

Aperiodic Order

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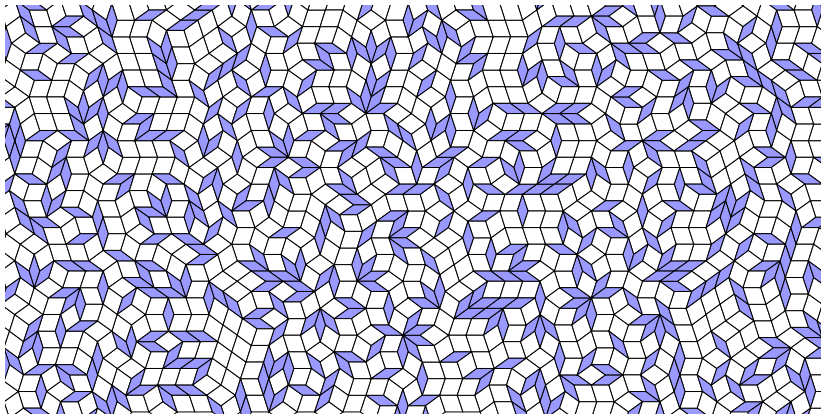
Outline

- 1 Quasiperiodicity
- 2 Local rules
- 3 Some issues

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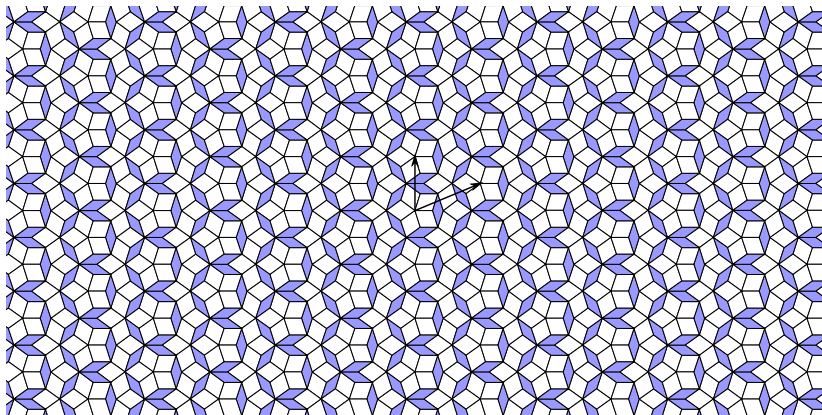
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The good, the bad and the ugly



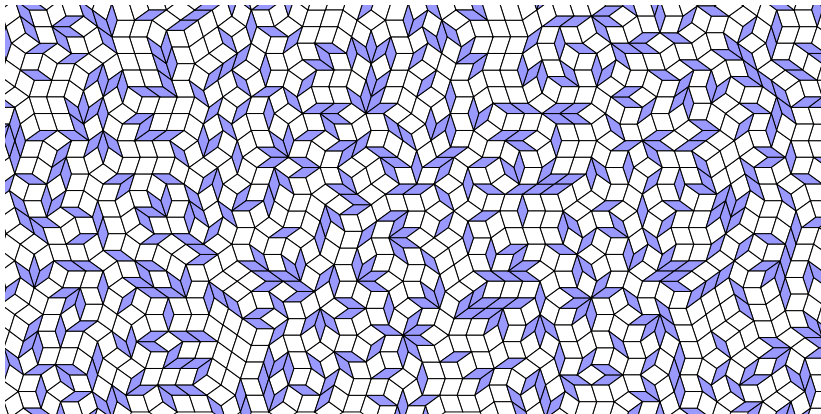
Tiling: covering of \mathbb{R}^n by interior-disjoint compact sets called **tiles**.

The good, the bad and the ugly



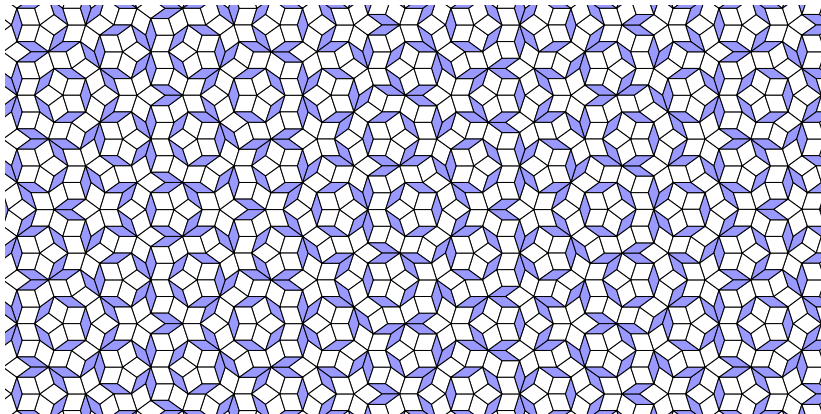
A tiling is **periodic** if it is invariant by a translation.

