A Self-Simulating Tileset

Thomas Fernique

Goal

We define step by step

- a finite set τ of decorated squares, where every edge is endowed with a (red,green,blue) triple of indices,
- a bijection ϕ from 3×3 squares of τ-tiles to τ-tiles,

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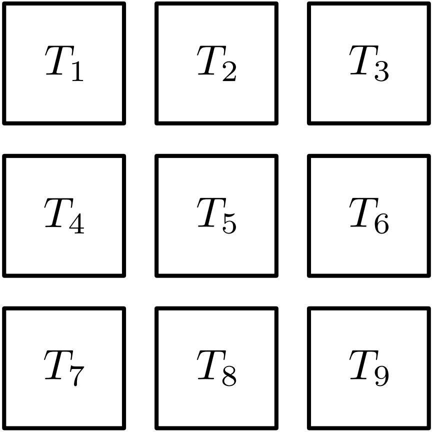
- a bijection ϕ from 3×3 squares of τ-tiles to τ-tiles,

such that:

- there exist τ-tilings,
- every τ-tiling can be uniquely partitioned into 3×3 squares,
- applying ϕ on these 3×3 squares (+scaling) yields a τ-tiling.

Step 1: macro-tiles



Start with tiles $T_1, \ldots, T_9$.

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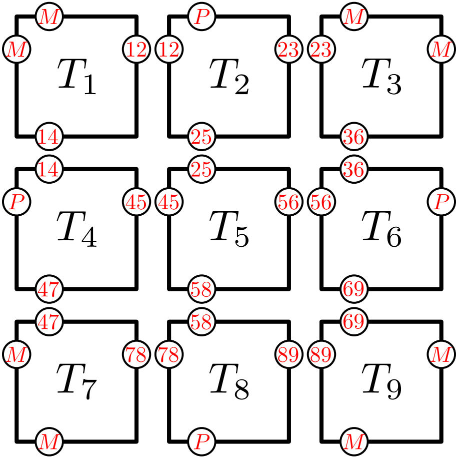

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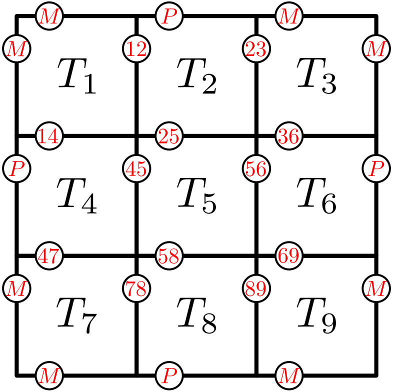

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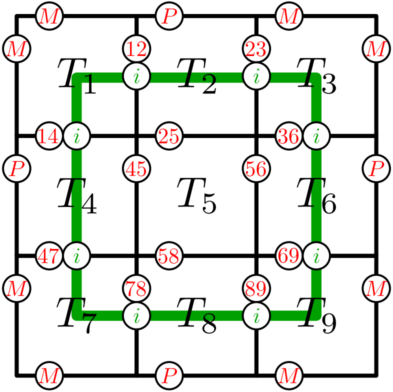
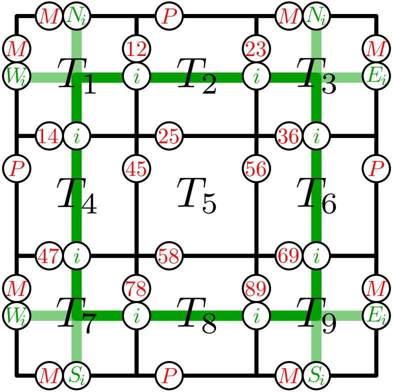
Step 2: rings and macro-macro-tiles



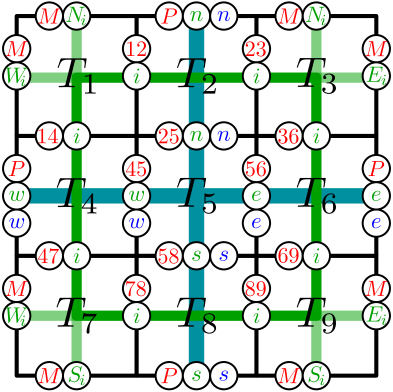
A green index $i \in \{1, \ldots, 9\}$ runs along a *ring* in every macro-tile.

Step 2: rings and macro-macro-tiles



Further green indices force rings to order as T_i's in a macro-tile.
(X_i denotes the red index on the X-edge of T_i, $X = N, W, S, E$)

Step 3: the network



A *network* carries green/blue indices from the central tile to *ports*.

