Brun expansions of stepped surfaces

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main result:
Action of dual maps of *free group morphisms* over stepped planes and surfaces (extends the substitutive case). Hidden tool: *flip*.

Application 1 (now):
Define Brun expansions of stepped planes and surfaces.

Application 2 (ask for details later):
Decide whether a given stepped surface is a stepped plane or not (and for patches too).

1. Stepped planes and stepped surfaces
2. Dual maps of free group morphisms
3. Brun expansions of stepped planes
4. Brun expansions of stepped surfaces

1 Stepped planes and stepped surfaces

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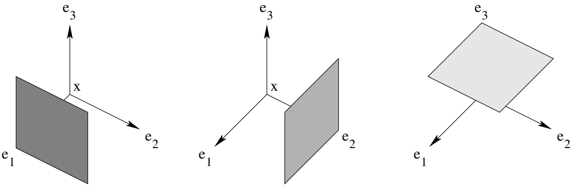
$(\vec{e}_1, \ldots, \vec{e}_d)$ basis of $\mathbb{R}^d$. $\vec{x} \in \mathbb{Z}^d$, $i \in \{1, \ldots, d\} \rightsquigarrow$ *face* $(\vec{x}, i^*)$:

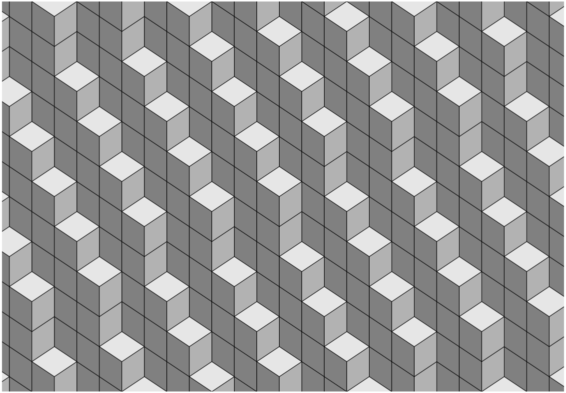


Definition

Stepped plane of normal vector $\vec{\alpha} \in \mathbb{R}^d_+ \backslash \{0\}$:

$$\mathcal{P}_{\vec{\alpha}} = \{(\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle \leq 0 < \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle\}.$$

Stepped plane



A stepped plane.

Let π be the orthogonal projection along $\vec{u} = \vec{e}_1 + \ldots + \vec{e}_d$.

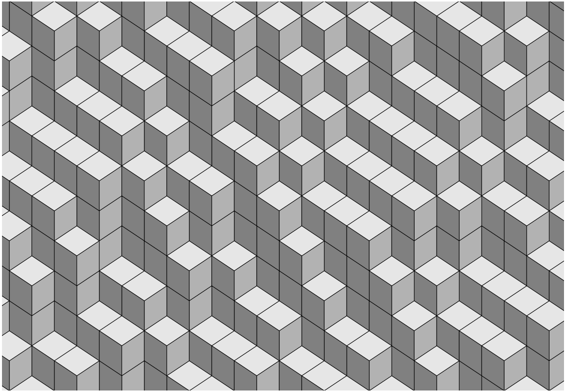
Proposition

Stepped planes are homeomorphic to $\vec{u}^{\perp}$ by π.

By extension:

Definition [Jamet]

Stepped surfaces : any set of faces homeomorphic to $\vec{u}^{\perp}$ by π.



A stepped surface.

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Morphism of the free group over $\{1,\ldots,d\}$ (here, $d=3$):

$$\sigma \ : \ \begin{cases} 1 & \mapsto & 3 \\ 2 & \mapsto & 3^{-1}1 \\ 3 & \mapsto & 3^{-1}2 \end{cases}$$

For example: $\sigma(1^{-1}312) = \sigma(1)^{-1}\sigma(3)\sigma(1)\sigma(2) = 3^{-2}21$.

Incidence matrix: $(M_\sigma)_{ij} = |\sigma(i)|_j - |\sigma(i)|_{j^{-1}}$. Here:

$$M_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

σ *unimodular* f. g. morph. $\rightsquigarrow$ dual map $E_1^*(\sigma)$ over *weighted* faces.

