedra

Theorem (Alexandrov, 1948)

If a metric space is geodesic, homeomorphic to a sphere, and locally Euclidean except for a finite number of cone points of positive angular defect, then it admits a unique convex polyhedral isometric embedding.

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Exercise: Fold a tetrahedron out of three hexagons.

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No C^2 isometric embedding for fullerenes.

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Theorem (Pogorelov, 1973)

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In practice