Aperiodic Order

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Outline

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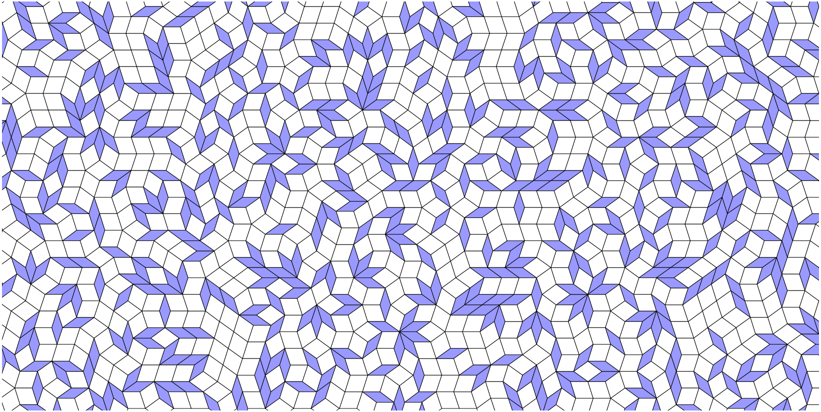
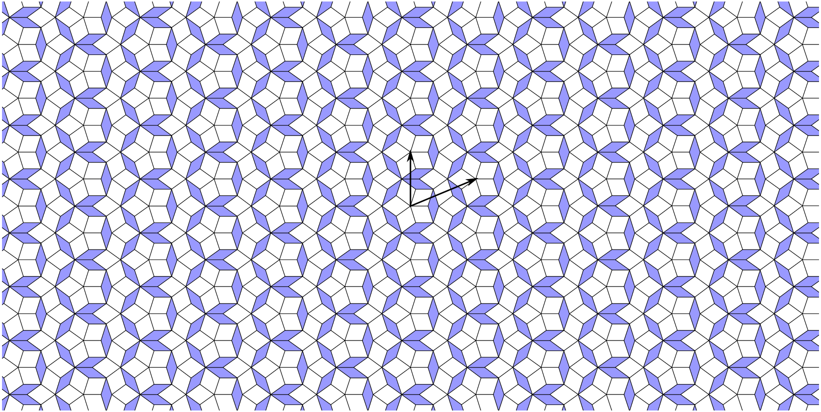
The good, the bad and the ugly



Tiling: covering of $\mathbb{R}^n$ by interior-disjoint compact sets called tiles.

The good, the bad and the ugly



A tiling is periodic if it is invariant by a translation.

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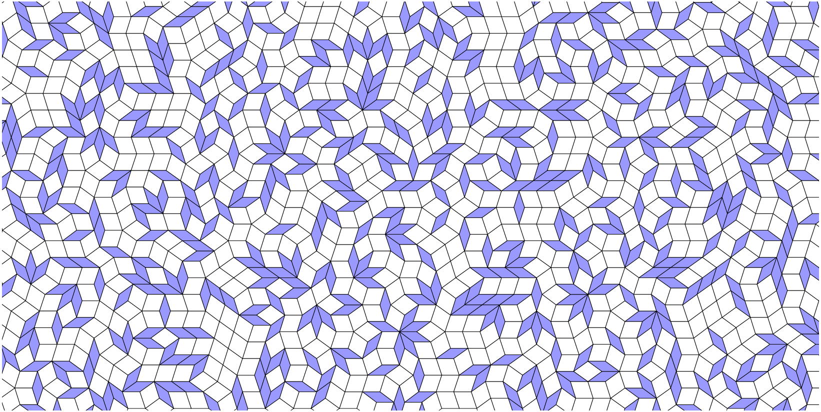

It is aperioric otherwise.

