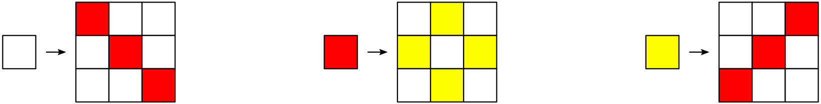
Enforcing 3 by 3 Substitutions by Matching Rules

Thomas Fernique

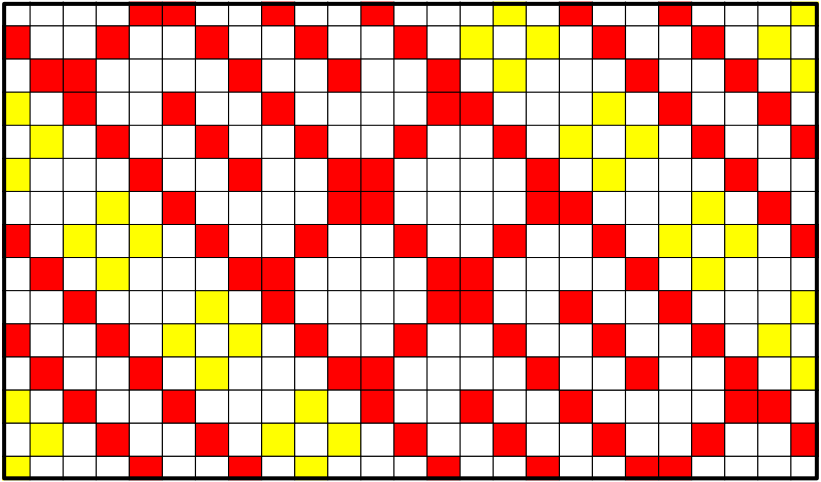
May 2023

3 by 3 substitutions



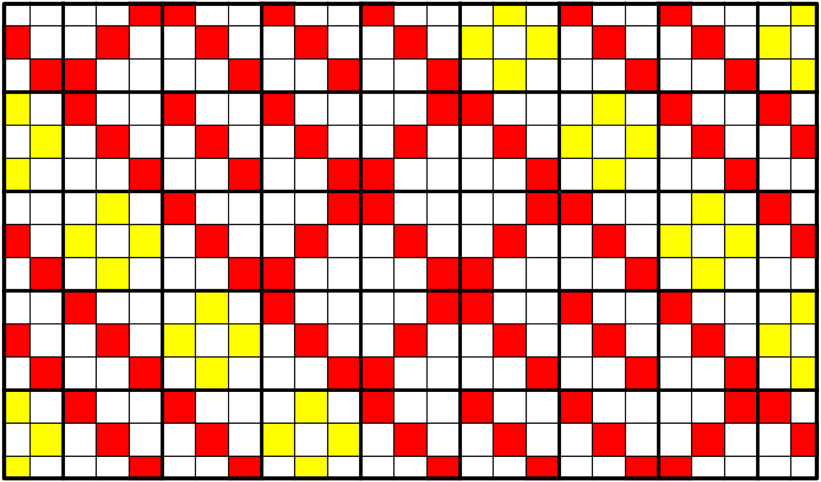
Map from *colored tiles* to 3×3 squares of them (finite colorset).

3 by 3 substitutions



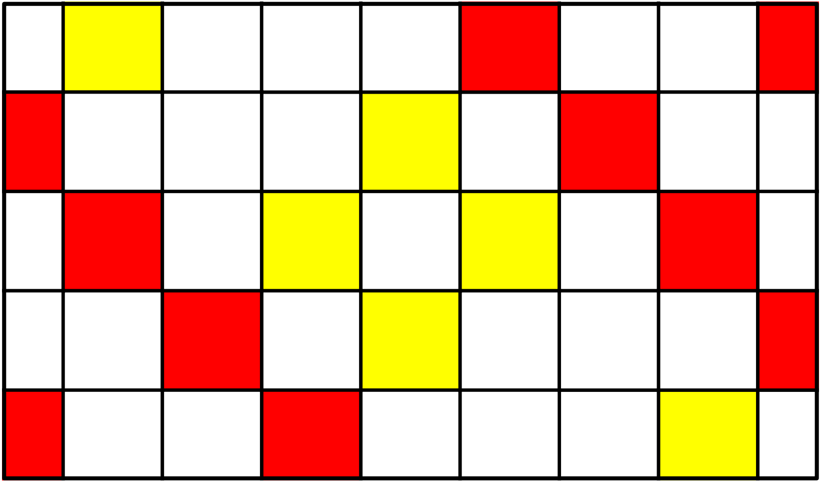
A *tiling* is a grid of colored tiles which covers the whole plane.

