# Stepped Planes, Stepped Surfaces and Generalized Substitutions

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## Introduction (1/3): Sturmian words

word: concatenation of letters (finite alphabet);

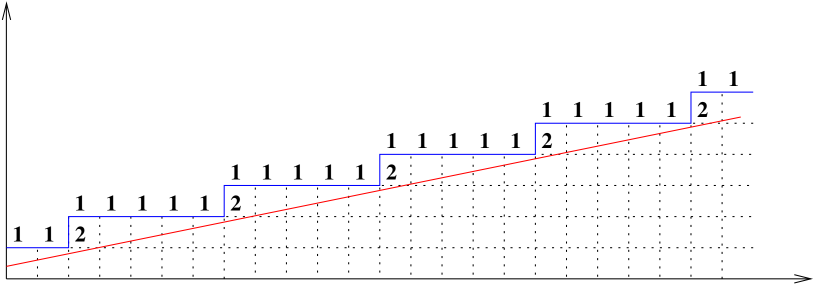
complexity: number $p(n)$ of factors of size $n$;

Sturmian words: aperiodic words of minimal complexity.

$$u = 12112121121121211212112\ldots \rightsquigarrow p(n) = n + 1.$$

# Introduction (2/3): Stepped lines

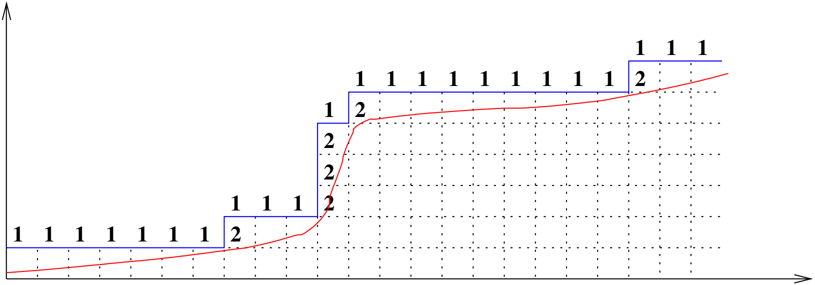
Straight half-line (red) $\rightsquigarrow$ stepped line (blue) $\rightsquigarrow$ 2-letter word:




Morse&Hedlund: Sturmian words $\equiv$ irrational slopes

# Introduction (3/3): Stepped curves

funct. curve (red) $\rightsquigarrow$ stepped curve (blue) $\equiv$ 2-letter word:



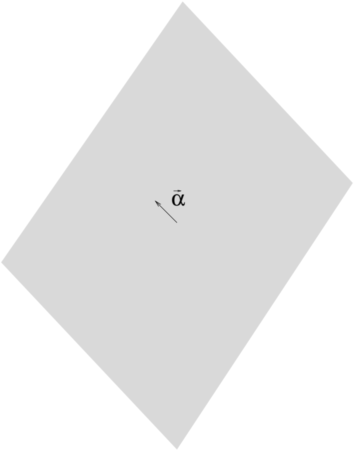

Sturmian words: aperiodic stepped curves of minimal complexity

# Outline

1. Stepped planes
   - Digitizations of real planes
   - Sturmian 2-dim. words
2. Stepped surfaces
   - Digitizations of real surfaces
   - Flips and shadows
3. Substitutions
   - Sturmian substitutions
   - Generalized substitutions

# Outline

- real plane normal to $\vec{\alpha}$
- union of unit cubes (below)
- *stepped plane* (boundary)
- lattice of rank 2
- 3-letter 2-dim. word



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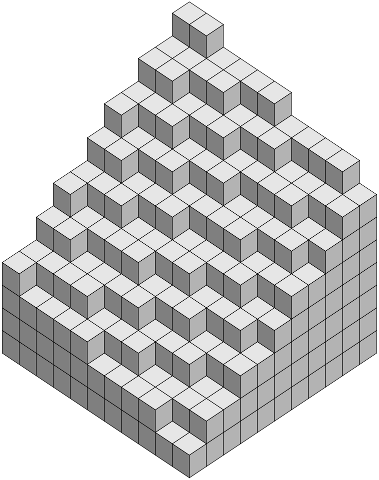