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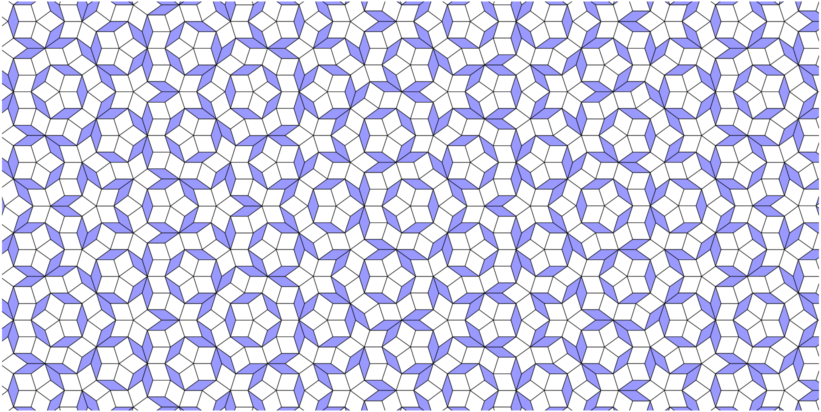
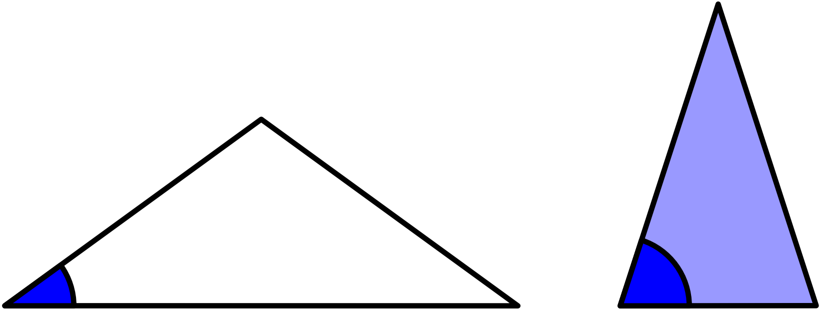


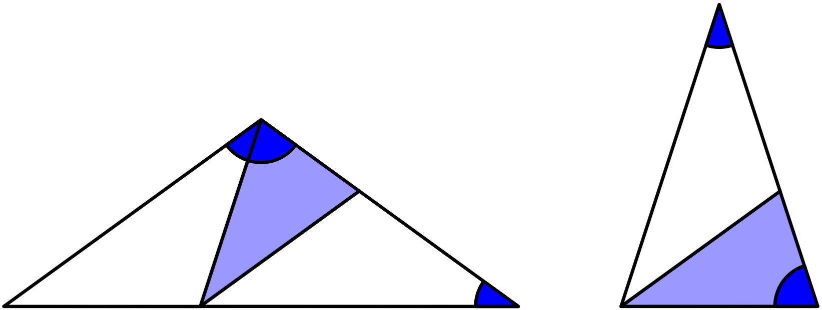
A tiling is quasiperiodic if each pattern reoccurs uniformly.

Substitutions



A method to define quasiperiodic tilings: substitutions.

Substitutions



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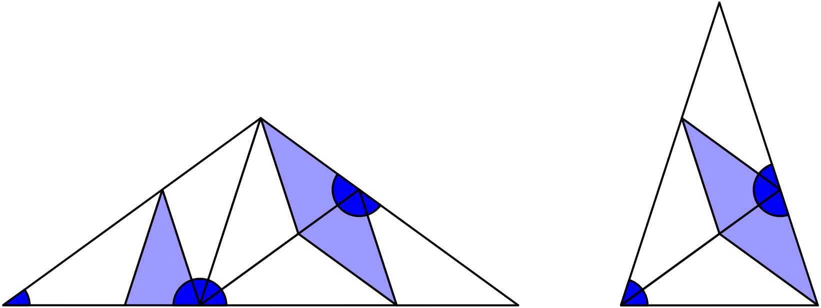

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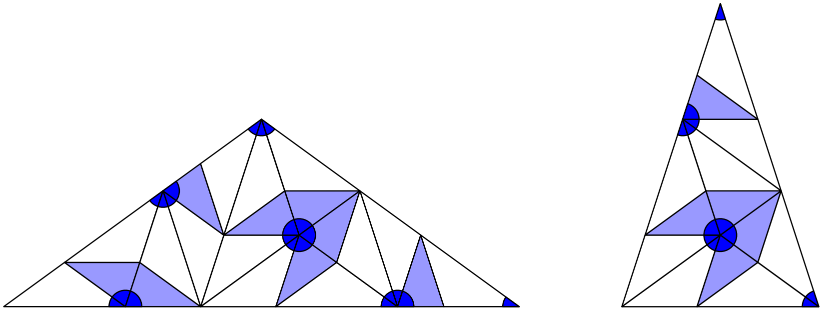