3 by 3 substitutions



The substitution is said to define the set of tilings which admit an infinite sequence of preimages (also called *hierarchical tilings*).

3 by 3 substitutions



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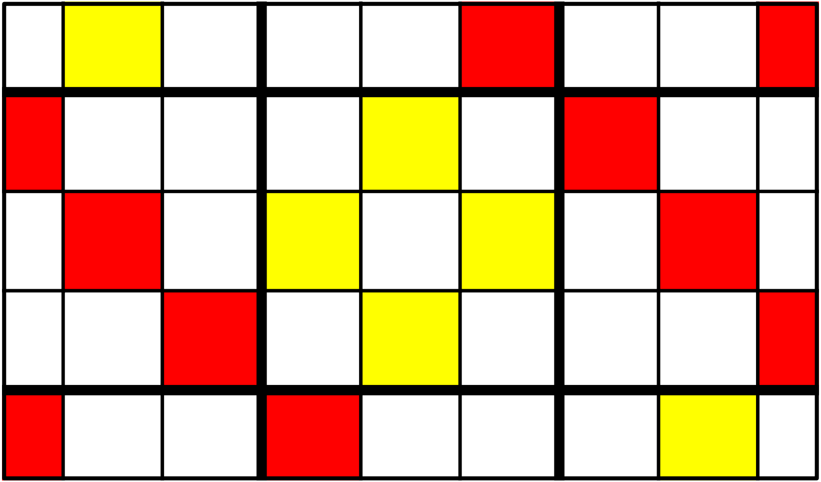

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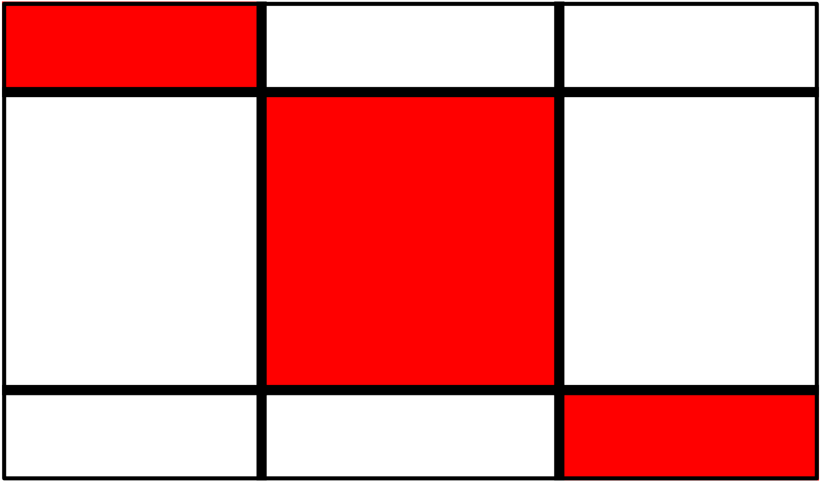

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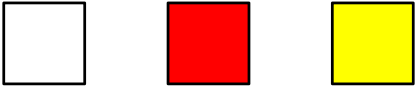
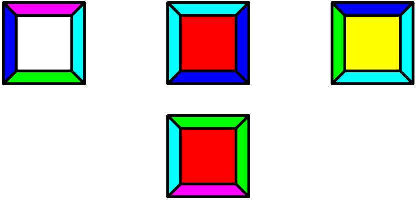
Matching rules