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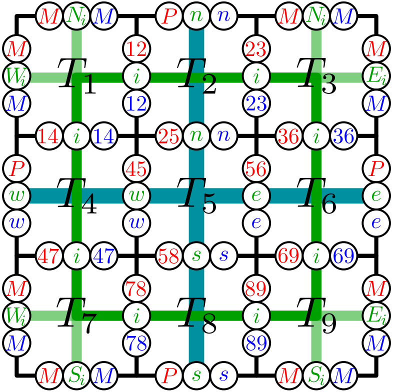

A *network* carries green/blue indices from the central tile to *ports*.
Off the network, each blue index replicates the red one.

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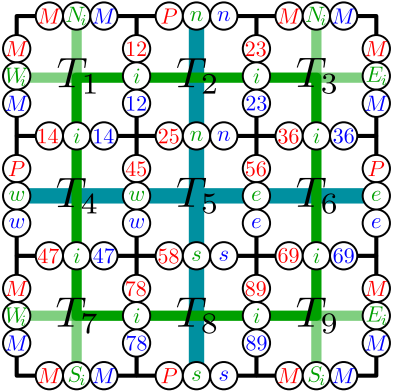
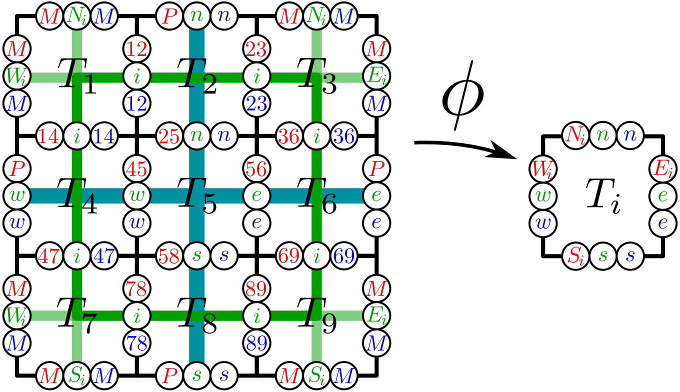


Duck Lemma

The green/blue indices of every $T_i \neq T_5$ determine the red indices.

Interlude: the map ϕ



ϕ maps the green indices forcing ring ordering onto red indices and copies the green/blue indices from network branches.

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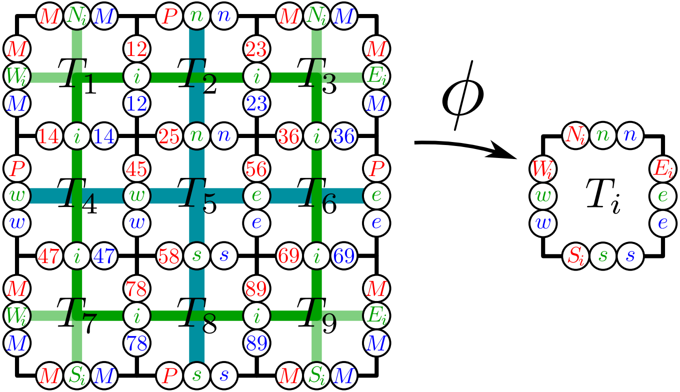

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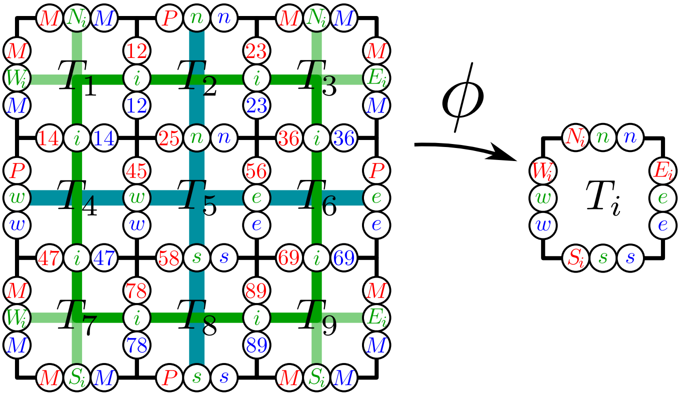
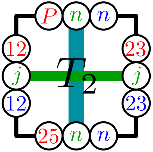


ϕ maps a tiling onto a tiling. Does it map a τ-tiling onto a τ-tiling?
The indices on the network will have to be chosen so that it holds.

Step 4: ring/network intersections

When a ring along which runs j crosses an X-branch, it checks that the pair carried by the branch is allowed on the X-edge of T_j.