For σ previously defined:

$$E_1^*(\sigma) \ : \ \begin{cases} (\vec{0}, 1^*) & \mapsto \quad (\vec{e}_1, 2^*) \\ (\vec{0}, 2^*) & \mapsto \quad (\vec{e}_1, 3^*) \\ (\vec{0}, 3^*) & \mapsto \quad (\vec{0}, 1^*) - (\vec{e}_1, 2^*) - (\vec{e}_1, 3^*). \end{cases}$$

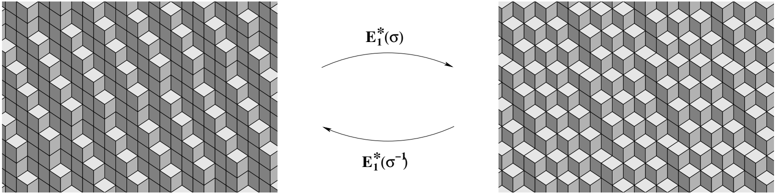
and, for $\lambda \in \mathbb{Z}$, $\vec{x} \in \mathbb{Z}^d$:

$$E_1^*(\sigma)(\lambda.(\vec{x}, i^*)) = M_\sigma^{-1}\vec{x} + \lambda.E_1^*(\sigma)(\vec{0}, i^*).$$

Theorem (B. F. 2007)

For σ unimodular free group morphism and $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \vec{0}$:

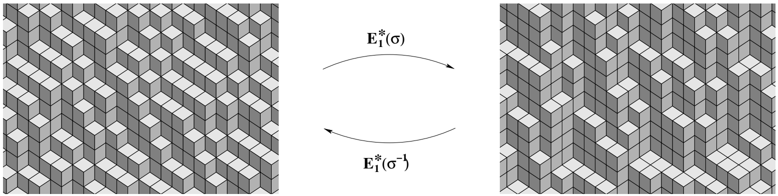
$$M_\sigma^\top \vec{\alpha} \in \mathbb{R}^d_+ \;\Rightarrow\; E_1^*(\sigma)(\mathcal{P}_{\vec{\alpha}}) = \mathcal{P}_{M_\sigma^\top \vec{\alpha}}.$$



Note: the action of $E_1^*(\sigma)$ depends only on M_σ (but not on σ).

Theorem (B. F. 2007)

For σ unimodular free group morphism: if the image by $E_1^*(\sigma)$ of a stepped surface has faces with weights in $\{0,1\}$, then it is a stepped surface. This holds, in particular, when $M_\sigma \geq 0$.



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Brun map T, defined for $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \backslash \{0\}$:

$$T(\alpha_1, \ldots, \alpha_d) = \left(\frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_d}{\alpha_i}\right),$$

where $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_\infty\}$. Matrix viewpoint:

$$(1, T(\vec{\alpha}))^\top \propto \begin{pmatrix} 0 & & 1 & \\ & \mathrm{I}_{i-1} & & \\ 1 & & -a & \\ & & & \mathrm{I}_{d-i} \end{pmatrix} (1, \vec{\alpha})^\top$$

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Brun expansion $(a_n, i_n)_{n \geq 0}$ of $\vec{\alpha}$:

$$a_n = \left\lfloor ||T^n(\vec{\alpha})||_\infty^{-1} \right\rfloor \quad \text{and} \quad i_n = \min\{j \mid \langle T^n(\vec{\alpha}) | \vec{e}_j \rangle = ||T^n(\vec{\alpha})||_\infty\}.$$

How to define Brun exp. of given stepped planes (unknown normal vectors) so that $\mathcal{P}_{(1,\vec{\alpha})}$ will have the Brun exp. of $\vec{\alpha}$?

Note: if $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_\infty\}$ and $a = \lfloor 1/\alpha_i \rfloor$ are known:

$$E_1^*(\beta_{a,i})(\mathcal{P}_{(1,\vec{\alpha})}) = \mathcal{P}_{(1,T(\vec{\alpha}))},$$

where $\beta_{a,i}$ has incidence matrix $B_{a,i}$ s.t. $B_{a,i}(1,\vec{\alpha}) = (1, T(\vec{\alpha}))$.

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Determining $(a,i) \rightsquigarrow$ *entries comparisons* and *floor computation*.

entries comparisons:

$(\vec{x},(i+1)^*),(\vec{x}+\vec{e}_{j+1},(i+1)^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}$ for some $\vec{x}$ yields $\alpha_i > \alpha_j$.