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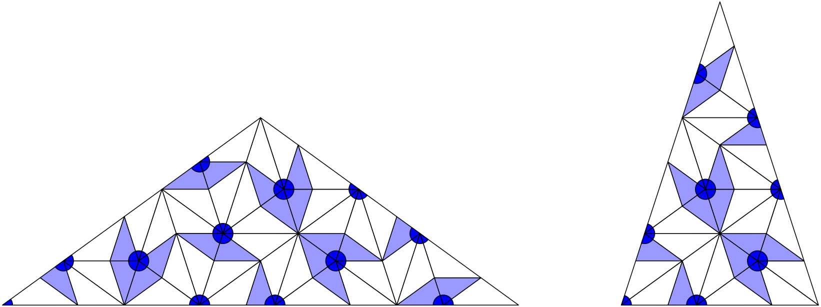

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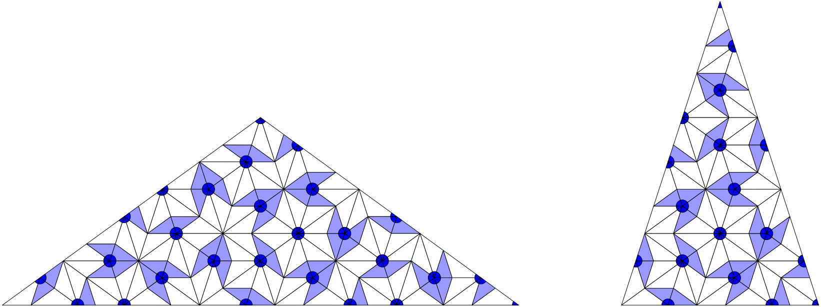
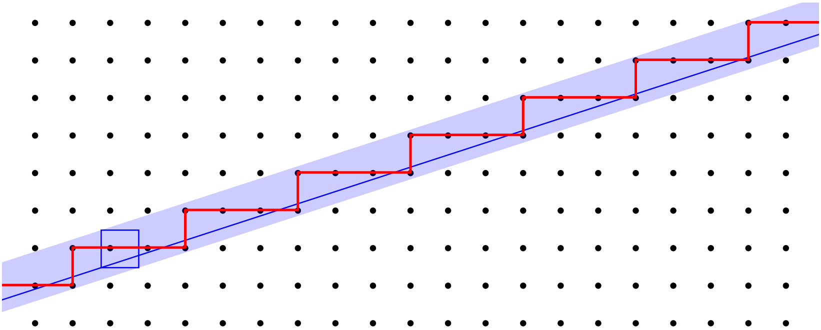


A method to define quasiperiodic tilings: substitutions.

Cut and projection



Another method: discretize linear subspaces of higher dim. spaces.

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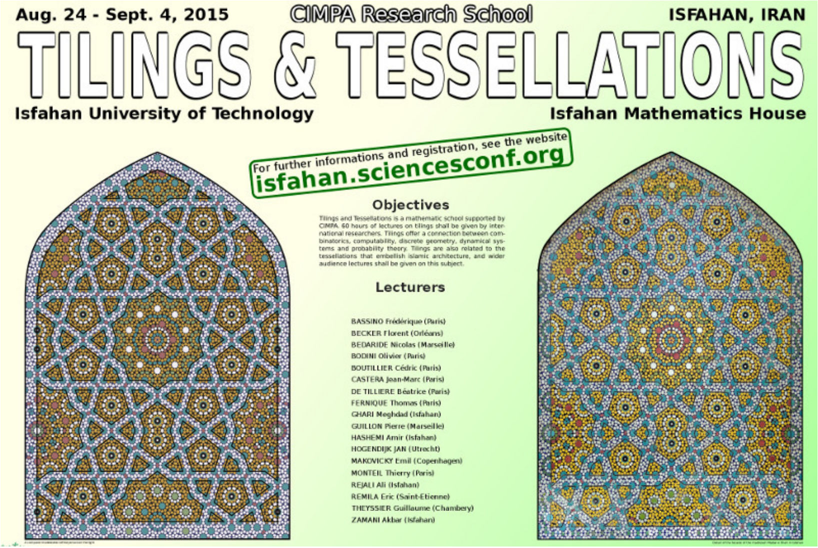
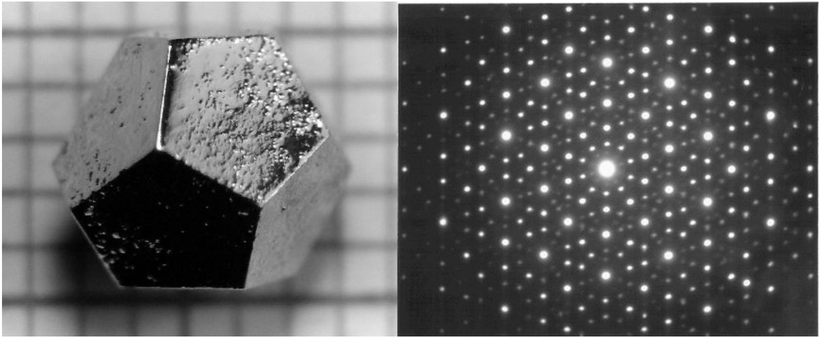
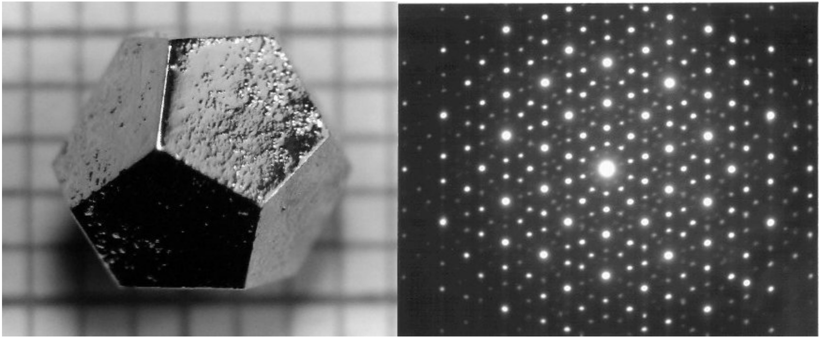


