Cut and Project Tilings 2:
Local Rules

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Characterizing slopes by forbidden patterns

Definition

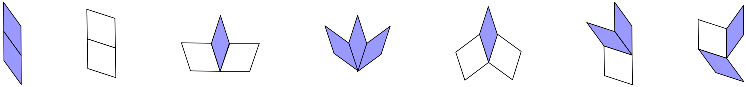
A d-plane E of $\mathbb{R}^n$ is said to be *characterized by forbidden patterns* if there is a finite set of patterns so that any $n \to d$ planar tiling without any of these patterns has a slope parallel to E.

Any rational plane. Some irrational planes. Which ones?

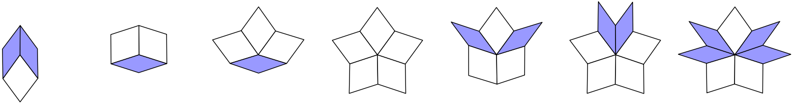
In symbolic dynamics: subshifts of finite type
In condensed matter theory: quasicrystals.

Examples

The Penrose tilings are characterized by the forbidden patterns:



Equivalently, they are the tilings with the following *vertex-atlas*:



The Ammann-Beenker tilings cannot be defined in such a way.

Coincidences

We shall assume in the following that the projection of $\mathbb{Z}^n$ is dense in the window of E.

Definition (coincidence)

A *coincidence* of a planar $n \to d$ tiling, this is $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in the window.

This is the smallest region corresponding to a pointed pattern.

Theorem (Bédaride-F.)

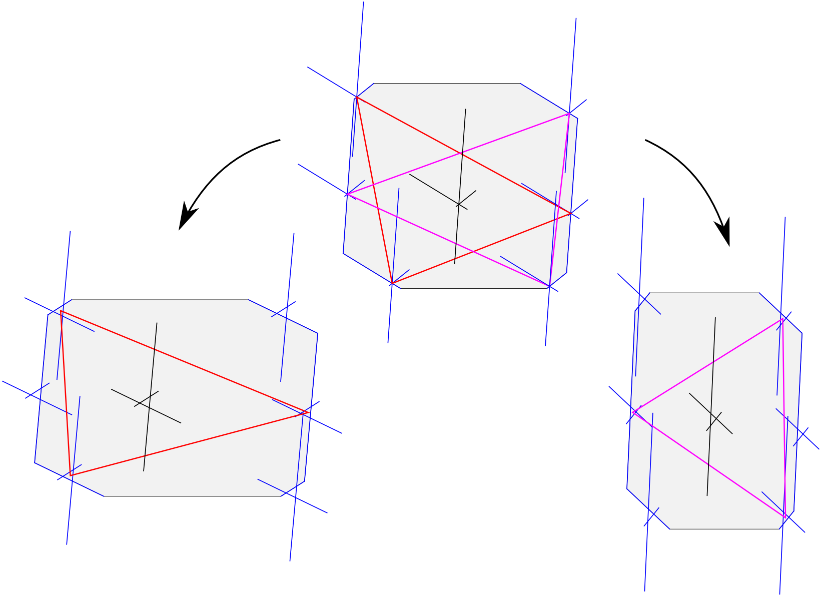
Coincidences and forbidden patterns characterize the same slopes.

From forbidden patterns to coincidences

Proof sketch:

- A forbidden pattern corresponds to an empty interior region.
- If the region is empty, then the pattern is useless!
- Otherwise, the region itself is a coincidence.
- These coincidences force each pattern to occur at most once.
- By shifting the slope, we can avoid all these forbidden pattern.
- The slope is therefore parallel to E.

From coincidences to forbidden patterns



In equations

Proposition (Bédaride-F.)

Each coincidence of a planar tiling corresponds to an algebraic equation on the Grassmann coordinates of its slope.

For an $n \to d$ tiling, this equation is homogeneous of degree $n - d$.

Corollary (Le, 1995)

If a slope is charaterized by forbidden patterns, then it is algebraic.

Examples

$$\vec{x}_0 = \begin{pmatrix} 1 \\ r_1 \\ 2 \\ 5 \end{pmatrix} \qquad \vec{x}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ r_2 \end{pmatrix} \qquad \vec{x}_2 = \begin{pmatrix} 4 \\ 1 \\ r_3 \\ 2 \end{pmatrix}$$

$$3\,G_{12}\,G_{13} + G_{14}\,G_{23} = G_{12}\,G_{14} + 2\,G_{13}\,G_{14} + 3\,G_{13}\,G_{24}.$$

$$\vec{x}_0 = \begin{pmatrix} r_1 \\ 1 \\ r_2 \\ 3 \\ 1 \end{pmatrix} \quad \vec{x}_1 = \begin{pmatrix} r_3 \\ 3 \\ 1 \\ 1 \\ r_4 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} 0 \\ r_5 \\ r_6 \\ 0 \\ 1 \end{pmatrix} \quad \vec{x}_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ r_7 \\ r_8 \end{pmatrix}$$

$$3\,G_{15}\,G_{23}\,G_{24} + 2\,G_{12}\,G_{23}\,G_{45} + G_{12}\,G_{24}\,G_{45} = 3\,G_{23}\,G_{24}\,G_{45} + 2\,G_{12}\,G_{34}\,G_{45}.$$

Local rules

Forbidden patterns do not necessarily ensure planarity!

Definition (local rules)

A d-plane E of $\mathbb{R}^n$ is said to have *local rules* if there is a finite set of patterns so that any $n \to d$ tiling without any of these patterns is planar and has a slope parallel to E.

At least $\frac{dn}{d+1}$ *linear coincidences* are necessary... (work in progress).