The good, the bad and the ugly



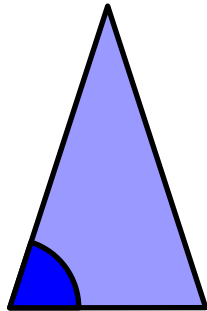
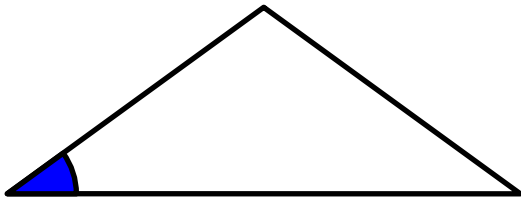
It is **aperioric** otherwise.

The good, the bad and the ugly



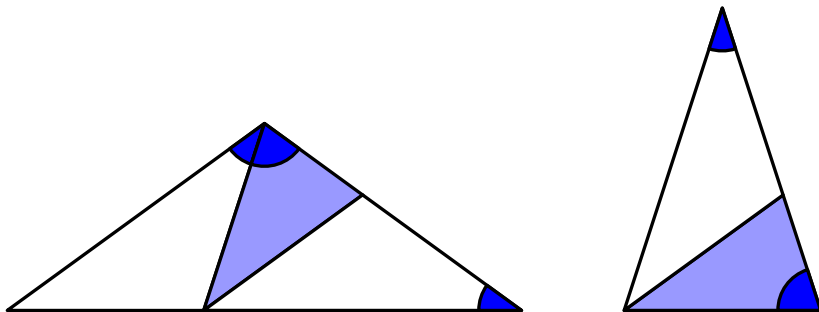
A tiling is **quasiperiodic** if each **pattern** reoccurs **uniformly**.

Substitutions



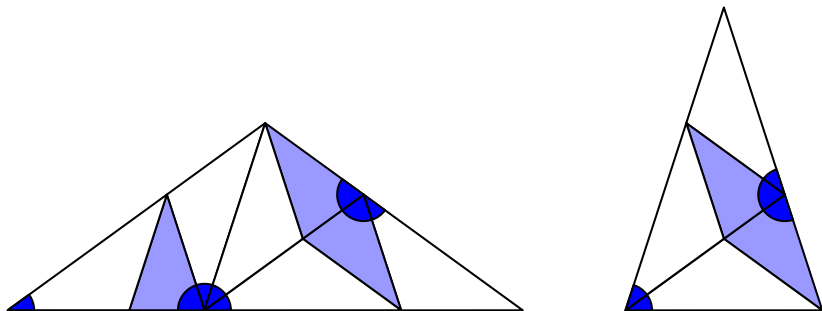
A method to define quasiperiodic tilings: [substitutions](#).

