Cut & Project Tilings of Finite Type

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Cut and project tilings

Definition (C & P tiling)

A *C & P tiling* is the projection onto a d-dim. subspace $E \subset \mathbb{R}^n$, called the *slope*, of the d-dim. facets of $\mathbb{Z}^n$ included in $E + [0,1]^n$.

n	d	example
2	1	Sturmian words
3	1	Billiard words
3	2	Discrete planes
4	2	Ammann-Beenker tilings
5	2	Penrose tilings
6	3	Icosahedral tilings
$\vdots$	$\vdots$	$\vdots$

Examples

A rhombille tiling in Saint-Étienne de Marmoutier (Alsace).

Examples

Michael Baake's Ammann-Beenker tiled floor in Tübingen.

Examples

My Penrose wooden floor in Paris.

Main question

Definition (Tiling space)

A *tiling space* is a set of tilings invariant by translation and closed.

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Main question (motivated by quasicrystal stabilization)

Which C & P tilings have a hull of finite type?

- How to characterize a slope by patterns?
- How to enforce planarity by patterns?

Studying patterns

Definition (Window)

The *window* W of a C & P tiling of slope $E \subset \mathbb{R}^n$ is the image of $[0,1]^n$ by the orthogonal projection π' onto $E^{\perp}$:

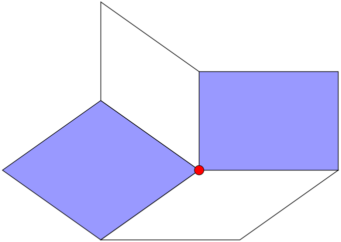
$$W := \pi'([0,1]^n).$$

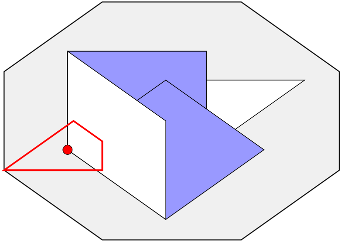
Proposition

To any pointed pattern P corresponds a subregion R of the window in which project the vertices which point this pattern in the tiling:

$$R := \bigcap_{\vec{x} \text{ vertex of } \mathcal{P}} \left(W - \pi'\vec{x}\right).$$

Examples





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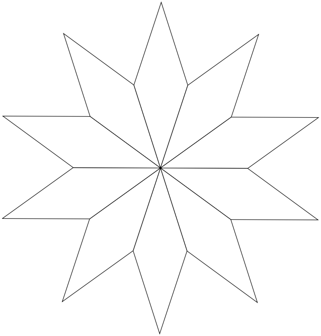

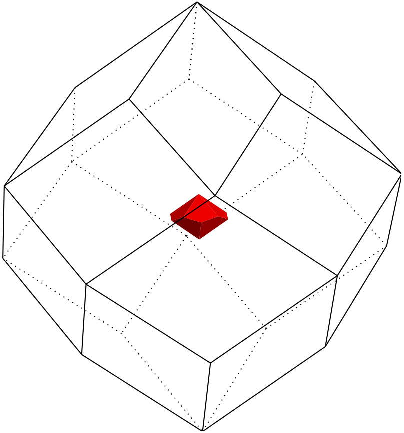

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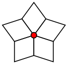
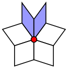
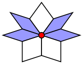

$13-8\varphi$


$13\varphi-21$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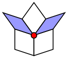
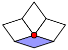
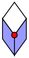
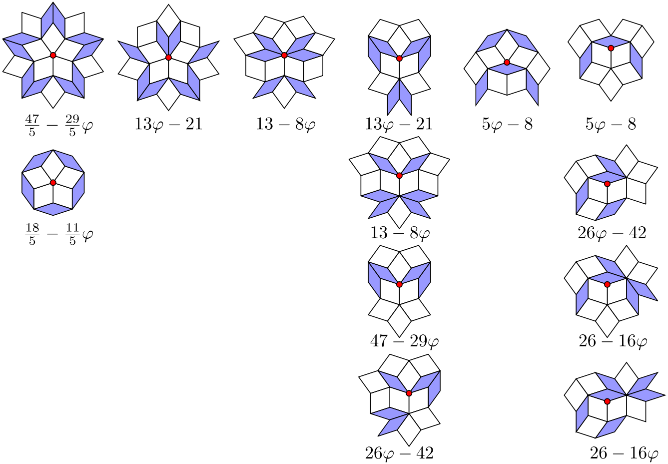