For e.g. a North-branch:



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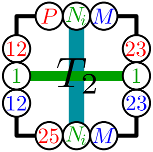

- $j \in \{1,3\} \rightsquigarrow \{M, P, 14, 25, 36, 47, 58, 69\} \times \{M\}$;

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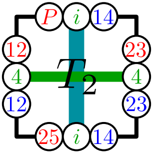

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- $j \in \{4, 6, 7, 9\} \rightsquigarrow \{1, \ldots, 9\} \times \{N_j\}$;

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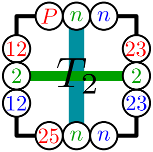

- $j \in \{1,3\} \rightsquigarrow \{M, P, 14, 25, 36, 47, 58, 69\} \times \{M\}$;
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- $j \in \{2,5,8\} \rightsquigarrow$ every pair already defined on any North-edge!

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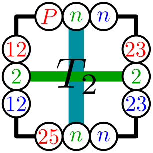

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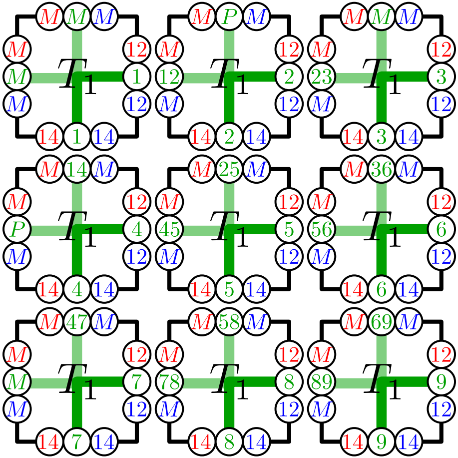
This yields $2 \times 8 + 4 \times 9 + 3 \times (2 \times 8 + 4 \times 9) = 208$ decorated T_2. Together with $T_{4,6,8}$ (208 each) and $T_{1,3,7,9}$ (9 each): 868 tiles.

Step 5: synchronizing network branches



Pairs on X- and Y-branches could be allowed on X- and Y-edges of <u>different</u> decorated T_j. The branches have to be *synchronized*.

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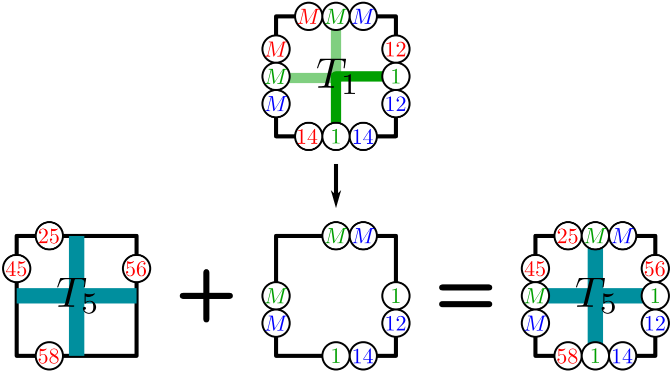

This is done by allowing on T_5 the pairs of every non-central tile.

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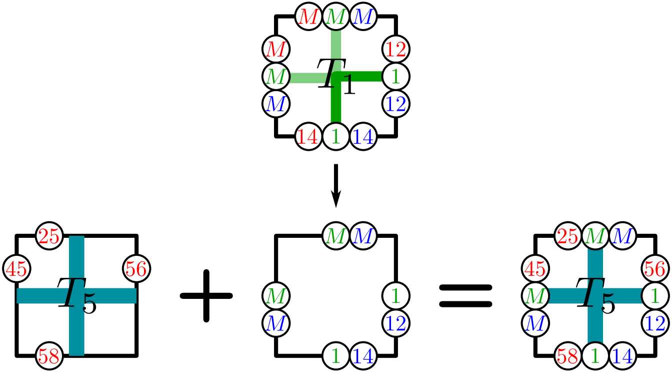
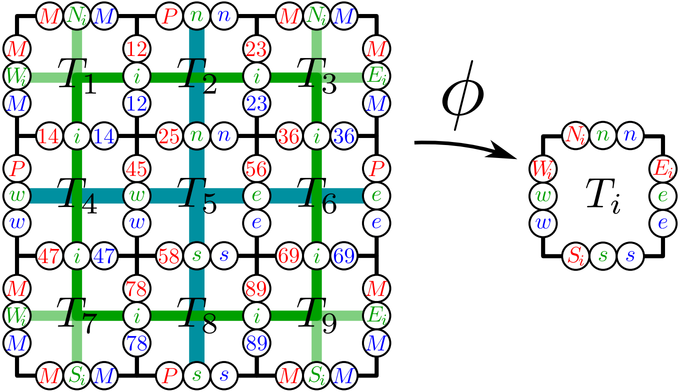


This is done by allowing on T_5 the pairs of every non-central tile.
This double the number of tiles: there are thus 1736 τ-tiles in all.