Quasicrystals (1982)



Quasicrystal: quasiperiodic material (aperiodic order).

Quasicrystals (1982)



Can stabilization be explained by short-range energetic interaction?

Outline

Dimension 1

Consider bi-infinite words over a finite alphabet $\mathcal{A}$.

- Subshift: the words avoiding a set $\mathcal{F}$ of *forbidden* finite words.
- Subshift of finite type (SFT): $\mathcal{F}$ can be chosen to be finite.
- Sofic subshift: letter-to-letter image of an SFT.

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Examples: the words over $\mathcal{A} = \{a, b\}$ with

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Examples: the words over $\mathcal{A} = \{a, b\}$ with

- alternating a's and b's? SFT
- at most one b? Sofic not SFT
- exactly one b? Not a subshift
- runs of b's all of the same length? Not sofic

Higher dimensions

Proposition

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Tiling space: higher dimensional extension of subshift.

Conjecture (Wang, 1960)

Any non-empty sofic tiling space contains a periodic tiling.

$\rightsquigarrow$ Algorithm to decide whether a given tile set does tile.

Undecidability

Theorem (Berger, 1964)

There is a non-empty sofic 2D tiling space without periodic tilings.

Explicit construction based on substitutions (play puzzle!).

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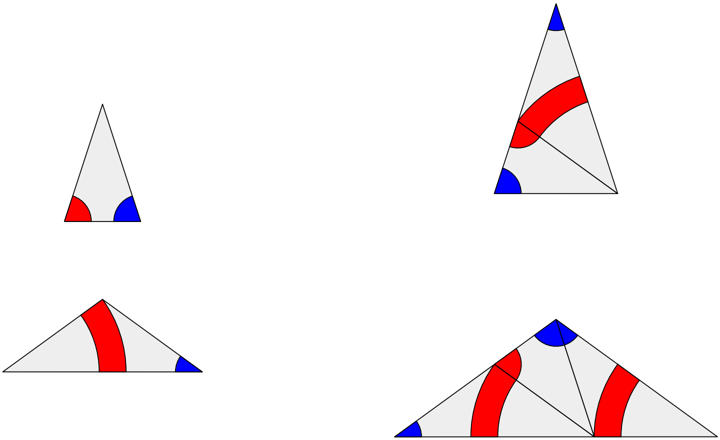
Proof sketch:

- simulate computation by square tiles (2D $\simeq$ space $\times$ time);
- run computation eveywhere for longer and longer time;
- rely on the undecidability of the Halting problem.

Penrose again



It does not mean that it cannot be proven for a given tile set!

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Method comparison

Can we compare the methods to generate quasiperiodic tiling?