Substitutions



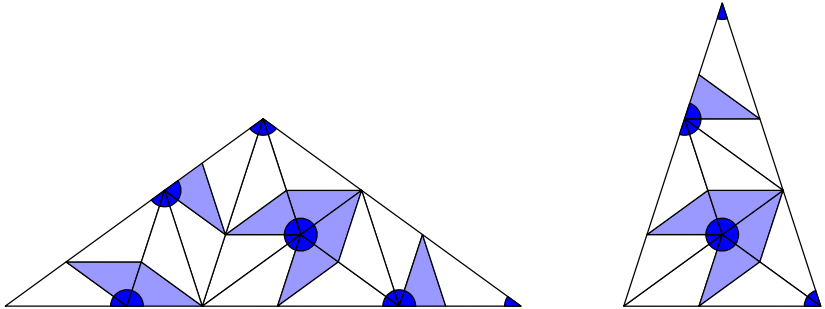
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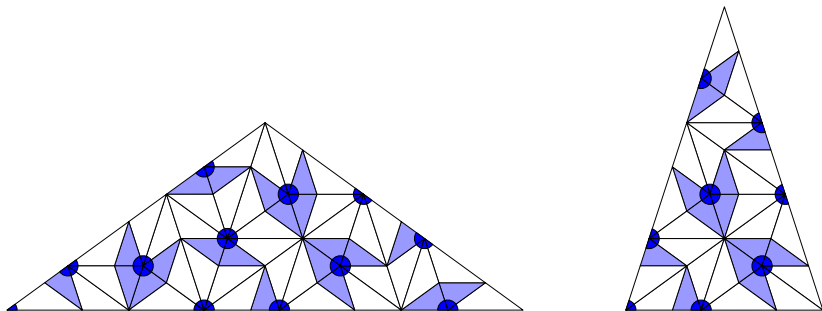
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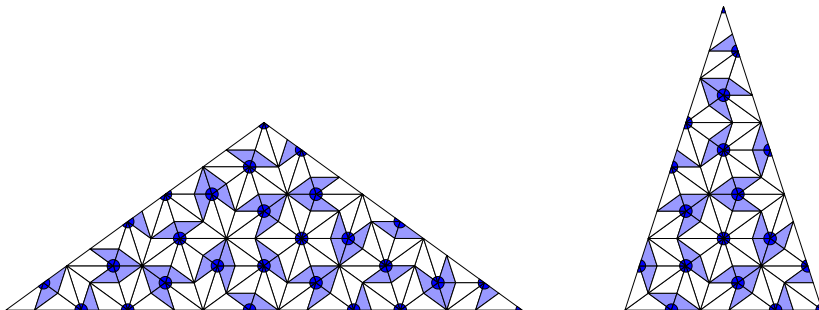
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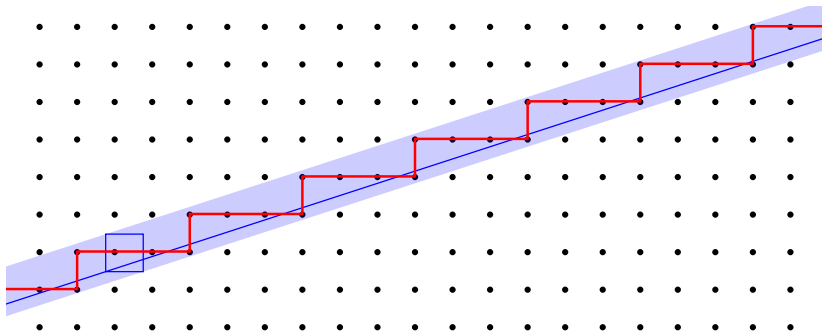
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Substitutions



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Cut and projection



Another method: discretize linear subspaces of higher dim. spaces.

Cut and projection



Cut and projection



Cut and projection



Cut and projection

Aug. 24 - Sept. 4, 2015

CIMPA Research School

ISFAHAN, IRAN

TILINGS & TESSELLATIONS

Isfahan University of Technology

Isfahan Mathematics House

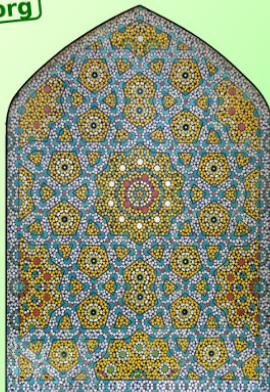
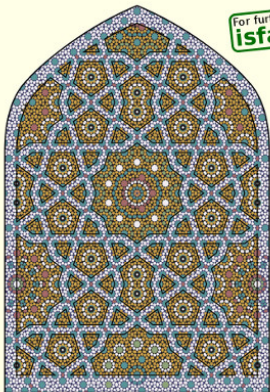
For further informations and registration, see the website
isfahan.sciencesconf.org