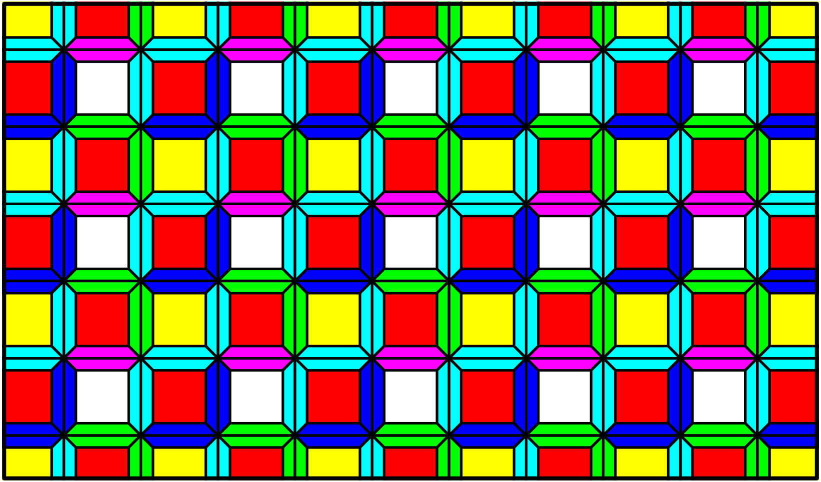
Consider a set of colored tiles.

Matching rules



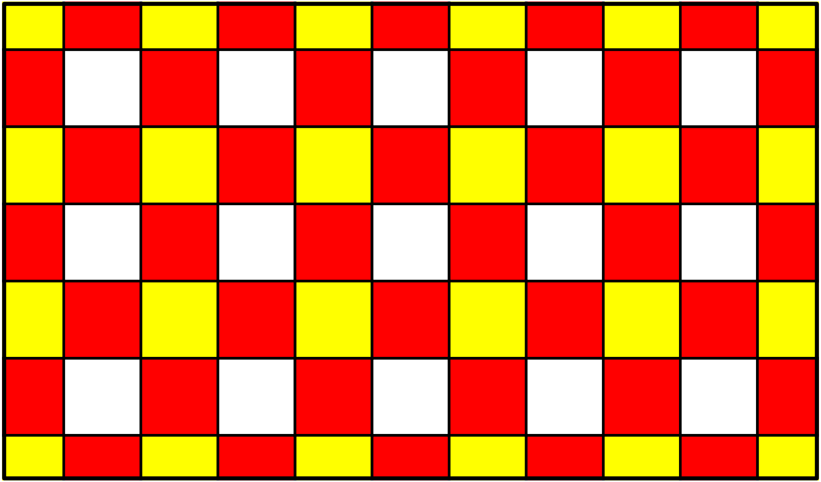
Tile's edges are *decorated*. A tile can yield several decorated tiles.

Matching rules



Consider the tilings by translated tiles whose decorations *match*.

Matching rules



Removing the decorations yields a set of tilings by colored tiles.
These tilings are said to be *enforced* by the set of decorated tiles.

Main result

Theorem

Given any 3 by 3 substitution, there exists a finite set of decorated tiles which enforces the set of tilings defined by this substitution.

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Several more or less convincing proofs (by chronological order):

- Shahar Mozes, *Tilings, substitution systems and dynamical systems generated by them*, J. Anal. Math. (1989), 48pp.
- Chaim Goodman-Strauss, *Matching rules and substitution tilings*, Ann. Math. (1998), 43pp/50+27pp.
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We here follow the last proof, corrected and improved with the help of Nikolay Vereshchagin and Nikita Andrusov.

Proof outline

We will define step by step:

- a finite set τ of decorated squares, where every edge is endowed with a (red,green,blue) triple of indices,
- a bijection ϕ from 3×3 squares of τ-tiles to τ-tiles,

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such that:

- every τ-tiling can be uniquely partitioned into 3×3 squares,
- applying ϕ on these 3×3 squares (+scaling) yields a τ-tiling,
- there exist τ-tilings.