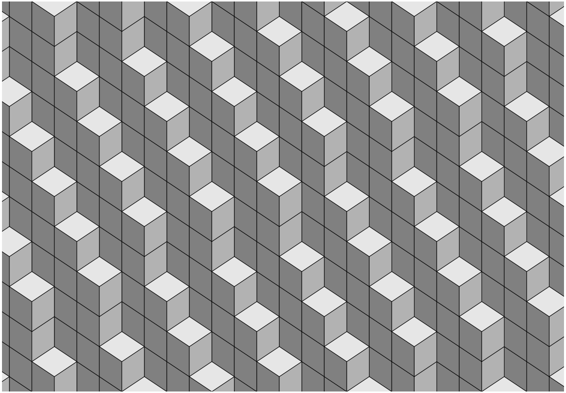
floor computation:

Let us introduce:

$$a_i(\mathcal{P}) = \max\{a \in \mathbb{N} \mid (\vec{x},(i+1)^*) \lhd \mathcal{P} \Rightarrow (\vec{x}-k\vec{e}_{i+1},1^*)_{0\leq k<a} \lhd \mathcal{P}\}.$$

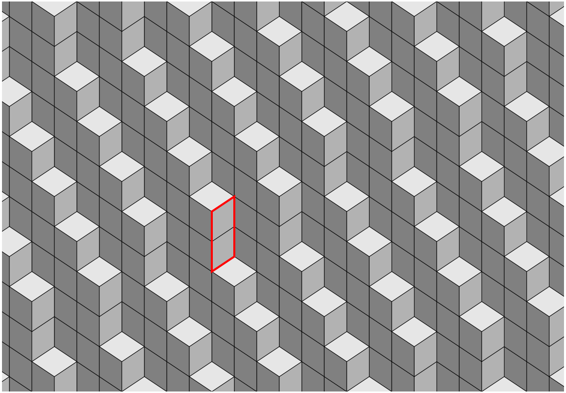
One shows:

$$a_i(\mathcal{P}_{(1,\vec{\alpha})}) = \left\lfloor \frac{1}{\alpha_i} \right\rfloor.$$



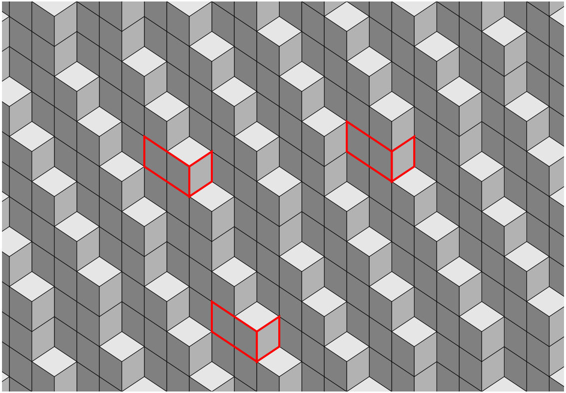
Stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$, with unknown $\vec{\alpha} = (\alpha_1, \alpha_2) \in [0,1]^2 \setminus \{0\}$.

Brun expansion of a stepped plane



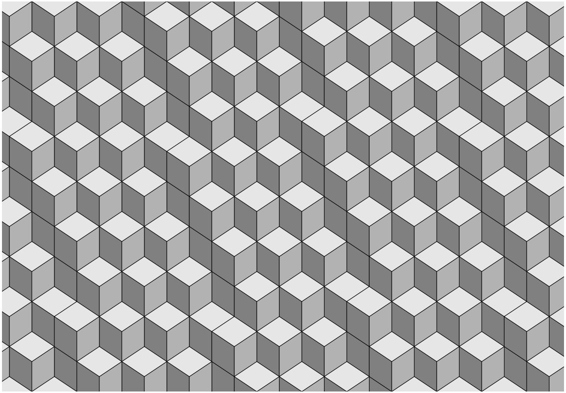
$(\vec{0}, 2^*), (\vec{e}_3, 2^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}$. Thus, $\alpha_1 > \alpha_2$.

Brun expansion of a stepped plane



$$(\vec{x}, 2^*) \lhd \mathcal{P}_{(1,\vec{\alpha})} \Rightarrow (\vec{x}, 1^*), (\vec{x} - \vec{e}_2, 1^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}.$$ Thus, $\lfloor 1/\alpha_1 \rfloor = 2$.

Brun expansion of a stepped plane



Finally: $\mathcal{P}_{(1,T(\vec{\alpha}))} = E_1^*(\beta_{2,1})(\mathcal{P}_{(1,\vec{\alpha})})$.