Objectives

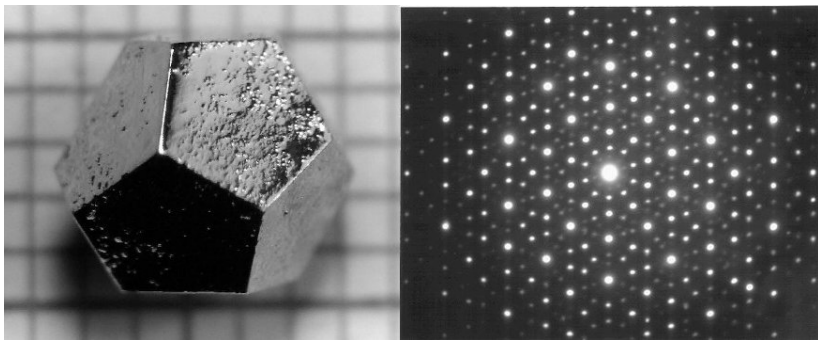
Tilings and Tessellations is a mathematic school supported by CIMPA. 60 hours of lectures on tilings shall be given by international researchers. Tilings offer a connection between combinatorics, computability, discrete geometry, dynamical systems and probability theory. Tilings are also related to the tessellations that embellish Islamic architecture, and wider audience lectures shall be given on this subject.

Lecturers

BASSINO Frédérique (Paris)
 BECKER Florent (Orléans)
 BEDARIDE Nicolas (Marseille)
 BODINI Olivier (Paris)
 BOUTILLIER Cédric (Paris)
 CASTERA Jean-Marc (Paris)
 DE TILLIERE Béatrice (Paris)
 FERNIQUE Thomas (Paris)
 GHARI Meghdad (Isfahan)
 GJILJON Pierre (Marseille)
 HASHEMI Amir (Isfahan)
 HOGENDEK JAN (Utrecht)
 MARONICKY Emil (Copenhagen)
 MONTEL Thierry (Paris)
 REJALI Ali (Isfahan)
 REMILA Eric (Saint-Étienne)
 THEYSSIER Guillaume (Chambéry)
 ZAMANI Akbar (Isfahan)

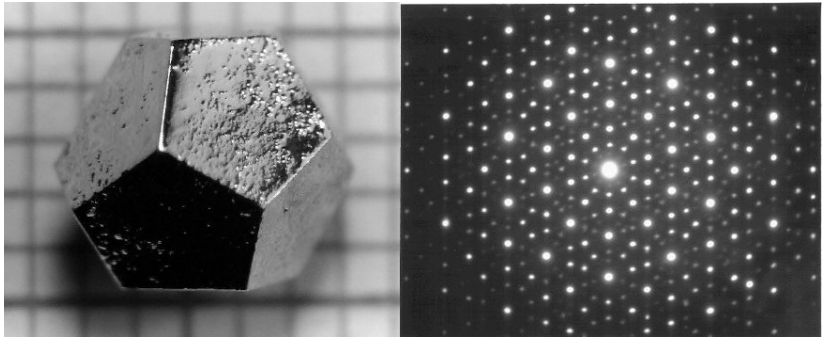


Quasicrystals (1982)



Quasicrystal: quasiperiodic material (aperiodic order).

Quasicrystals (1982)



Can stabilization be explained by short-range energetic interaction?

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- 2 Local rules
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Dimension 1

Consider bi-infinite words over a finite alphabet \mathcal{A} .

- **Subshift**: the words avoiding a set \mathcal{F} of *forbidden* finite words.
- Subshift of **finite type** (SFT): \mathcal{F} can be chosen to be finite.
- **Sofic** subshift: letter-to-letter image of an SFT.

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Examples: the words over $\mathcal{A} = \{a, b\}$ with

- alternating a 's and b 's?
- at most one b ?
- exactly one b ?
- runs of b 's all of the same length?

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Examples: the words over $\mathcal{A} = \{a, b\}$ with

- | | |
|--|----------------|
| • alternating a 's and b 's? | SFT |
| • at most one b ? | Sofic not SFT |
| • exactly one b ? | Not a subshift |
| • runs of b 's all of the same length? | Not sofic |

Higher dimensions

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Tiling space: higher dimensional extension of subshift.

Conjecture (Wang, 1960)

Any non-empty sofic tiling space contains a periodic tiling.

↔ Algorithm to decide whether a given tile set does tile.

Undecidability

Theorem (Berger, 1964)

There is a non-empty sofic 2D tiling space without periodic tilings.

Explicit construction based on substitutions (play puzzle!).

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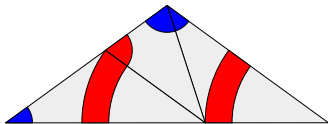
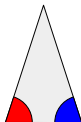
Theorem (Berger, 1964)

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Proof sketch:

- simulate computation by square tiles ($2D \simeq \text{space} \times \text{time}$);
- run computation everywhere for longer and longer time;
- rely on the undecidability of the Halting problem.