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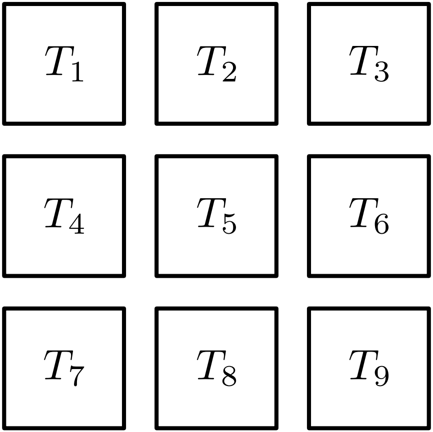
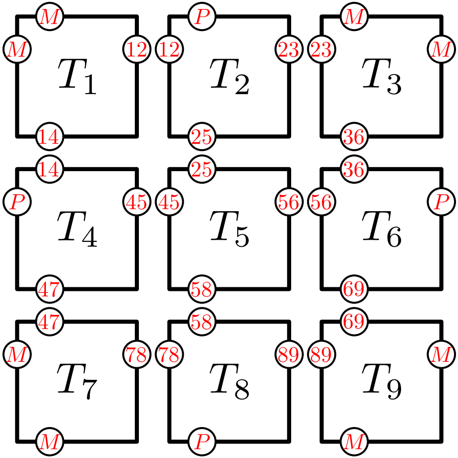
The main theorem will then easily follow.

Step 1: macro-tiles



Start with tiles $T_1, \ldots, T_9$.

Step 1: macro-tiles



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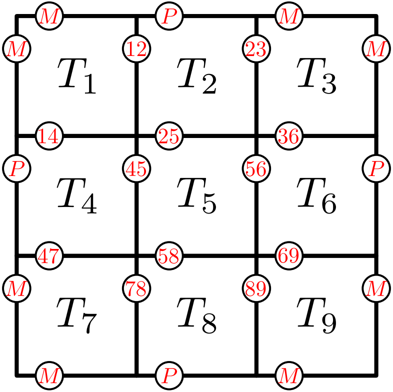

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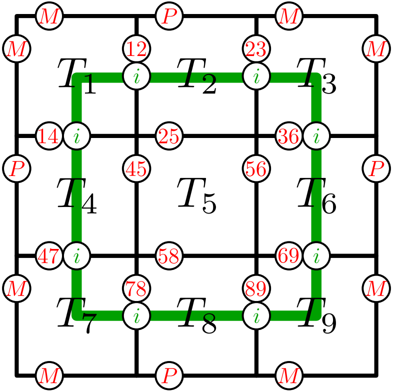
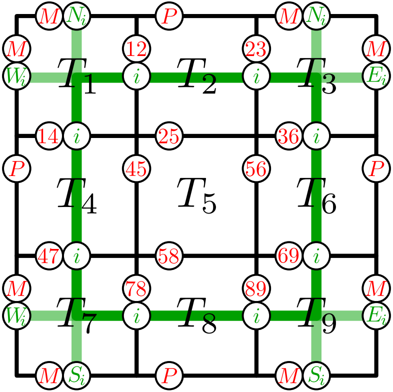
Step 2: rings and macro-macro-tiles



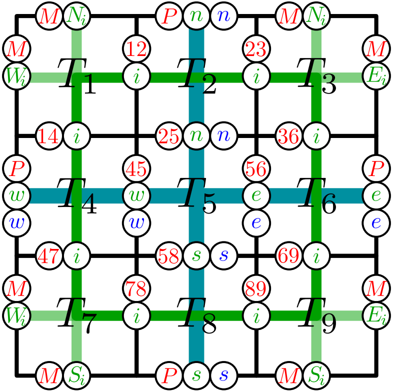
A green index $i \in \{1, \ldots, 9\}$ runs along a *ring* in every macro-tile.

Step 2: rings and macro-macro-tiles



Further green indices force rings to order as T_i's in a macro-tile.
(X_i denotes the red index on the X-edge of T_i, $X = N, W, S, E$)

Step 3: the network



A *network* carries green/blue indices from the central tile to *ports*.

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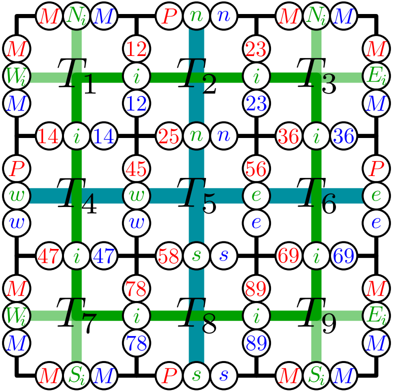
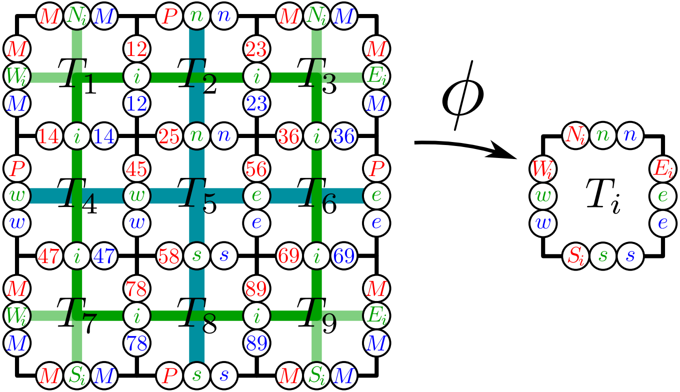