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Dual maps and “information grabbing” defined for stepped surfaces

$\rightsquigarrow$ natural extension of Brun expansions for stepped surfaces.

Dual maps and "information grabbing" defined for stepped surfaces

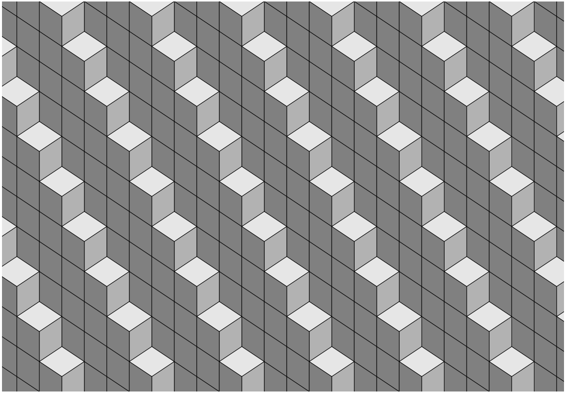
$\rightsquigarrow$ natural extension of Brun expansions for stepped surfaces.

Theorem (B. F. 2007)

Stepped surfaces having the Brun expansion of $\vec{\alpha} \in \mathbb{R}_+^d \setminus \{0\}$ are:

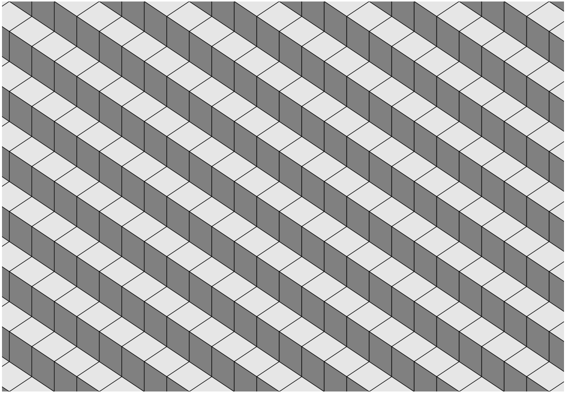
- *the stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$ (finite or infinite expansion);*
- *some stepped surfaces almost equal to $\mathcal{P}_{(1,\vec{\alpha})}$ (idem);*
- *some non-plane stepped surfaces (only finite expansion).*

The stepped plane case (finite or infinite expansion)



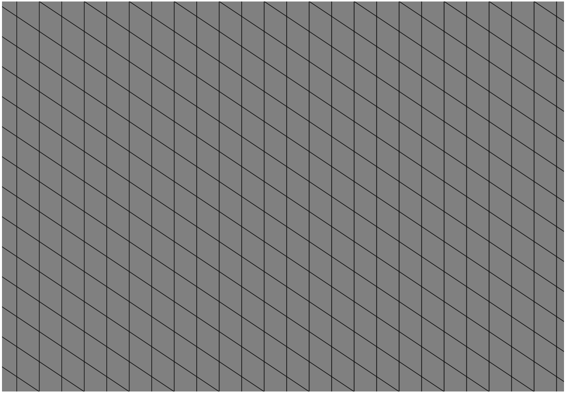
$$(a, i) = (4, 1)$$

The stepped plane case (finite or infinite expansion)



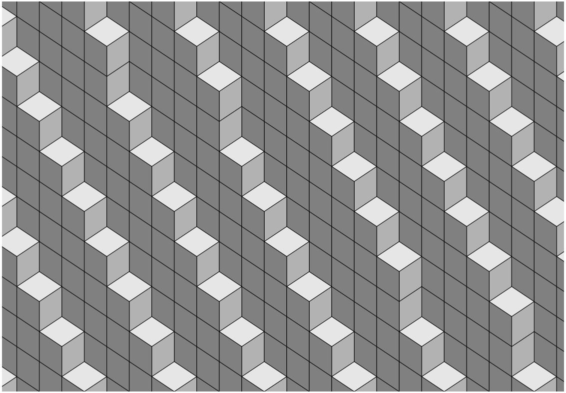
$$(a, i) = (1, 2)$$

The stepped plane case (finite or infinite expansion)



$a = \infty$. Stepped plane recognized.