- substitutions (and S-adic extension);
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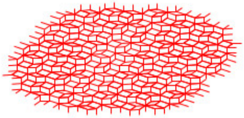
Some results:

- Kari-Čulik is not substitutive (Monteil);
- substitution ⇒ local rules (Mozes, Goodman-Strauss, F.-Ollinger);
- for cuts: computable ⇔ local rules (F.-Sablik);
- some algebraic cuts ⇒ substitution (Harriss, Arnoux-Ito, F.,. . .).

Growth

Dead ends: the existence of tilings does not mean it is easy to tile!

Growth



Idea: add a tile only where there is no choice (intuitive & realist).

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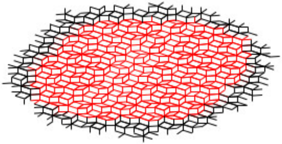

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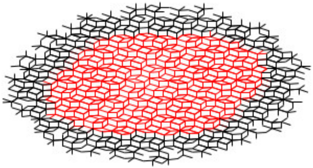

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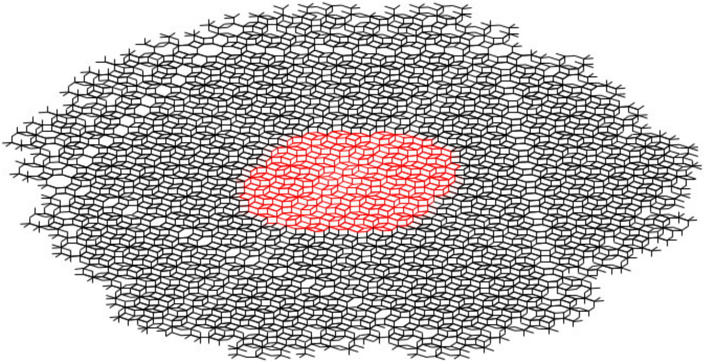

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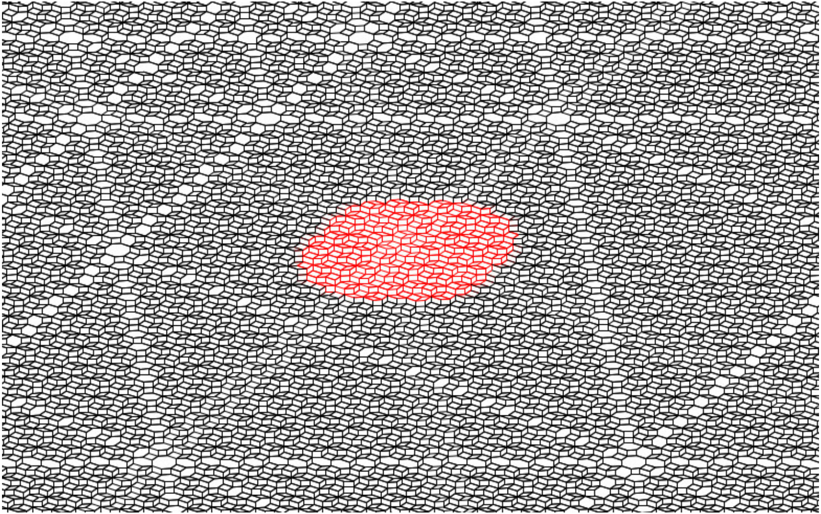
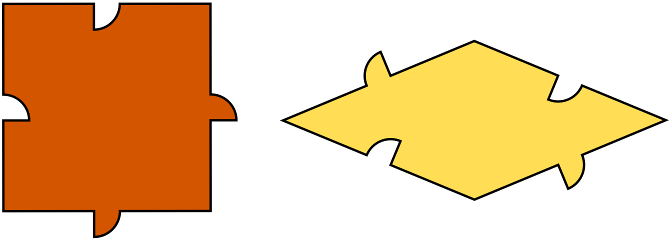


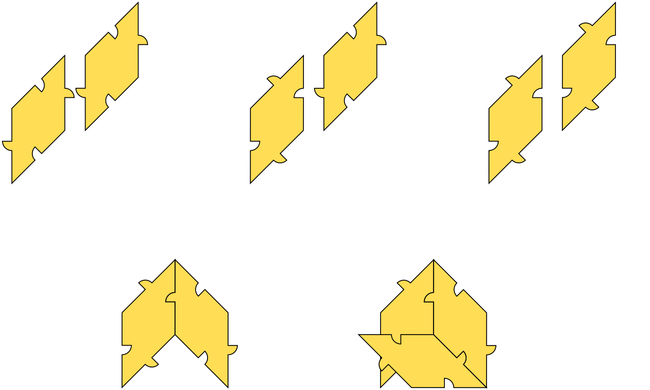
Works in some cases (quadratic cuts in $\mathbb{R}^4$, work in progress).