Penrose again



It does not mean that it cannot be proven for a given tile set!

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Method comparison

Can we compare the methods to generate quasiperiodic tiling?

- substitutions (and S-adic extension);
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Some results:

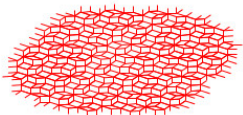
- Kari-Čulik is not substitutive (Monteil);
- substitution \Rightarrow local rules (Mozes, Goodman-Strauss, F.-Ollinger);
- for cuts: computable \Leftrightarrow local rules (F.-Sablik);
- some algebraic cuts \Rightarrow substitution (Harriss, Arnoux-Ito, F.,...).

Growth



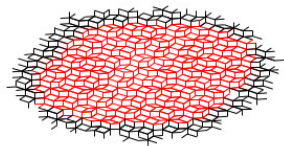
Dead ends: the existence of tilings does not mean it is easy to tile!

Growth



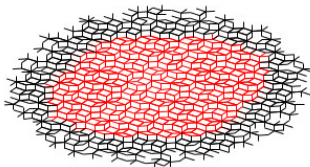
Idea: add a tile only where there is no choice (intuitive & realist).

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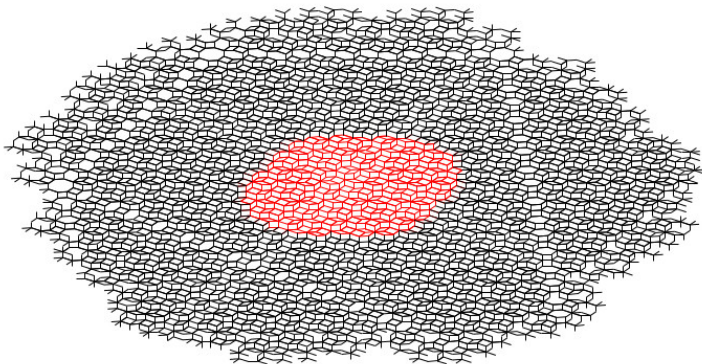
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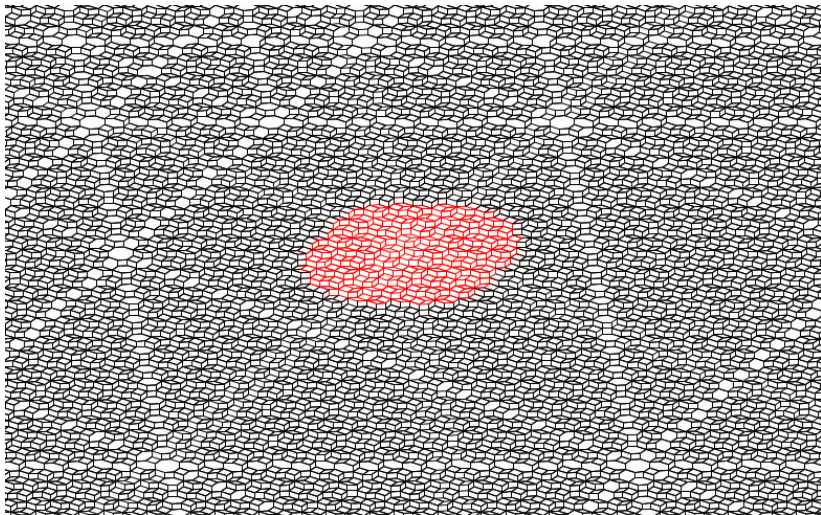
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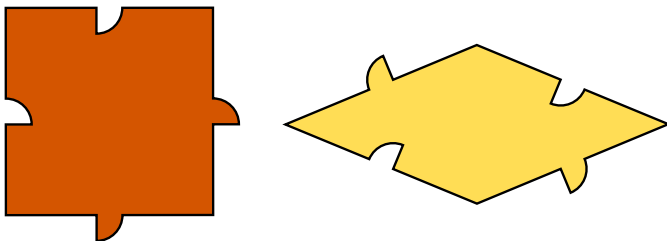
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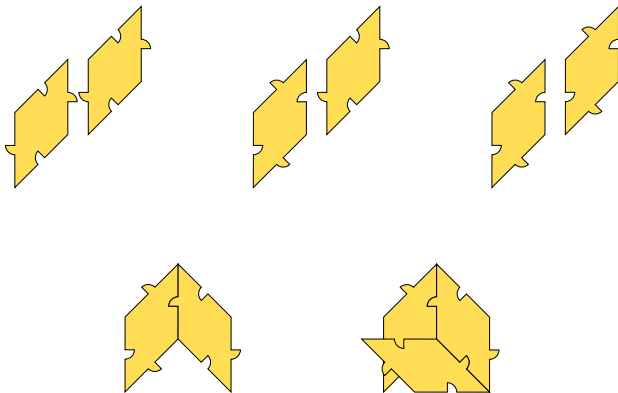
Works in some cases (quadratic cuts in \mathbb{R}^4 , work in progress).