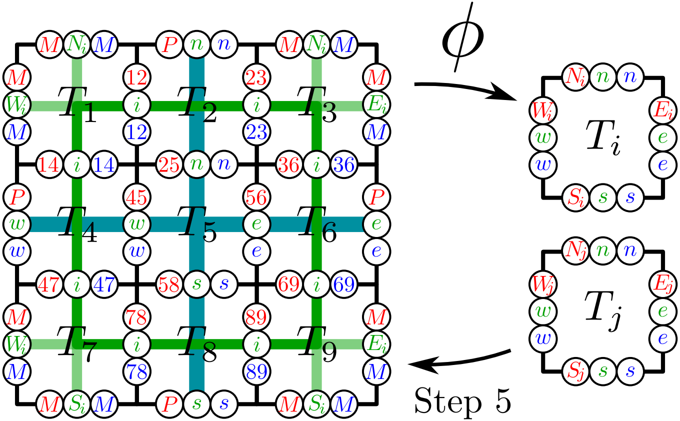
Back to the map ϕ



ϕ maps any τ-macro-tile with i on the ring onto a decorated T_i.

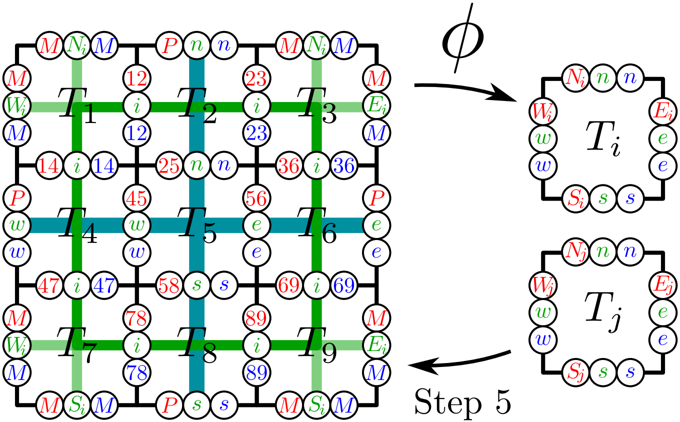
But why should it be a τ-tile?

Back to the map ϕ



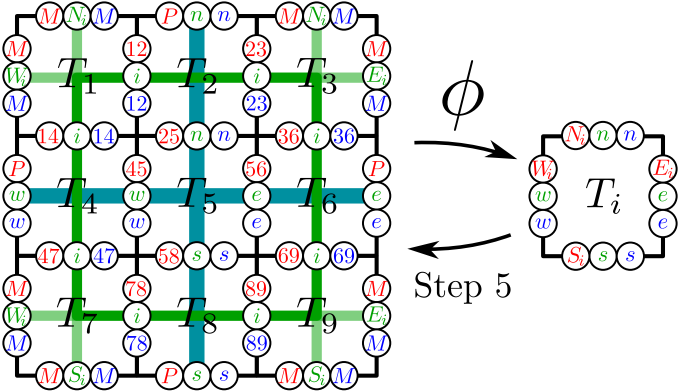
The decorated T_j the central tile is derived from (Step 5) is in τ.
We claim that it is one and the same tile, except if $i = 5$.

Back to the map ϕ



When the ring intersects an X-branch of the network, it forces the green/blue pair to be allowed on some decorated T_i (Step 4).

Back to the map ϕ



The green/blue indices of $T_{i\neq 5}$ determine i (Duck lemma) $\rightsquigarrow$ $i = j$.

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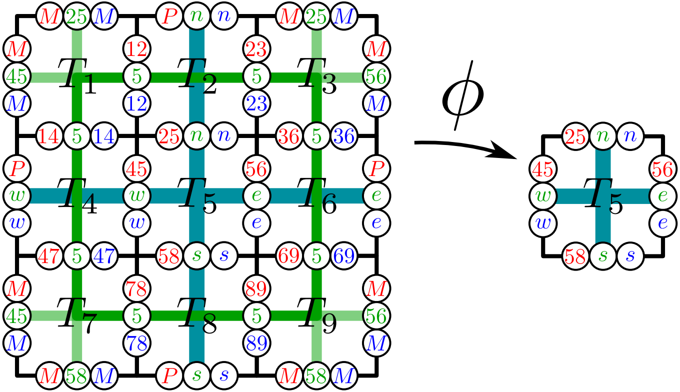

For $i = 5$, ϕ simply maps the macro-tile onto its central tile!

