Around the Nivat's conjecture

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Seminar "Faktor Jaziky"
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Januar 28, 2008

Imja: Thomas (Tomá, ili Fomá).
Ochestvo: Francisovich (mozhno bez).
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In this talk (in english): some questions that I would like to examine during my stay, with anybody who is interested in. . .

Infinite word: sequence of letters indexed by $\mathbb{N}$.

Subwords of length n of a word w: $L_n(w)$.

(Factor) complexity of a word w: $p_w : n \mapsto \mathrm{Card}(L_n(w))$.

(Eventually) periodic word w: for $k \geq d$, $w_{k+p} = w_k$.

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Theorem (Morse-Hedlund)

A word w is (eventually) periodic iff, for some n, $p_w(n) \leq n$.

Thus, min. complexity of aperiodic words: $n+1$.

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Such words exist: *Sturmian words*. Many equivalent definitions:

1. aperiodic words of minimal complexity;
2. coding of irrational discrete lines;
3. coding of irrational rotations on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$;
4. ...

Bi-infinite word: sequence of letters indexed by $\mathbb{Z}$.

Subwords and (factor) complexity: *idem*.

Periodic bi-infinite word w: for any $k \in \mathbb{Z}$, $w_{k+p} = w_k$.

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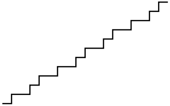
Bi-infinite Sturmian words: similar, but some “degenerated cases”:



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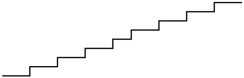

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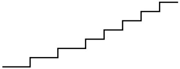

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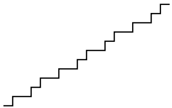
Up to the action of Sturmian morphisms (Ali Aberkane):



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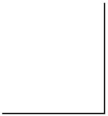

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Bidim. word: sequence of letters indexed by $\mathbb{Z}^2$.

Subwords: set $L_{(m,n)}(w)$ of $m \times n$ rectangular factors.

(Rectangular) complexity: $p_w : (m,n) \mapsto \mathrm{Card}(L_{(m,n)}(w))$.

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Conjecture (Nivat'97)

A bidim. word w is periodic if, for some m, n, $p_w(m,n) \le mn$.

Note: this cannot be an equivalence (easy counter-ex.).

Some partial results:

1. holds if $p_w(m,n)$ is bounded (easy);
2. does not holds in higher dimensions (Cassaigne'00);
3. holds if $p_w(2,n) \leq 2n$ (Sander-Tijdeman'02);
4. holds if $p_w(m,n) \leq mn/144$ (Epifanio-Koskas-Mignosi'03);
5. holds if $p_w(m,n) \leq mn/16$ (Quas-Zamboni'04).

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Seems hard, but J. Cassaigne did not found a counter-example. . .

If it holds, min. complexity of aperiodic bidim. words: $mn + 1$.

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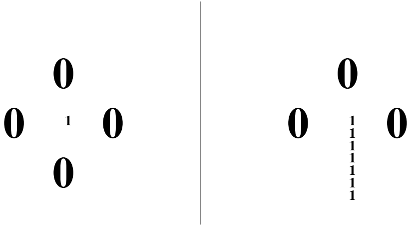
Such words exist, Cassaigne described them:

A single 1 or a half-line of 1's (up to some rescaling):



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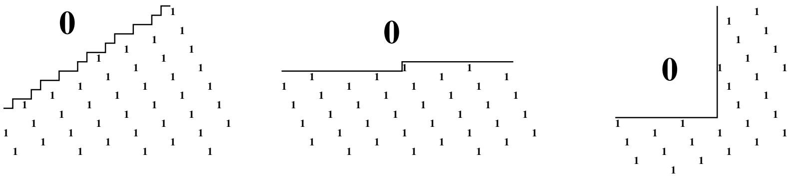
or Sturmian bi-infinite outline of a lattice (up to some rescaling):



However, contrary to classic Sturmian words:

1. they are not uniformly recurrent;
2. no known equivalent definition (rotation, discrete line. . .).

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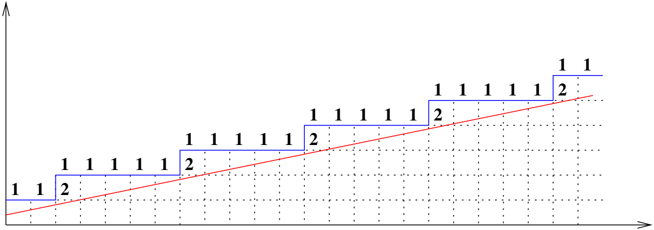
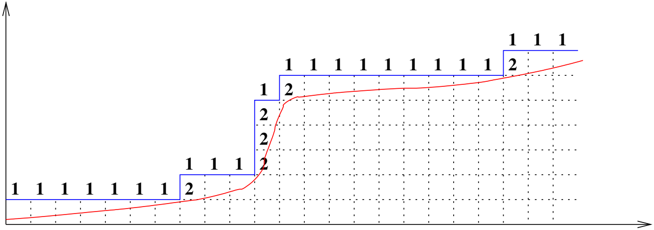
Note also that natural extensions of the Nivat's conjecture to words indexed by $\mathbb{Z}^d$ do not hold (counter-examples by J. Cassaigne).