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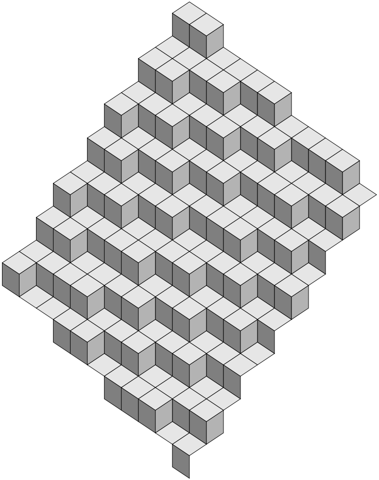

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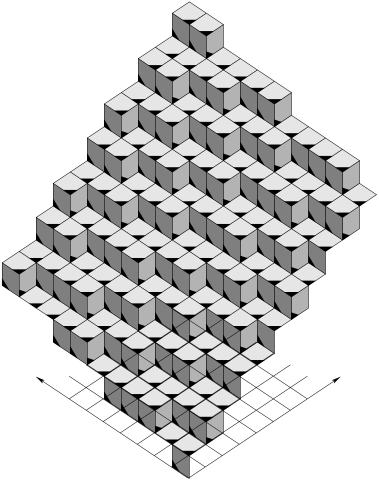

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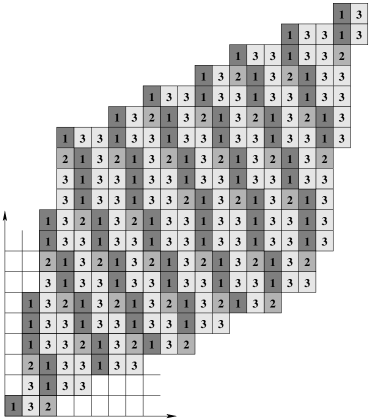

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Recall: aperiodic digitizations of lines ≡ Sturmian words.

**Definition (Vuillon,98)**

Sturmian 2-dim. words ≡ aperiodic digitizations of planes

Recall: Sturmian words ≡ aperiodic words of minimal complexity.

But: aperiodic 2-dim. words of minimal "complexity": 2 letters

⇝ restriction to a subset of the 2-dim. words?

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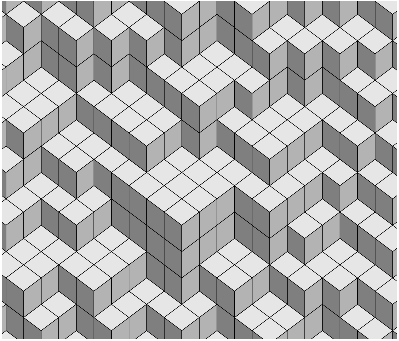
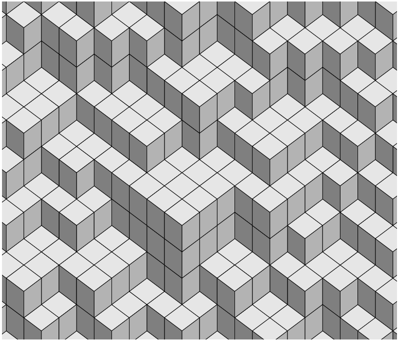
funct. surface $\rightsquigarrow$ stepped surface $\rightsquigarrow$ 3-letter 2-dim. word (not all):



Note: stepped surfaces $\equiv$ lozenge tiling of $\mathbb{R}^2$ + origin funct. surface $\rightsquigarrow$ stepped surface $\rightsquigarrow$ 3-letter 2-dim. word (not all):



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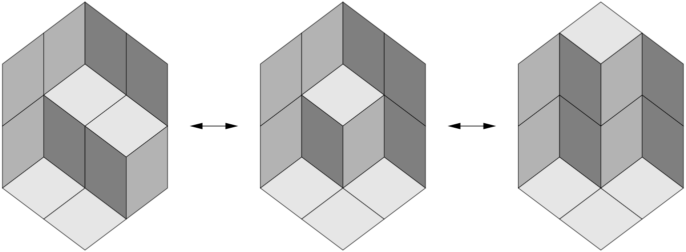
Recall: lozenge tilings of a finite domain are connected by *flips*:

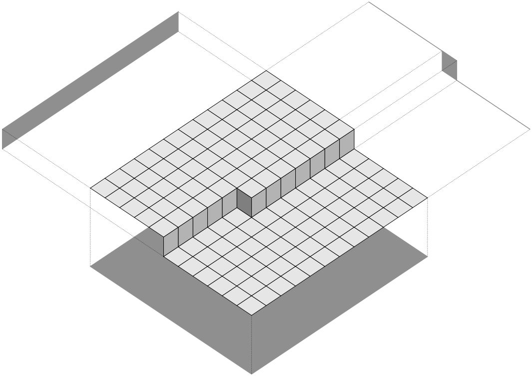


For stepped surfaces: flips $\equiv$ adding/removing unit cubes.