Maxing rules



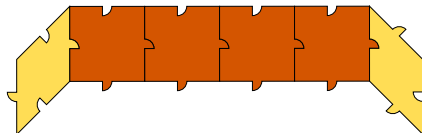
How to tile with as few square tiles as possible?

Maxing rules



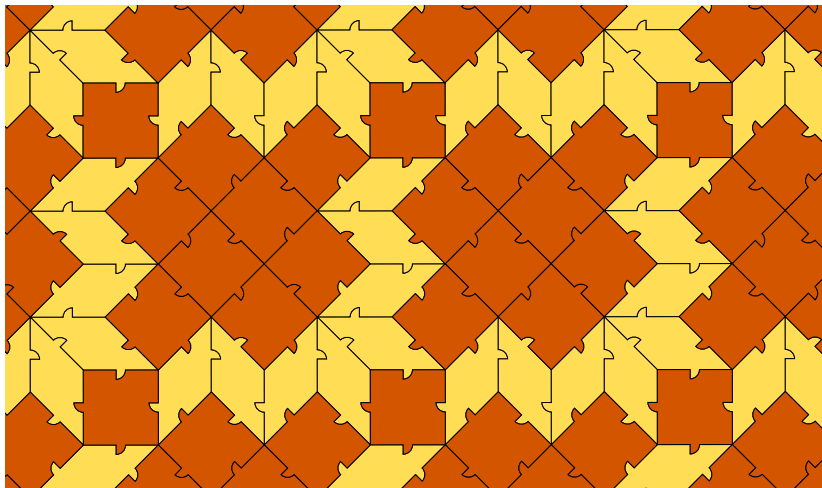
Lozenges only cannot tile the plane. Squares are needed!

Maxing rules



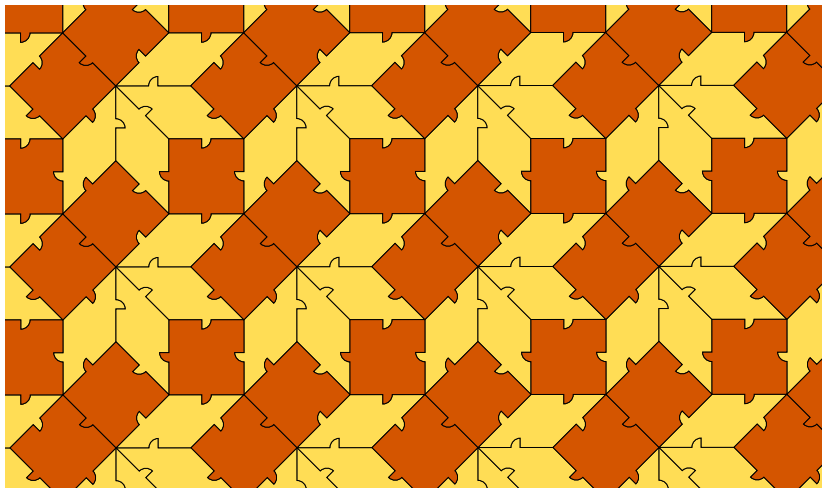
Squares **transfer information** between lozenges.

