Checking Complicated Inequalities over Compact Sets with Interval Arithmetic

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1 Precision issue

What does the following program?

```
a=1.0
while a+1-a==1:
    a=a*2
```

It stops after a **finite number of iterations** because only **finitely many real numbers are machine-representable** (see, e.g., norm IEEE 754).

2 Interval arithmetic

Principle:

1. represent every real number by a **machine-representable** interval which **contains** it;

2. perform computations so that the result of $f(I_1, \ldots, I_k)$ is a **machine-representable interval** which **contains** $\{f(x_1, \ldots, x_n) \mid x_i \in I_i\}$.

For example, the result of $[1, 2] + [3, 4]$ can be $[4, 6]$ (optimal) or, e.g., $[3, 7]$ (not optimal). Some other optimal examples:

$$\begin{aligned}
[1, 2] - [3, 4] &= [-3, -1] \\
[1, 2] \times [3, 4] &= [3, 8] \\
[-1, 2] \times [-3, 4] &= [-6, 8].
\end{aligned}$$

Finding the optimal interval may be not that simple, e.g., $\sin([1, 2])$?

3 Equivalent expressions

Often we want to use interval arithmetic to evaluate as precisely as possible a function f in a real number x. However, there are **infinitely many different equivalent expressions** for f. For example, if we evaluate $f(x) = x - x$ in $[1, 2]$ with interval arithmetic:

$$f([1, 2]) = [1, 2] - [1, 2] = [-1, 1].$$

Here f can clearly be simplified - and every computer algebra software do it automatically - but it may not be that easy. In general, computer algebra software use heuristics to **simplify expressions**, but it is not clear what is the best expression for evaluation with interval arithmetic. The usual rule of thumb is to **minimize the number of occurrences of variables**. For example, the simple following rewriting saves two variable occurrences:

$$x + xy + y \longrightarrow (x + 1)(y + 1) - 1.$$

For $x = y = [1, 2]$ both expression yield $[3, 8]$. For $x = y = [-2, -1]$ the former yields $[-3, 2]$ while the latter yields $[-1, 0]$.

4 Checking inequalities

The "classic" usage of interval arithmetic is to bound real numbers by small interval. However, it can also be interesting, to use the **large intervals to handle a continuum of numbers**. For example, can you prove that the following function is positive over $[1, 2]$?

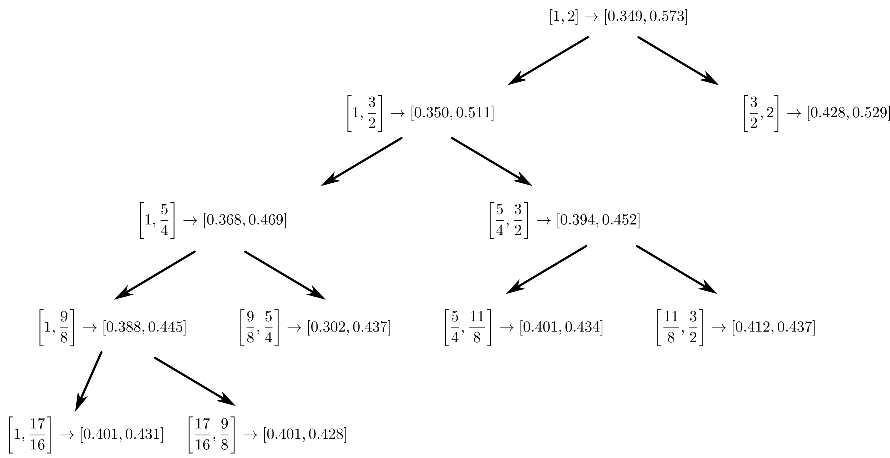
$$f : x \to \frac{\arctan(\ln(1 + x))}{\sin(x)\sqrt{3 - \cos^7(x)}}.$$

If you plot f, you "sees" that it is clearly positive and seem even to above 0.4. But this is not a proof... Do you really want to compute the derivative and find its roots?

Actually, it suffices to compute $f([1, 2])$ with your favourite interval arithmetic software. For example, SageMath yields the interval $[0.349, 0.573]$, which allows to conclude on the positivity over $[1, 2]$. Let us stress that this interval is very likely to be not optimal (its "quality" depends on the software implementation).

5 Interval recursive subdivision

How to prove that the previous function f is larger than 0.4 over $[1,2]$, as suggested by the plot? Since $f([1,2]) = [0.349, 0.573]$ contains 0.4, we cannot yet conclude. Idea: **halve the interval to increase the precision**:



6 Infinite recursion

How to prove that a function f with $f(0) = 0$ is nonnegative over $[0,1]$? For any $\varepsilon > 0$, $f([0,\varepsilon])$ will (usually) strictly contain 0, so that the recursion will be infinite around 0. Idea: use the **Mean value theorem** (Lagrange theorem):

$$\forall x > 0, \quad \exists c \in [0, x], \quad f(x) = f(0) + x f'(c).$$

Namely, $\forall x \in [0, \varepsilon]$:

$$f(x) = x f'(c) \subset [0, \varepsilon] \times f'([0, \varepsilon]).$$

This latter interval thus contains the optimal $f([0,\varepsilon])$. If $f'(0) > 0$, then there exists $\varepsilon > 0$ small enough such that $f'([0,\varepsilon])$ has a **positive left endpoint**. This ensures that f is nonnegative[1] over $[0,\varepsilon]$. An **explicit value** of ε can be found by dichotomy.

If $f'(0) = 0$, use the Taylor theorem at a **larger order**.

Remark: this may also be used to **improve the precision** for **small enough** intervals: on the one (bad) hand f' has usually more variables than f, on the other (good) hand we multiply $f'([0,\varepsilon])$ by the small interval $[0,\varepsilon]$.

[1] Actually positive except in 0.