$2\varphi-3$


$5\varphi-8$


$2-\varphi$


$5-3\varphi$

Examples



$\frac{47}{5} - \frac{29}{5}\varphi$
$13\varphi - 21$
$13 - 8\varphi$
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$\frac{18}{5} - \frac{11}{5}\varphi$
$13 - 8\varphi$
$26\varphi - 42$
$26 - 16\varphi$
$47 - 29\varphi$
$26 - 16\varphi$
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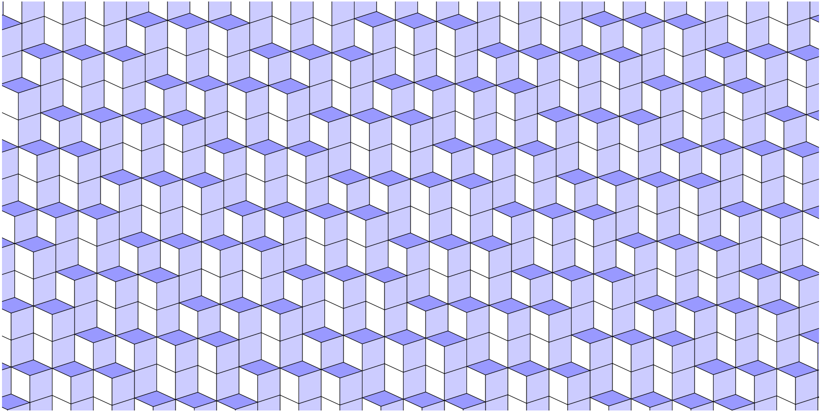
Codimension 1

Theorem (Levitov, 1988)

Aperiodic codim. 1 C & P tilings do not have hull of finite type.



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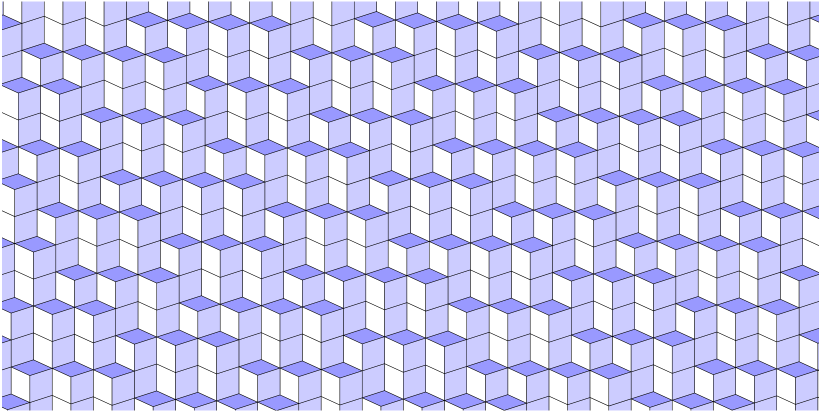

Shifting a slope yields "isolated flips" that can be "decorrelated".

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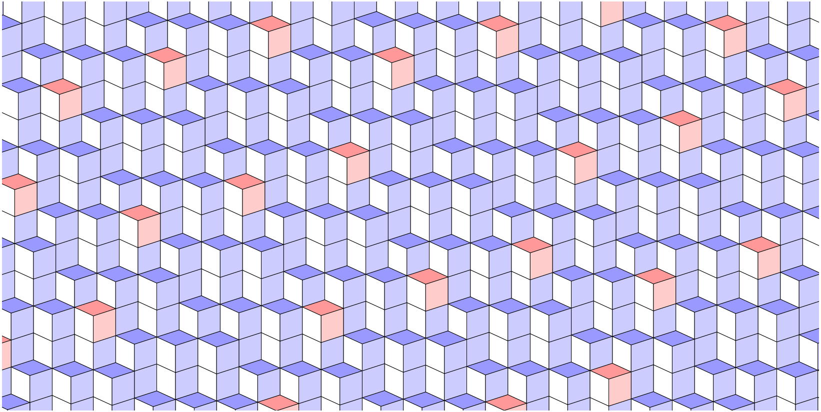