A *network* carries green/blue indices from the central tile to *ports*.
Outside the network, the blue index just replicate the red one.

Interlude: the map ϕ



ϕ maps the green indices forcing ring ordering onto red indices and copies the green/blue indices from network branches.

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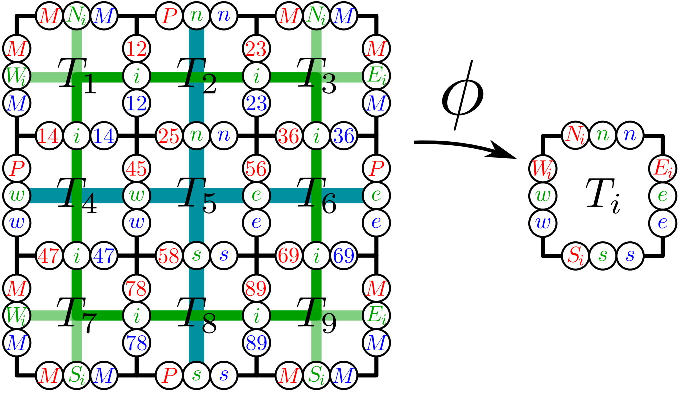

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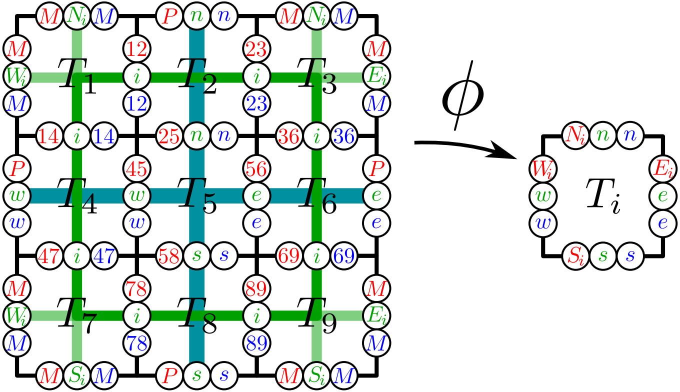
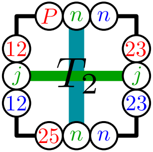


ϕ maps a tiling onto a tiling. Does it map a τ-tiling onto a τ-tiling?
The indices on the network will have to be chosen so that it holds.

Step 4: ring/network intersections

When a ring along which runs j crosses an X-branch, it checks that the pair carried by the branch is allowed on the X-edge of T_j.

For e.g. a North-branch:



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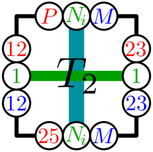

- $j \in \{1, 3\} \rightsquigarrow \{M, P, 14, 25, 36, 47, 58, 69\} \times \{M\}$;

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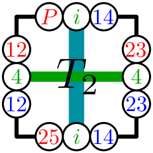

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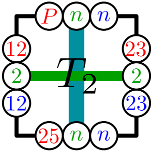

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- $j \in \{2, 5, 8\} \rightsquigarrow$ <u>every</u> pair already defined on <u>any</u> north-edge!