$\rightsquigarrow$ restriction to subsets of n-dim. words (*e.g.*, unif. rec.)?

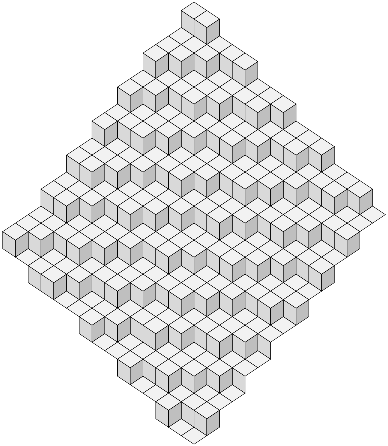
Sturmian word: coding of an irrational *stepped line*:



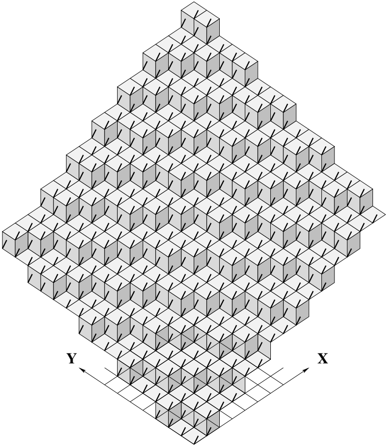
Two-letter word: coding of a *stepped curve*:

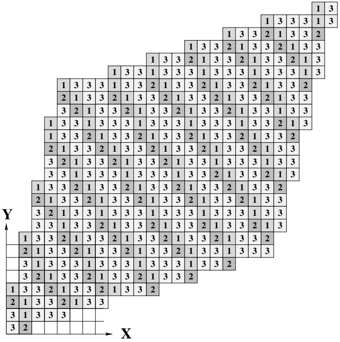


Stepped plane

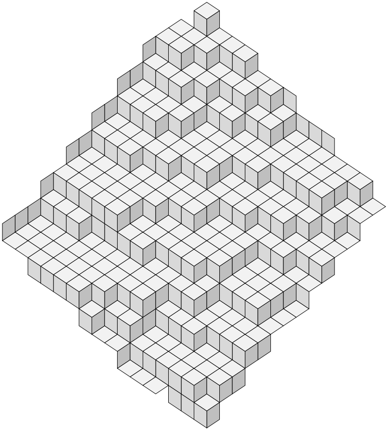


Stepped plane $\rightsquigarrow$ coding by a three-letter bidim. word:





Stepped surface



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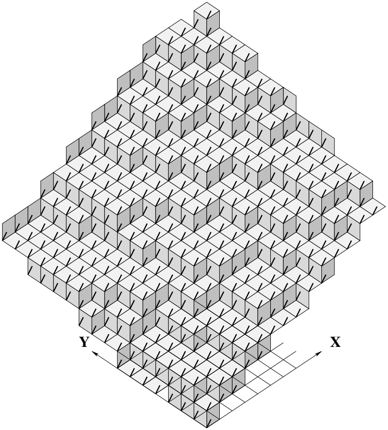

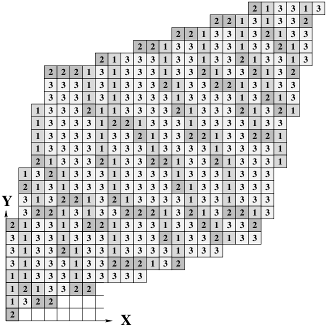

According to Berthé-Vuillon, coding of *irrational* stepped planes:

- correspond to coding of two rotations on $\mathbb{T}^1$;
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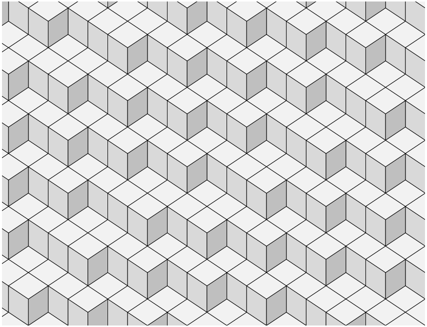
Some stepped surfaces of rectangular complexity $mn + m + n$:



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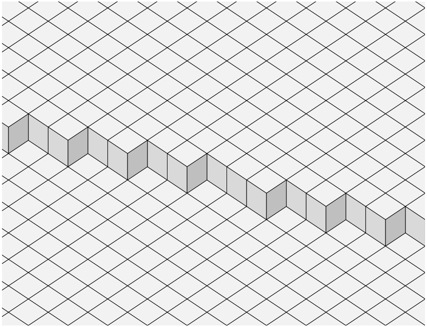

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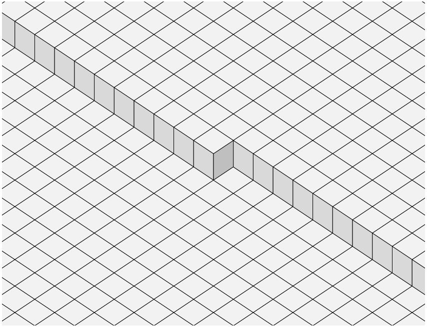

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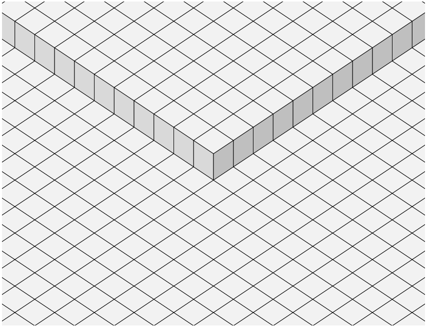

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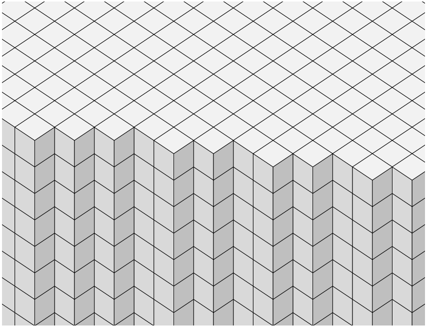

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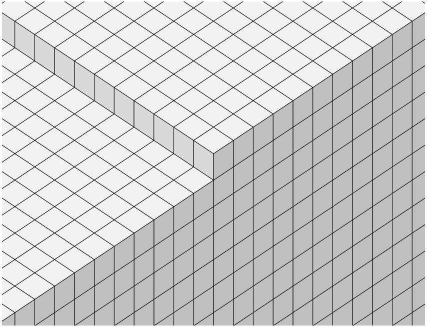

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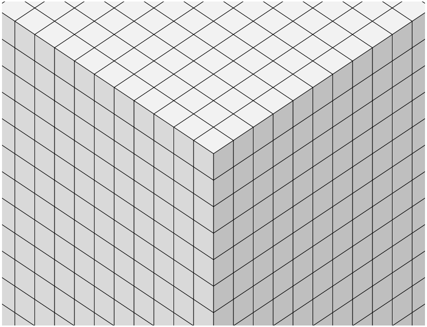

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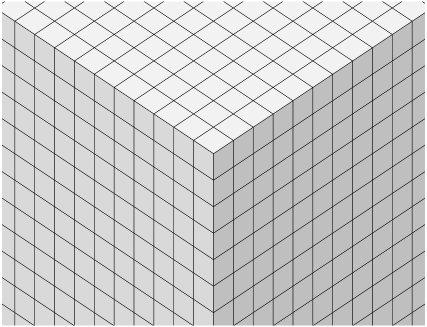
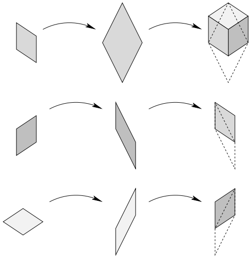


Which transformations preserve rectangular complexity?

Generalized substitution are defined on unit faces of cubes. They act as discretizations of unimodular matrices.



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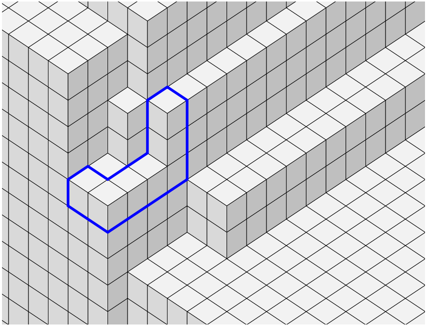

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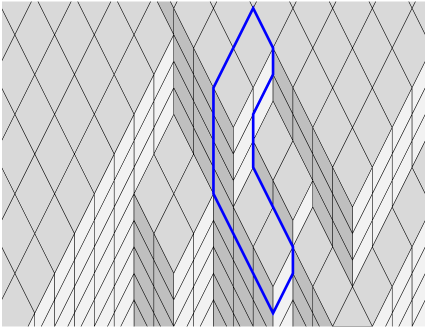