Maxing rules



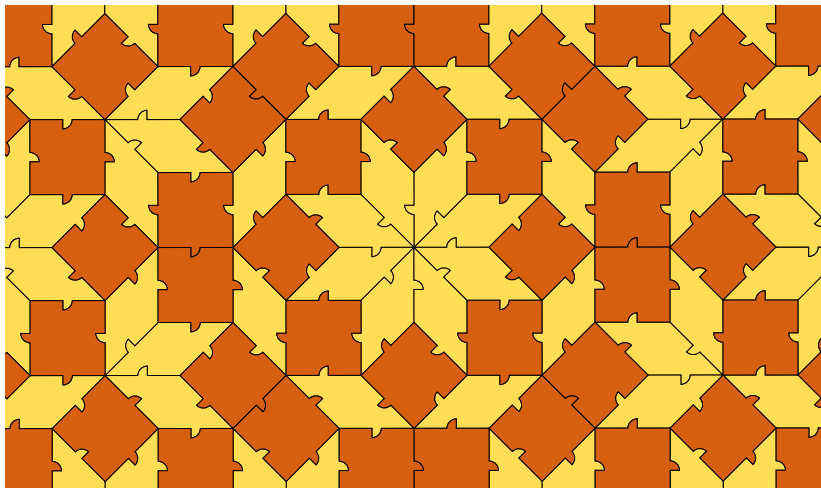
The possible tilings discretize the planes of a hyperbola in $\text{Gr}(4, 2)$.

Maxing rules



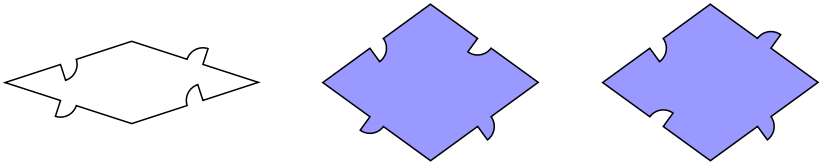
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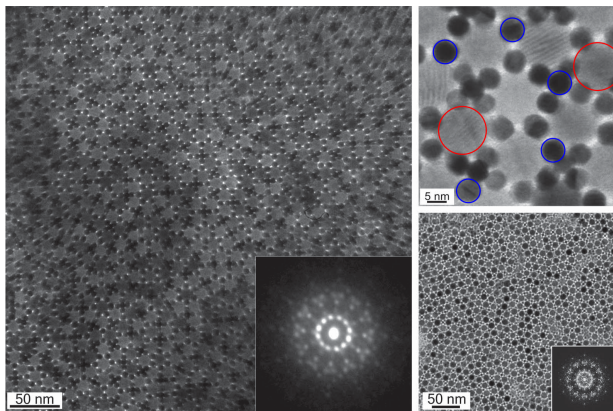
Square minimization yields aperiodicity (Ammann-Beenker tilings).

Maxing rules



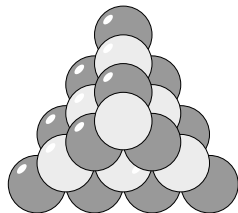
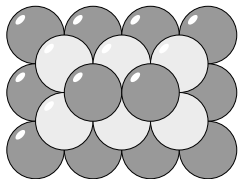
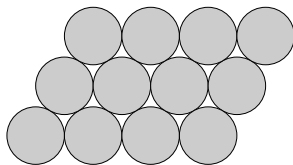
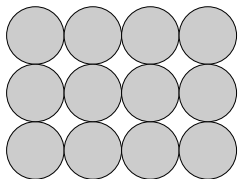
Which other aperiodic tilings can be described this way? Penrose?

Sphere packings



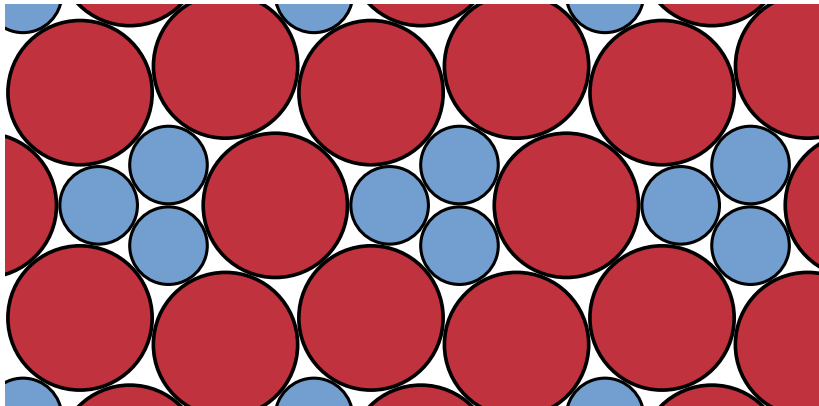
Find k different spheres in \mathbb{R}^n whose densest packing is aperiodic.