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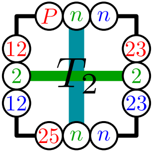

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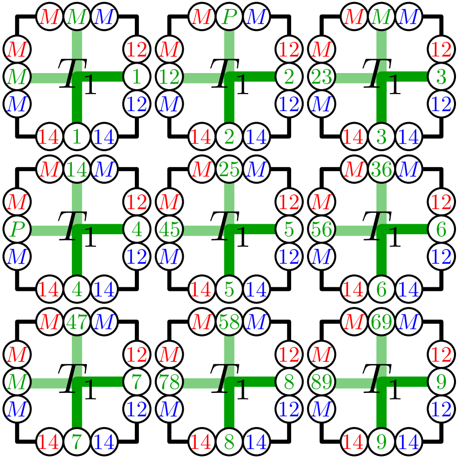
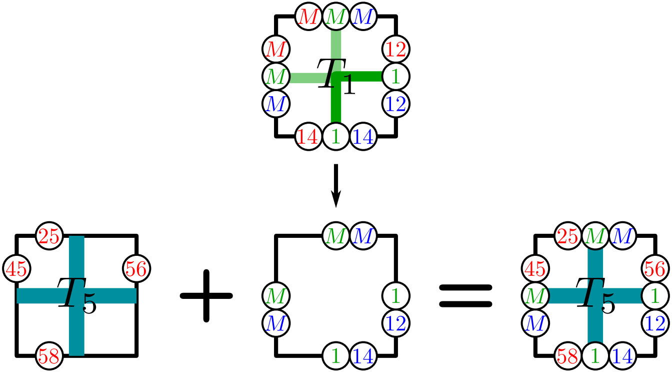
This yields $2 \times 8 + 4 \times 9 + 3 \times 6 \times 9 = 214$ decorated T_2.
Together with $T_{4,6,8}$ (214 each) and $T_{1,3,7,9}$ (9 each): 892 tiles.

Step 5: synchronizing network branches



Pairs on X- and Y-branches could be allowed on X- and Y-edges of <u>different</u> decorated T_j. The branches have to be *synchronized*.

Step 5: synchronizing network branches



This is done by allowing on T_5 the pairs of every non-central tile.

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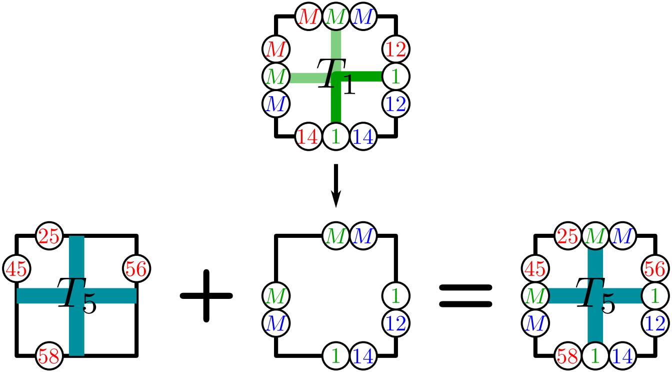
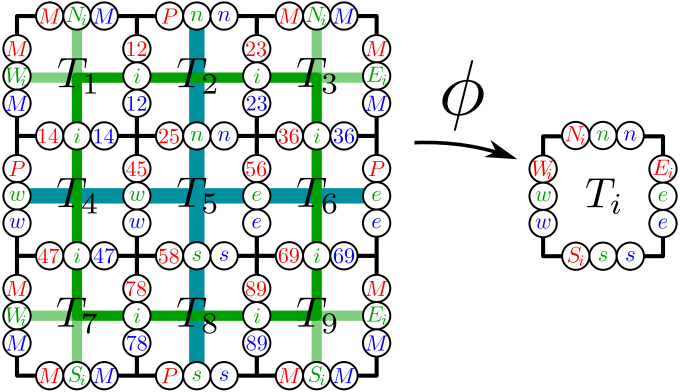


This is done by allowing on T_5 the pairs of every non-central tile.
This double the number of tiles: there are thus 1784 τ-tiles in all.

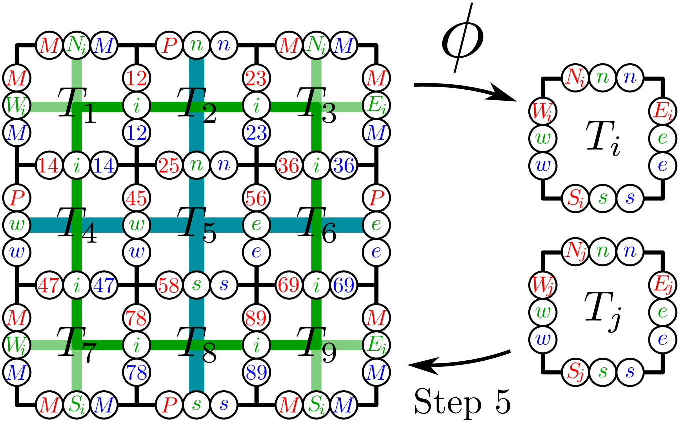
Back to the map ϕ



ϕ maps any τ-macro-tile with i on the ring onto a decorated T_i.