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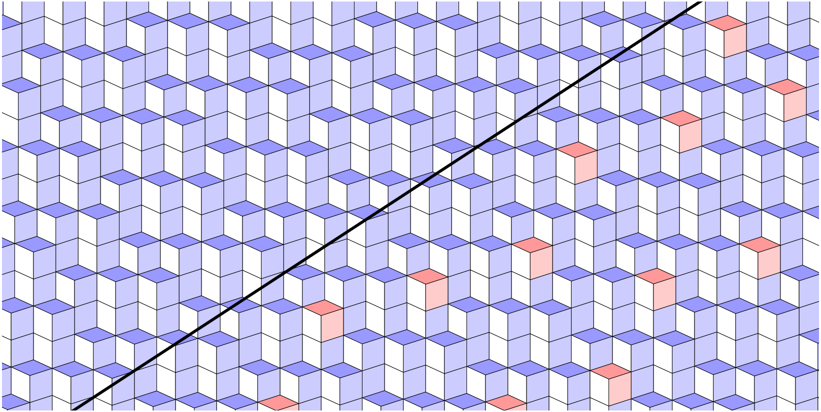

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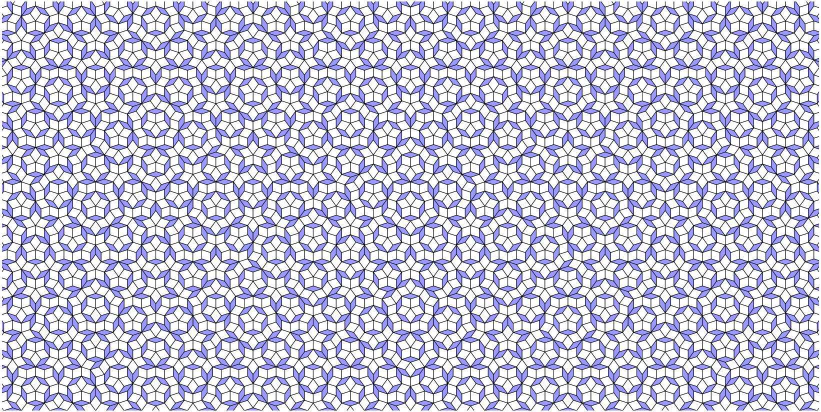
Higher codimensions

Theorem (De Bruijn, 1981)

Penrose tilings are $5 \to 2$ *C & P tilings whose hull has finite type.*



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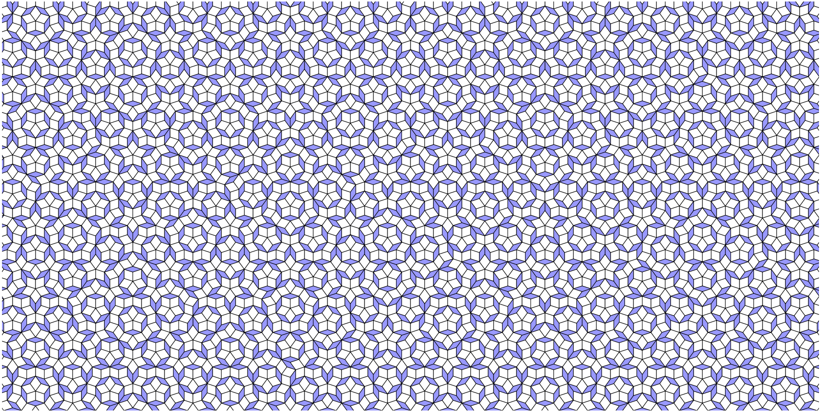

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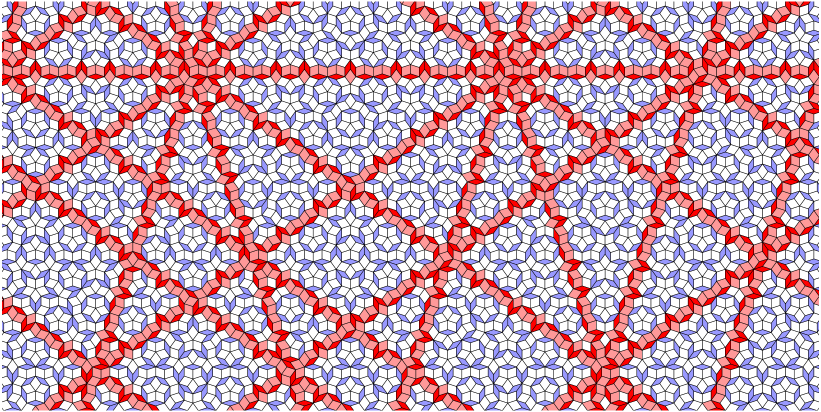