Maxing rules



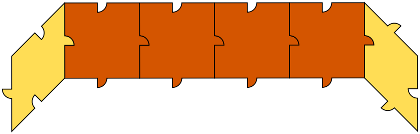
How to tile with as few square tiles as possible?

Maxing rules



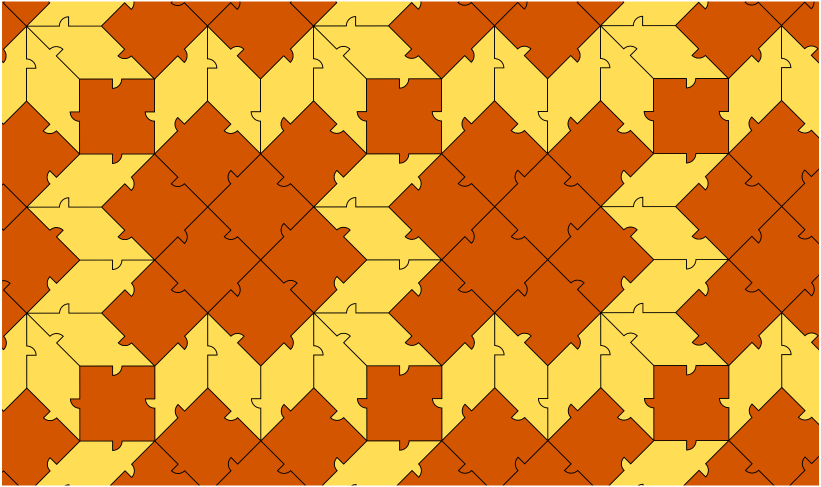
Lozenges only cannot tile the plane. Squares are needed!

Maxing rules



Squares transfer information between lozenges.

Maxing rules



The possible tilings discretize the planes of a hyperbola in $\mathrm{Gr}(4,2)$.

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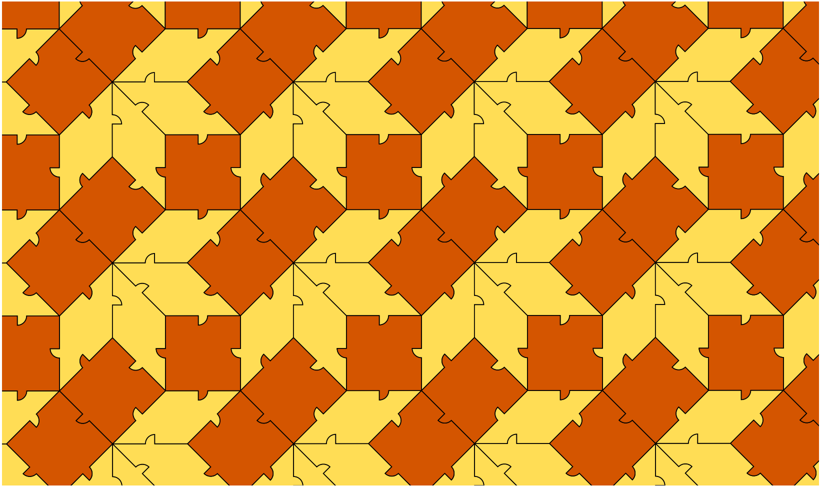