Connectivity?

# Outline

$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ canonical basis of $\mathbb{R}^3$, $\pi_i$ orthogonal projection along $\vec{e}_i$.
*Shadows* of a stepped surface $\mathcal{S}$: $\pi_1(\mathcal{S})$, $\pi_2(\mathcal{S})$, $\pi_3(\mathcal{S})$.



Note: invariant by performing flips.

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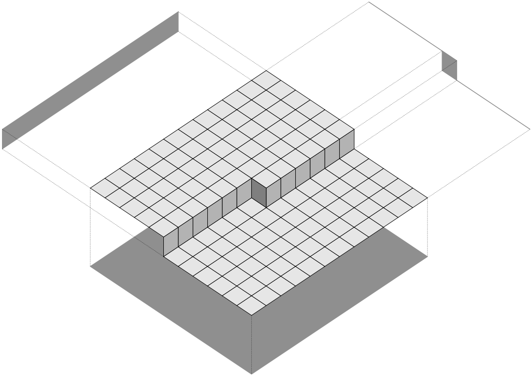

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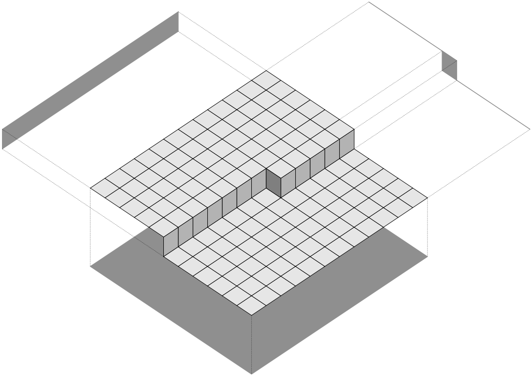

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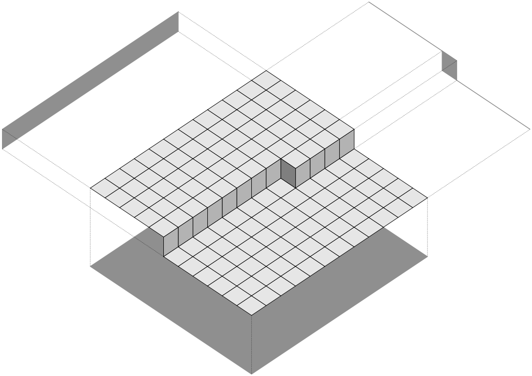

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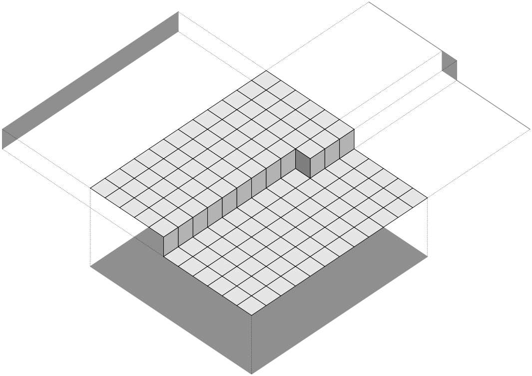

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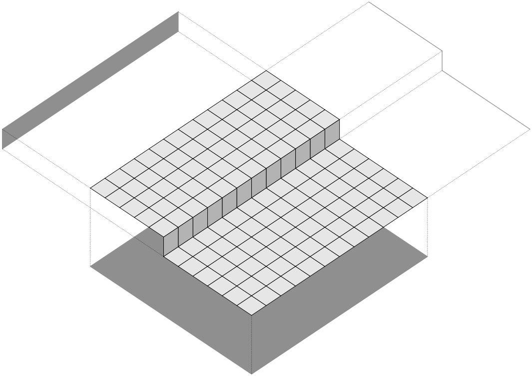
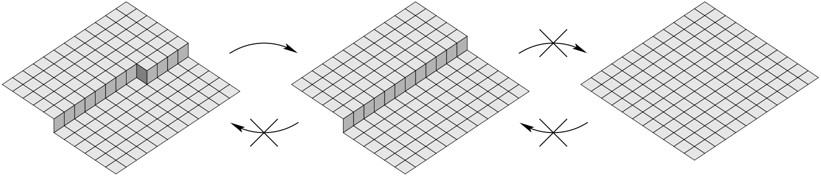


Note: invariant by performing flips finitely many times!