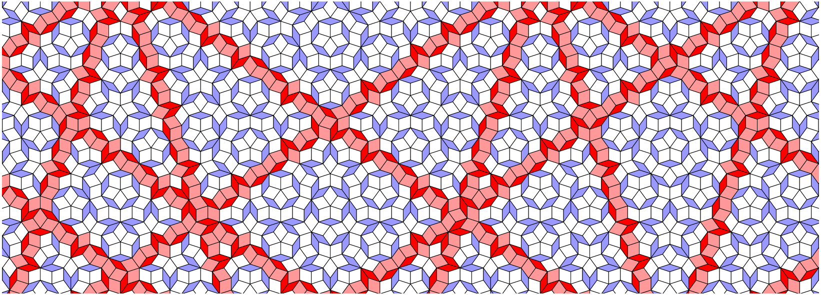
Shifting the slope yields “lines of flips” that cannot be decorrelated.

Subperiods

These lines of flips are directed by “hidden periodicities”:

Definition (Subperiod)

A *subperiod* of a d-plane E is a vector with $d+1$ integer entries.



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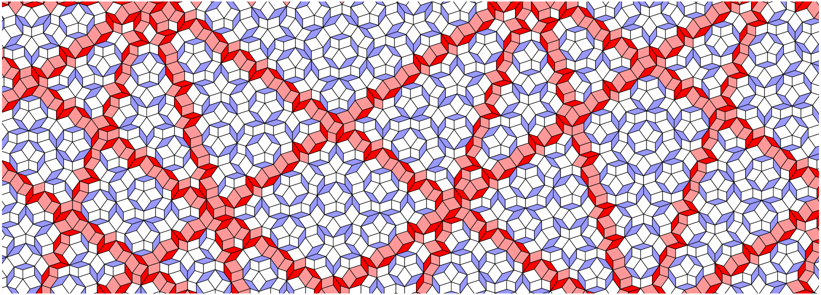

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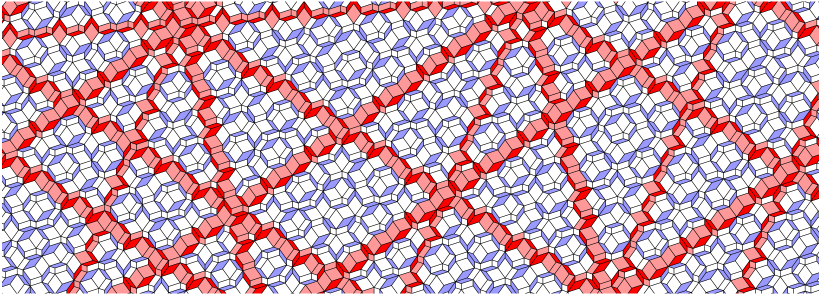

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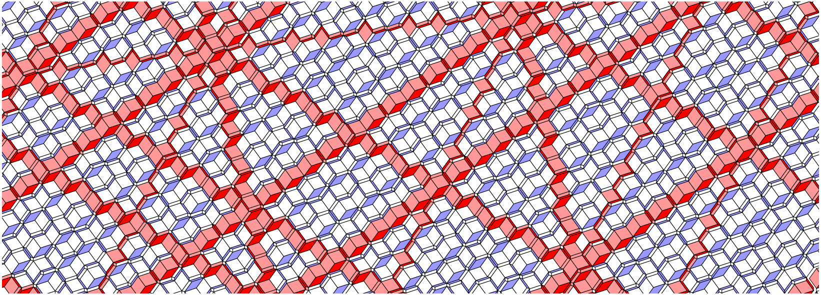

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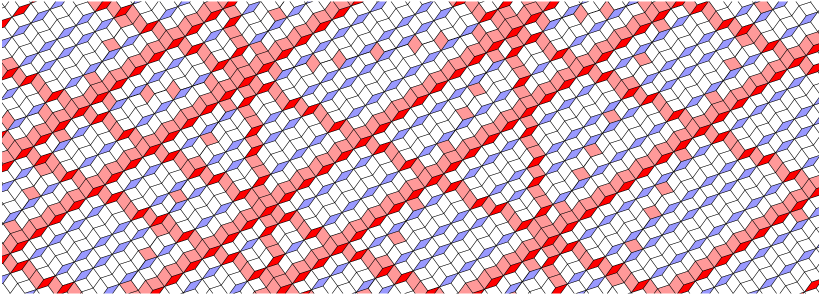