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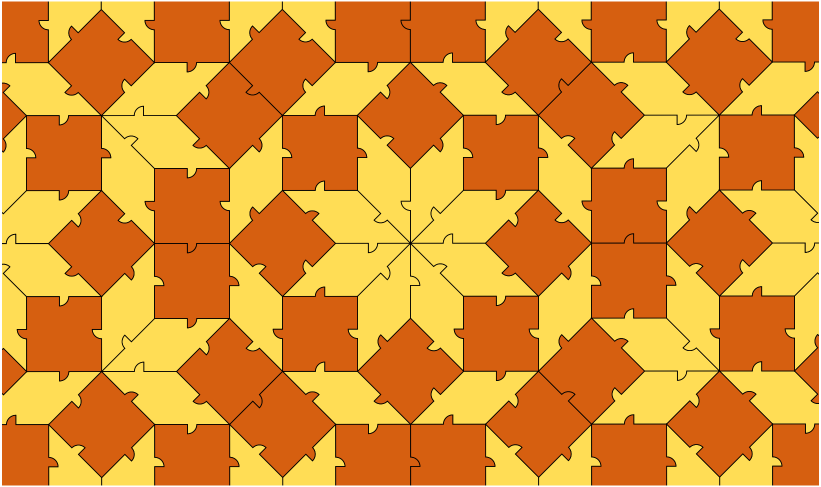
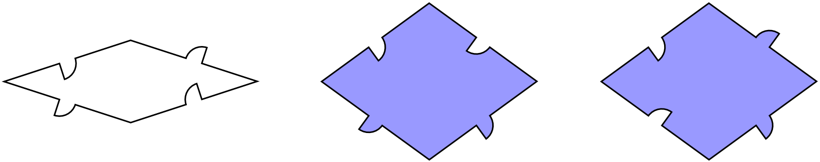


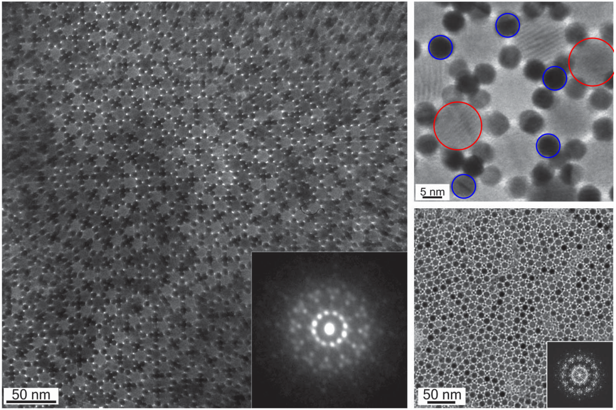
Square minimization yields aperiodicity (Ammann-Beenker tilings).

Maxing rules



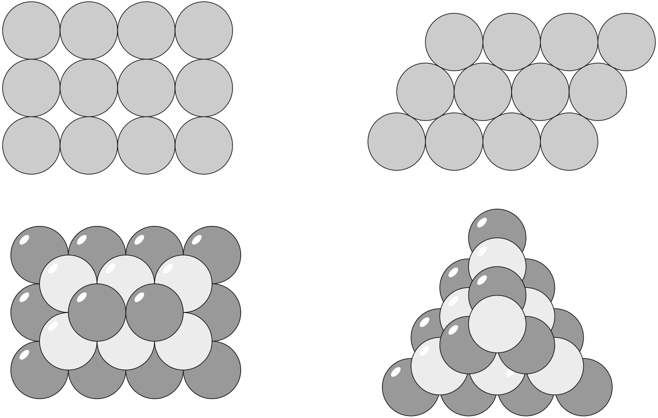
Which other aperiodic tilings can be described this way? Penrose?

Sphere packings



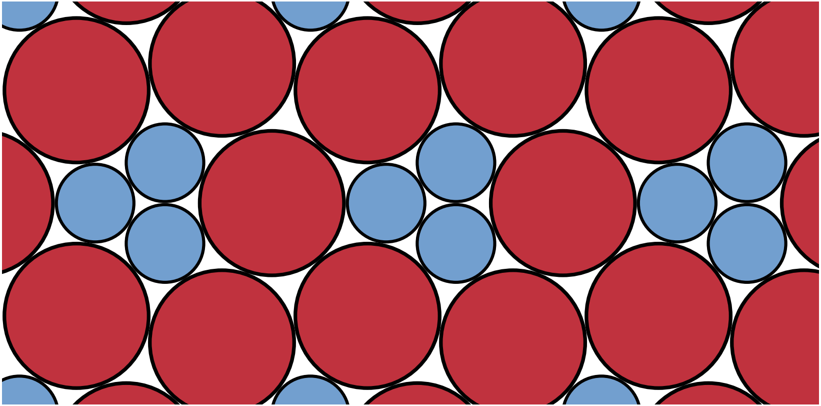
Find k different spheres in $\mathbb{R}^n$ whose densest packing is aperiodic.