## Theorem (Arnoux, Berthé, Jamet, F.)

$\mathcal{S} \rightsquigarrow \mathcal{S}'$ *by a sequence of flips iff*, $\forall i$, $\pi_i(\mathcal{S}') \subset \pi_i(\mathcal{S})$.



## Corollary

*Stepped surface* $\equiv$ *stepped plane* $+$ *flips.*

*proof*: (non-degenerated) stepped planes have shadows $\mathbb{R}^2$.

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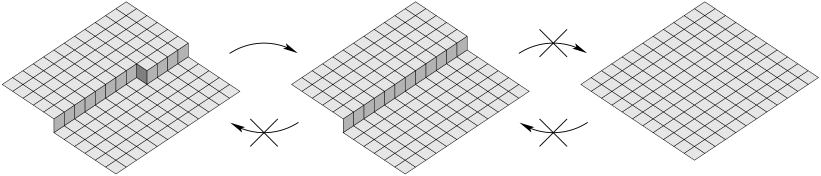

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substitution: non-erasing morphism: $\sigma(u\cdot v) = \sigma(u)\cdot\sigma(v)$;

Sturmian substitution: maps Sturmian words to Sturmian words.

$$\sigma : \begin{array}{l} 1 \to 12 \\ 2 \to 1 \end{array} \quad \rightsquigarrow \quad \sigma(12112\ldots) = 12112121\ldots$$

$\rightsquigarrow$ useful for *generating* and *classifying*

# Outline

Generalized substitution: map on unit faces of $\mathbb{R}^3$ (Arnoux-Ito).

**Theorem**

*Generalized substitions map stepped planes to stepped planes.*

*proof*: relies on the analytic expression of a stepped plane.

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- stepped surface = stepped plane + flips;
- flips are just "moved".

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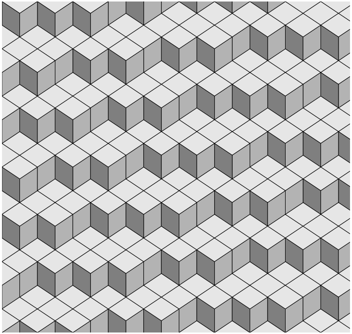

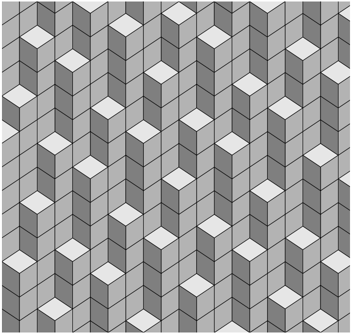

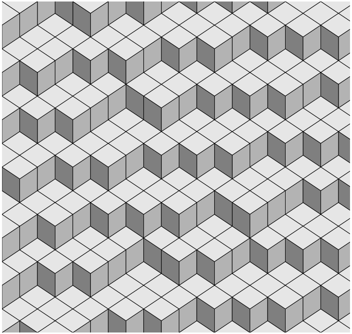

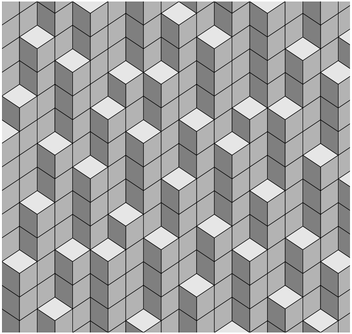

# Conclusion

| | | |
|---|---|---|
| 2-letter words | $\longrightarrow$ | stepped surfaces |
| Sturmian words | $\longrightarrow$ | aperiodic stepped planes |
| Sturmian substitutions | $\longrightarrow$ | generalized substitutions |
| complexity $n+1$ | $\longrightarrow$ | ? |
| substitutions | $\longrightarrow$ | ? |