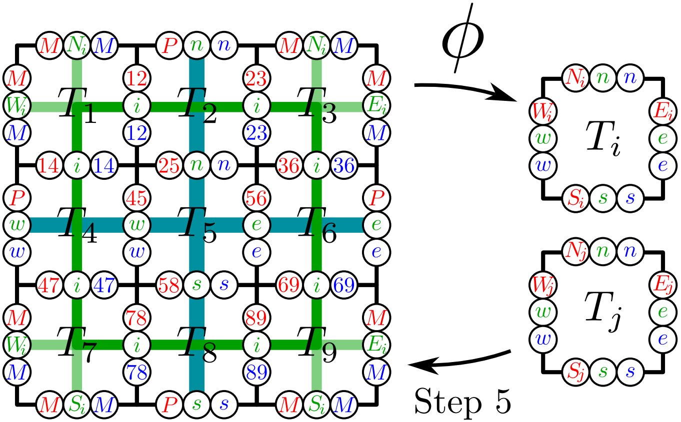
But why should it be a τ-tile?

Back to the map ϕ



The decorated T_j the central tile is derived from (Step 5) is in τ.
We claim that it is one and the same tile, except if $i = 5$.

Back to the map ϕ



When the ring intersects an X-branch of the network, it forces the green/blue pair to be allowed on some decorated T_i (Step 4).

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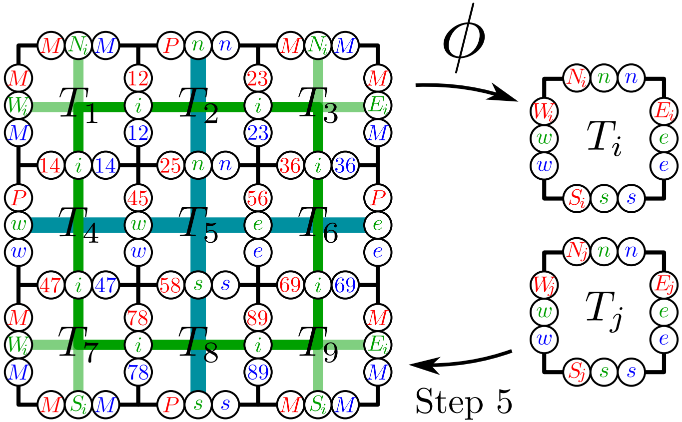

If the X-edge of T_i is not on the network of the macro-tile, then its blue index replicates its red one (Step 3).

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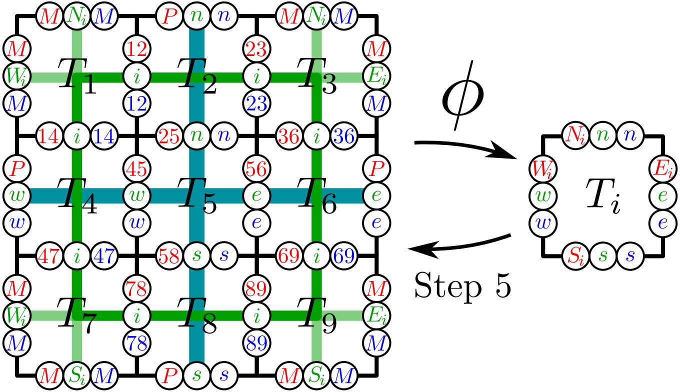

A red index (other than M) determines i (Step 1), whence $i = j$.