Sphere packings



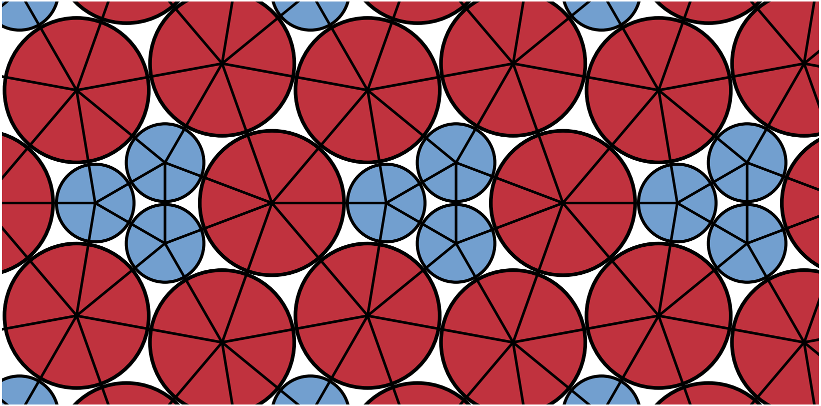
For $k = 1$, $n \notin \{2, 3, 8, 24\}$. Maybe for some $n \geq 10$?

Sphere packings



Good candidates to prove maximimal density: compact packings.

Sphere packings



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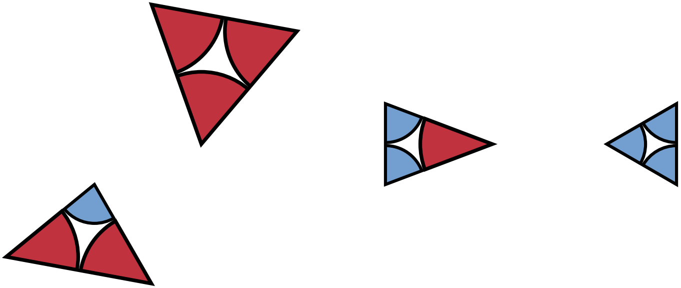

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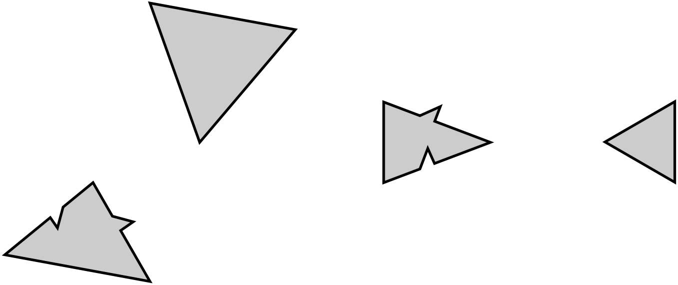

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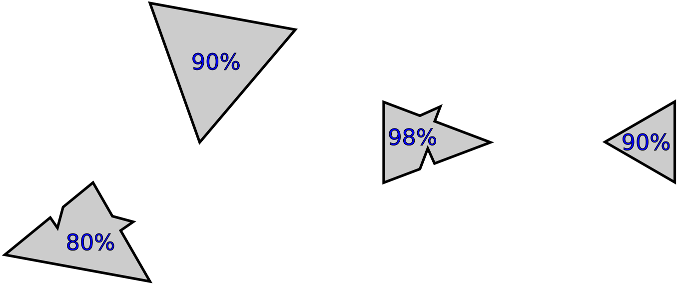

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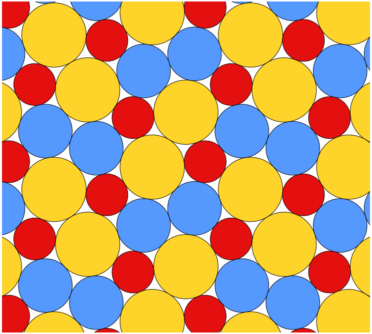

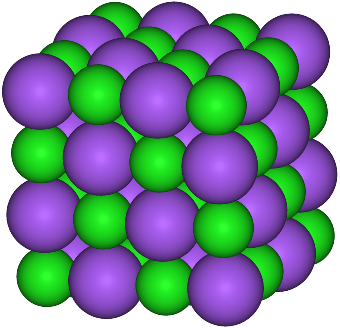

For 2 discs (9 cases), 3 discs (164 cases) or 2 balls (1 case):
The densest compact packings always contain a periodic packing.