Sphere packings



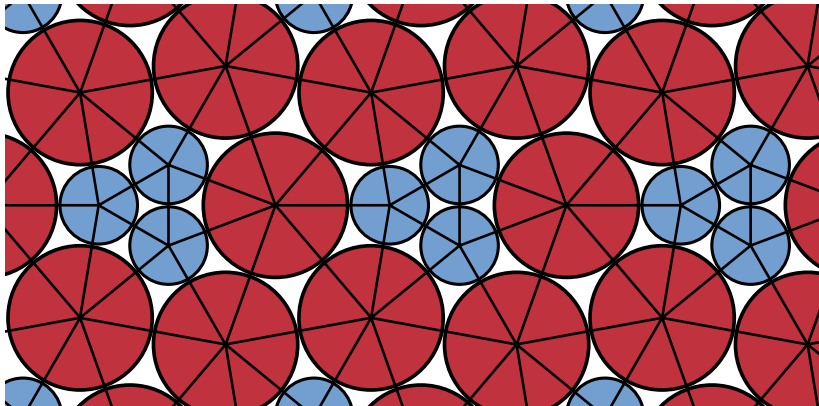
For $k = 1$, $n \notin \{2, 3, 8, 24\}$. Maybe for some $n \geq 10$?

Sphere packings



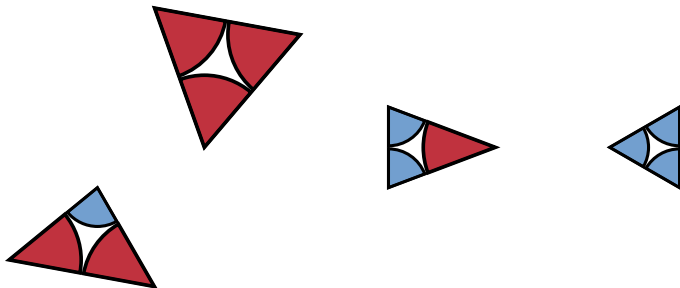
Good candidates to prove maximal density: [compact packings](#).

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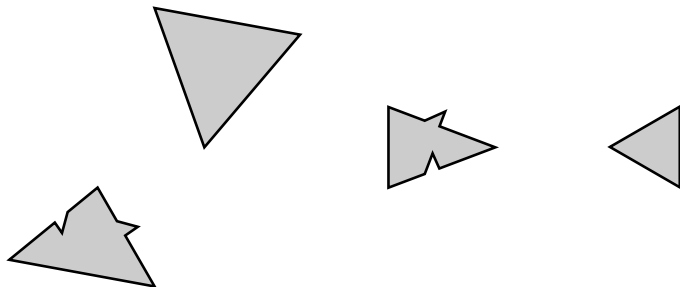
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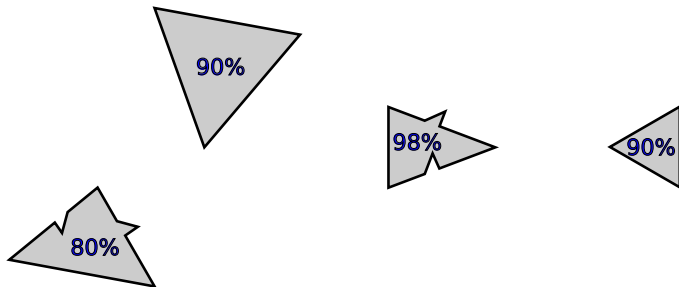
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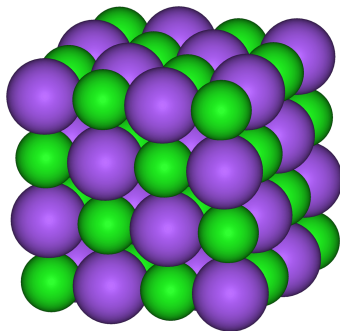
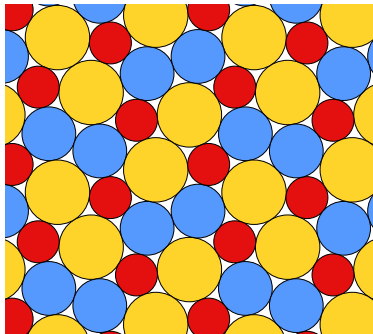
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Sphere packings



For 2 discs (9 cases), 3 discs (164 cases) or 2 balls (1 case):
The densest compact packings always contain a periodic packing.