The stepped quasi-plane case (finite or infinite expansion)



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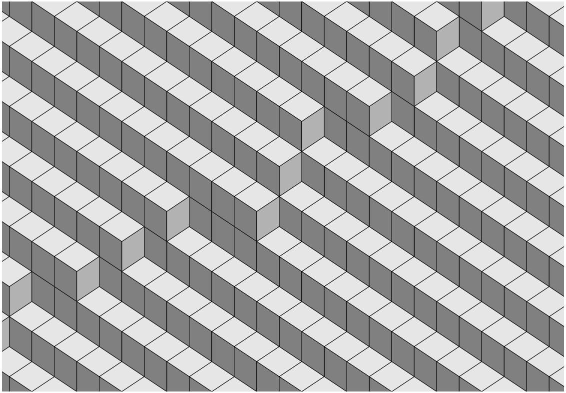
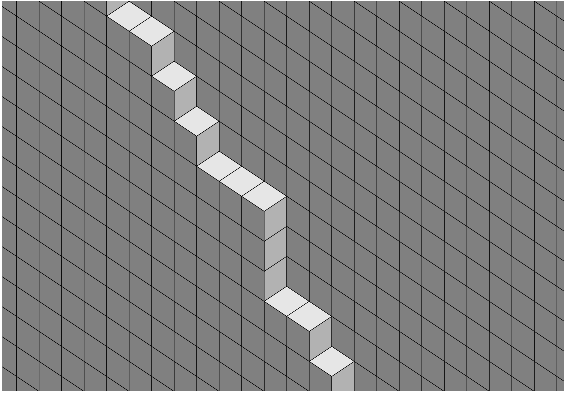
The stepped quasi-plane case (finite or infinite expansion)

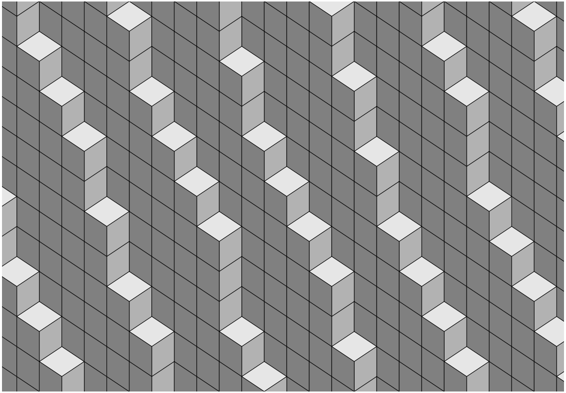


$$(a, i) = (1, 2)$$



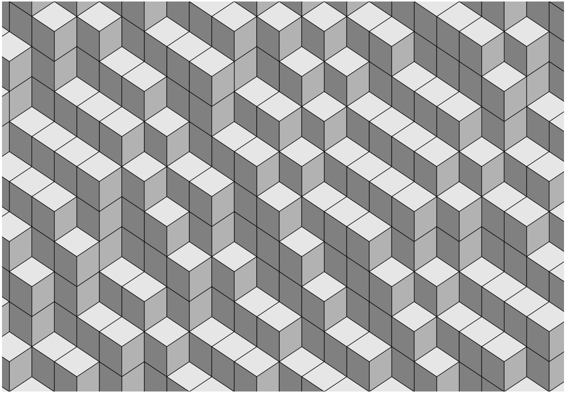
$a = \infty$. Not a stepped plane... but almost.

The stepped surface case (only finite expansion)



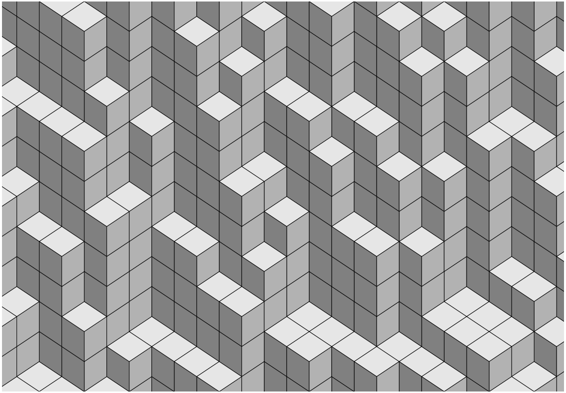
$$(a, i) = (4, 1)$$

The stepped surface case (only finite expansion)



$$(a, i) = (1, 2)$$

The stepped surface case (only finite expansion)



a undefined. Not at all a stepped plane.

Where is "digital plane recognition"?

Finite expansions $\rightsquigarrow$ stepped planes recognized from the last obtained stepped surface.

Patches (finite subset of stepped surfaces) $\rightsquigarrow$ finite expansions. . .