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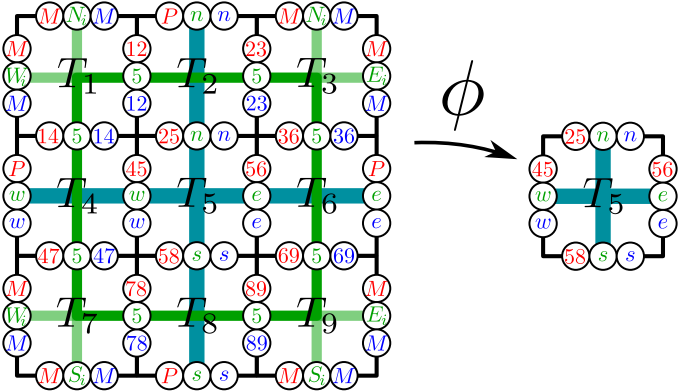

This fails for $i = 5$ because the network crosses every edge of T_5.
But in this case, ϕ simply maps the macro-tile onto its central tile!

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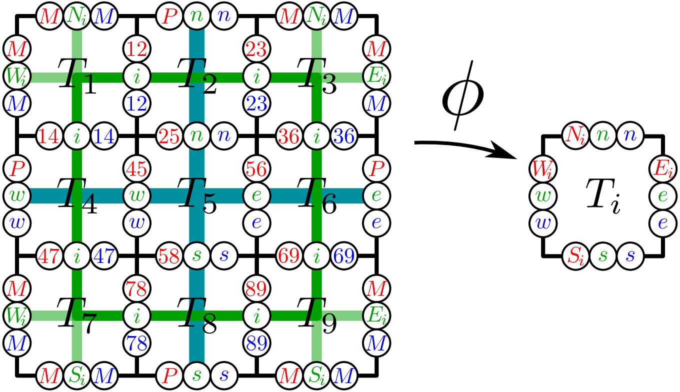

Whatever i on the ring, ϕ thus maps the τ-macro-tile onto a τ-tile.
It is moreover a bijection: the inverse function is straightforward.

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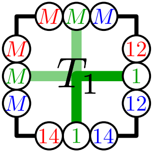

In particular, applying *ad infinitum* ϕ^{-1} to any τ-tile yields arbitrarily large τ-patches, hence a τ-tiling by compacity.

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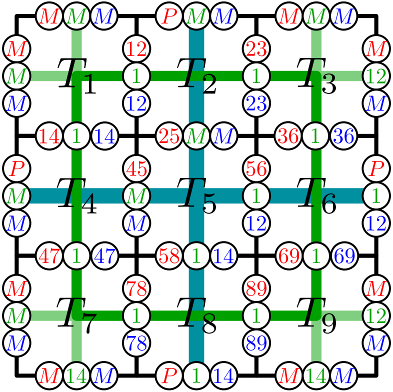

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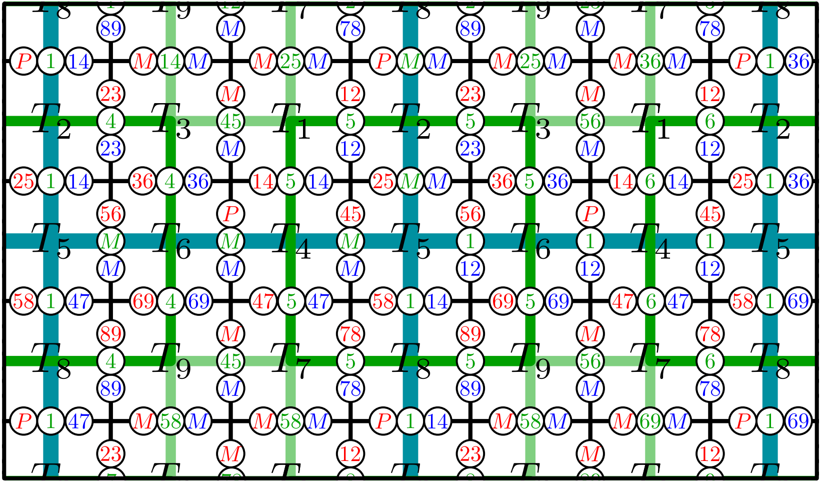

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Proof of the theorem

We use τ tiles to enforce a given 3×3 substitution.

The T_i's come in colors (those appearing in the substitution).

The ring indices as well.

The color of a τ-tile is determined w.r.t. the substitution by

- its position in the macro-tile (given by its red indices)
- the color on the ring (green index if $i \neq 5$, blue ones if $i = 5$).