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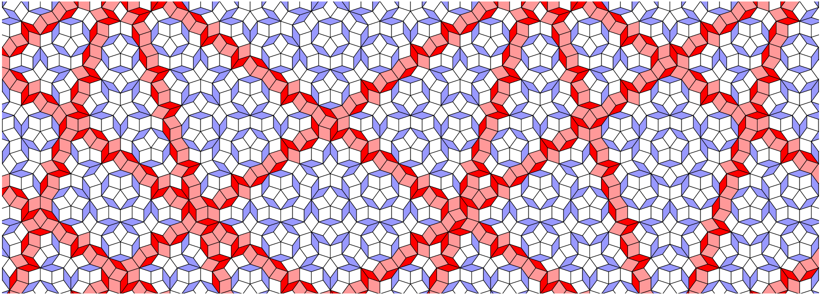

Proposition (Subperiods and expansivity)

If a C & P tiling has subperiods for any choice of $d+1$ entries, then these subperiods give the non-expansive directions of its hull. Otherwise, there is no expansive direction.

A conjecture

Conjecture (Bédaride-Fernique, $\sim$ 2012)

An aperiodic $n \to d$ C & P tiling have a hull of finite type iff

1. *it has subperiods for any choice of entries;*
2. *these subperiods characterize its slope.*

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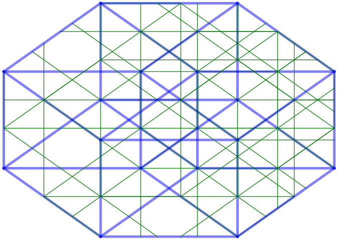
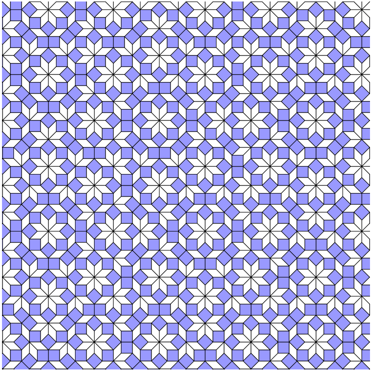
- It holds if planarity is assumed (Bédaride-F., 2020)

Ammann-Beenker

Ammann-Beenker tilings are not characterized by their subperiods.
Patterns of given size are preserved by suitably modifying the slope.





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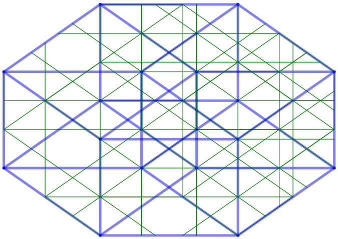

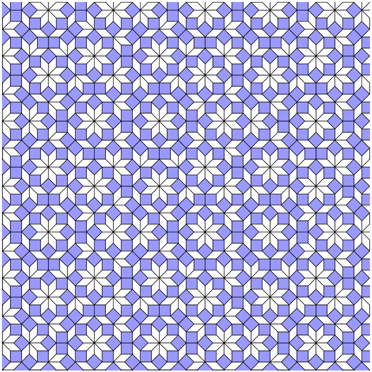

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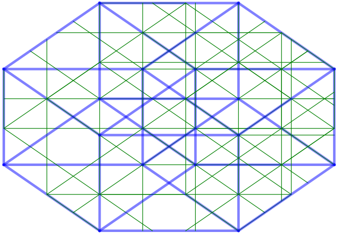

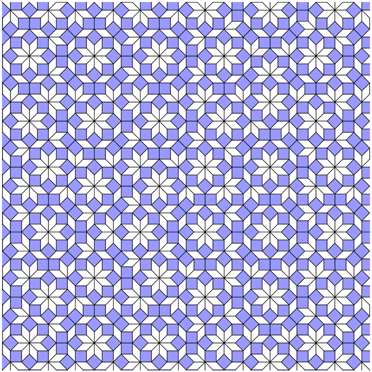

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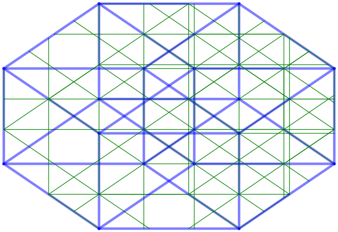

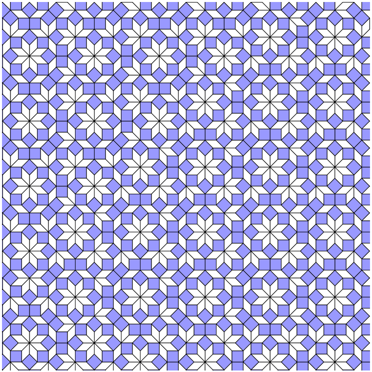

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