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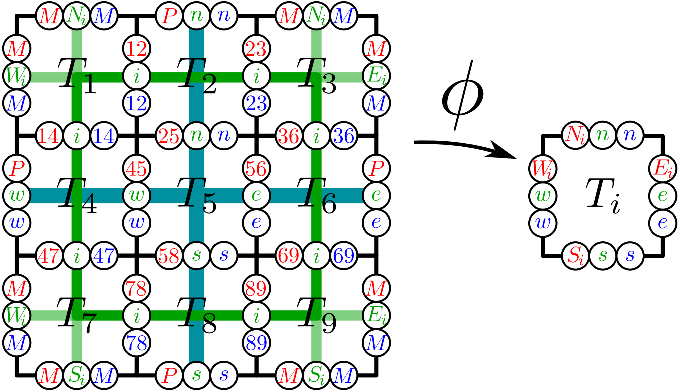

Whatever i on the ring, ϕ thus maps the τ-macro-tile onto a τ-tile.
It is moreover a bijection: the inverse function is straightforward.

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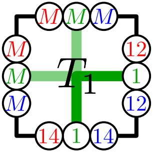

In particular, applying *ad infinitum* ϕ^{-1} to any τ-tile yields arbitrarily large τ-patches, hence a τ-tiling by compacity.

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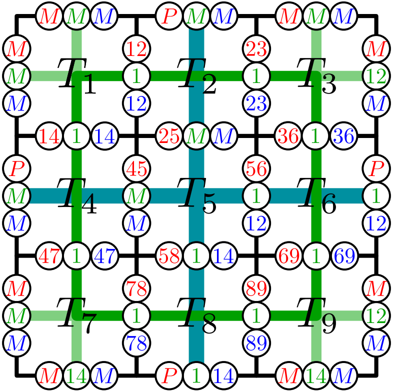

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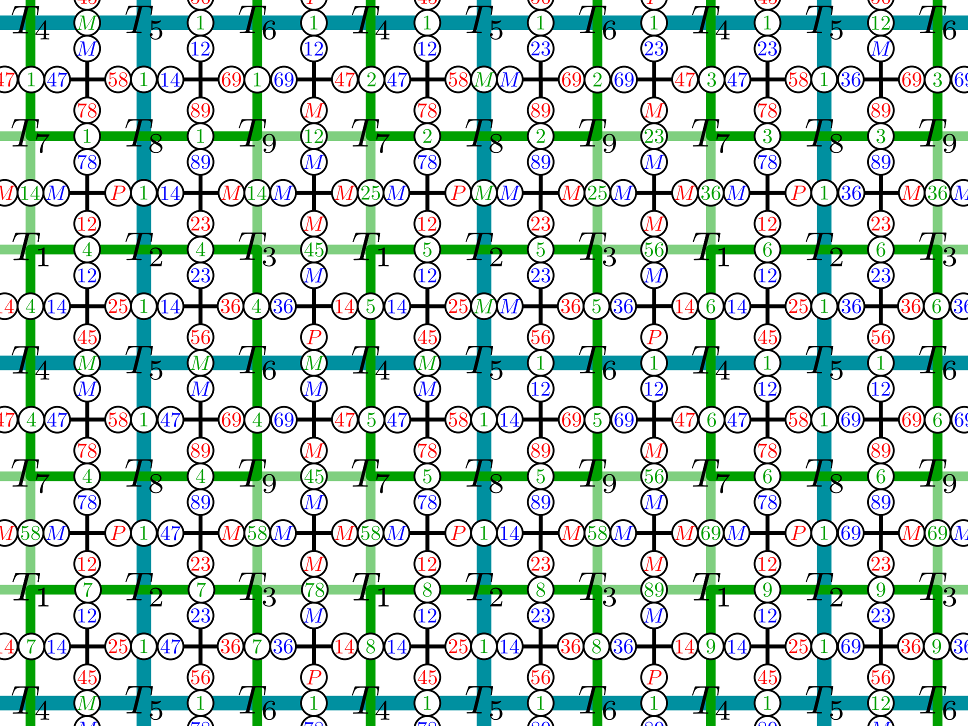

T4 T5 T6 T4 T5 T6 T4 T5 T6
T7 T8 T9 T7 T8 T9 T7 T8 T9
T1 T2 T3 T1 T2 T3 T1 T2 T3
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