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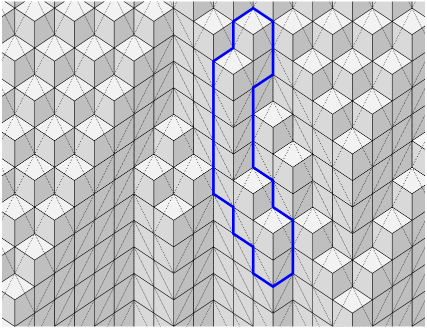
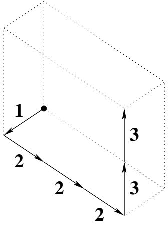


Three-letter word $w \rightsquigarrow$ stepped line $\gamma(w)$ of $\mathbb{R}^3$:



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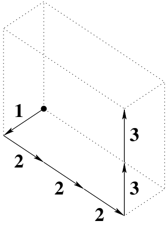

Substitution $\sigma \rightsquigarrow$ map $E_1(\sigma)$ over unit segments s.t.:

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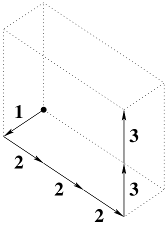

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Duality segment/face $\rightsquigarrow$ dual map $E_1^*(\sigma)$ over unit faces.

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- true for irrational stepped planes ;
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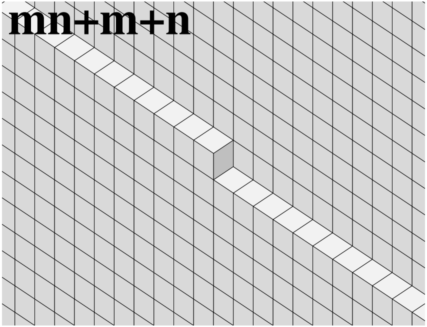
- true for irrational stepped planes ;
- but not always for degenerated cases. . .

How could we fix this?

- Consider only particular generalized substitutions?
- Modify the notion of complexity?
- . . .

Some examples:



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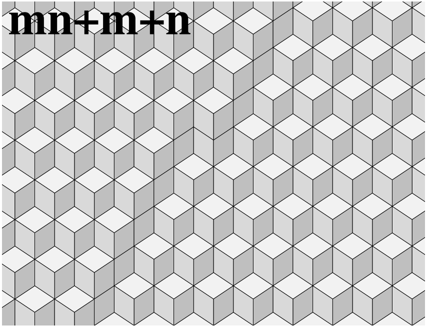

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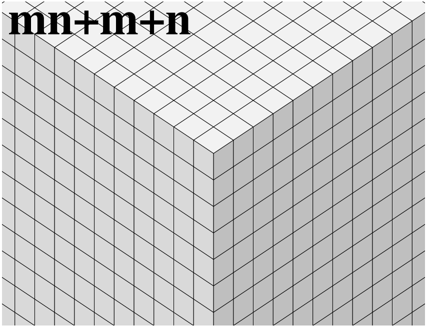

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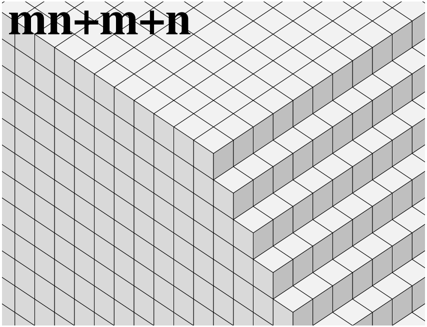

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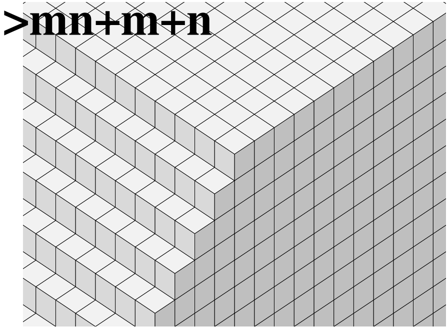

Problem: symmetry of stepped surfaces is broken by codings.

Question 1 (Aberkane/Cassaigne-like classification)

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Question 2 (Nivat-like conjecture)

Is it true that codings of stepped surfaces having complexity less than $mn + m + n$ are periodic?

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Question 3 (Nivat-completness)

Bijection ϕ from bidim. words to coding of stepped surfaces s.t.:

$$p(u) < mn+1 \;\Leftrightarrow\; p(\phi(u)) < mn+m+n?$$

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