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# MULTIDIMENSIONAL STURMIAN SEQUENCES AND GENERALIZED SUBSTITUTIONS

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ABSTRACT

This paper is devoted to a study on the way *generalized substitutions* - a multi-dimensional extension of substitutions - act on *multi-dimensional Sturmian sequences.* We give a sufficient condition under which these multi-dimensional Sturmian sequences are obtained by iterated compositions of generalized substitutions. This condition relies on Brun expansions - a multi-dimensional extension of continued fraction expansions.

*Keywords:* Multi-dimensional Sturmian Sequence; Generalized Substitution; Multi-dimensional Continued Fractions; Brun Expansion; Substitutive Sequence.

## 1. Introduction

The general purpose of this paper is to contribute to the extension of the theory of Sturmian sequences in a multi-dimensional framework. Here, we are particularly interested in the links between substitutions and continued fraction expansions. We will first review, in the one-dimensional case, some basic notations, definitions and results, whose multi-dimensional extension will then be studied in this paper.

### *1.1. Sturmian Sequences*

Let $\mathcal{A}$ be an *alphabet*, *i.e.* a finite set of symbols. A *word* (resp. *sequence*) over $\mathcal{A}$ is a finite (resp. infinite) concatenation of letters of $\mathcal{A}$. For example, 121211 is a word of length 6 over $\{1,2\}$. A sequence $w$ is *ultimately periodic* if it can be written $w = uvv\ldots v\ldots$ for two words $u$ and $v$, and *aperiodic* otherwise. A *factor* of a sequence $w$ is a word which occurs in $w$. For example, the factors of length 2 of 121211 are exactly 12, 21 and 11. The complexity function $p_w$ of $w$ is then defined as follows: $p_w(k)$ is the number of different factors of length $k$ of $w$ . It is known that a sequence $w$ is ultimately periodic if and only if $p_w(k) \leq k$ for some

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$k$. Hence, a sequence $w$ with complexity $p_w(k) = k + 1$ for all $k$ is aperiodic, and no aperiodic sequence with lower complexity exists. These sequences are so-called *Sturmian* sequences.

Sturmian sequences have been widely studied for their interactions with complexity theory, discrete geometry, dynamical systems, number theory, quasicrystals and tilings (see [32] and the references inside). There are many other equivalent definitions of Sturmian sequences. The reader is referred to Chap.2 of [27] or Chap.6 of [32] for a complete and detailed presentation. We here just review two of them:

- a *cutting sequence* is the discretization of a real line $y = \alpha x + \rho$; $\alpha$ is called the *slope* of the sequence and $\rho$ its *intercept*. Sturmian sequences correspond to irrational slopes;
- a *rotation sequence* encodes the trajectory of an element $\rho$ of the unit circle under the action of a rotation $R_\alpha$ of angle $\alpha$. The circle is split into two intervals, and the $k$-th letter of the rotation sequence depends on which interval contains $R_\alpha^k(\rho)$. Sturmian sequences correspond to irrational angles.

### *1.2. Substitutions and Continued Fractions*

A *substitution* (or *morphism*) $\sigma$ over an alphabet $\mathcal{A}$ maps each letter of $\mathcal{A}$ onto a non-empty words over $\mathcal{A}$. This definition is extended to words and sequences over $\mathcal{A}$ according to the rule $\sigma(uv) = \sigma(u)\sigma(v)$. Then, starting from an initial word $u_0$, a substitution $\sigma$ allows one to generate sequences $\sigma^n(u_0)$, for $n \geq 0$. In particular, if $u_0$ is a prefix of $\sigma(u_0)$ and if the length of $\sigma^n(u_0)$ tends to infinity with $n$, one can define:

$$u = \lim_{n\to\infty} \sigma(u_0)^n.$$

Such a sequence $u$ satisfies $\sigma(u) = u$ and it is said to be *invariant*. One then defines the *substitutive* sequences as images by morphisms of the free monoid of invariant sequences. They are algorithmically easily generated and have a strongly ordered structure, although not necessarily periodic. For example, the substitution $\sigma : 1 \mapsto 12, 2 \mapsto 1$ yields a sequence $(\sigma^n(1))_n$ of unbounded length words such that $\sigma^n(1)$ is a prefix of $\sigma^{n+1}(1)$. Thus, infinitely many applications of $\sigma$ on the letter 1 lead to an invariant sequence:

$$1 \to 12 \to 121 \to 12112 \to \cdots \to 1211212112112112112 \cdots$$

Sturmian sequences and substitutions are strongly linked via continued fraction expansions. Indeed, if $u$ is a Sturmian sequence of slope $\alpha$ and intercept $\rho = 0$, and if $[\alpha] = [a_0, a_1, \ldots]$ denotes the continued fraction expansion of $\alpha$, then it is known that:

1. $u$ is *S-adic*, *i.e.* there are two substitutions $\sigma_0$, $\sigma_1$ and a sequence $(u_k)_k$ of Sturmian sequences such that (see *e.g.* [7]):

$$\forall k \geq 0, \qquad u = \sigma_0^{a_0} \circ \sigma_1^{a_1} \circ \sigma_0^{a_2} \circ \cdots \circ \sigma_{k \bmod 2}^{a_k}(u_k);$$

2. $u$ is substitutive if and only if $[\alpha]$ is ultimately periodic (see *e.g.* [10, 18, 43]).

Note that the case of a non-zero intercept $\rho$ admits a similar characterization (see *e.g.* [8, 10]).

### *1.3. Toward a Multi-dimensional Extension*

In this paper, we would like to extend what we recalled above to a multi-dimensional case. The first step is to define a notion of *multi-dimensional Sturmian sequence*. In fact, such a notion already exists and has been defined in the 2-dimensional case in [41] by discretizations of real planes. This discretization corresponds, in discrete geometry, to the notion of *standard arithmetic plane* (see [33]). An equivalent definition by rotations has also been given (see [11, 12]). We recall in the multi-dimensional case both definitions in Section 2, where we also discuss the problem of a definition in terms of aperiodic sequence of minimal complexity.

Defining a notion of *multi-dimensional substitution* is more difficult. In the one-dimensional case of sequences, the definition of a substitution $\sigma$ is extended from letters to sequences according to the rule $\sigma(uv) = \sigma(u)\sigma(v)$. But there is no canonical extension of such a rule in the multi-dimensional case. Thus, various notions of substitution have been proposed (see *e.g.* [5, 6, 23, 35]). Here we use the notion of *generalized substitution* introduced in [6]. Generalized substitutions are obtained by *duality* from classic substitutions, and it is proven in [23] that they are a particular case of *local rules substitutions* introduced in [5, 23]. In Section 3, we recall the definition of generalized substitutions. We show precisely the way they act over multi-dimensional Sturmian sequences (Theorem 1). We also provide a way to *effectively* generate multi-dimensional sequences (Theorem 2).

Finally, Section 4 investigates links between generalized substitutions and multi-dimensional Sturmian sequences. More precisely, we would like to characterize multi-dimensional Sturmian sequences which are *S-adic* or *substitutive*, naturally generalizing the corresponding notions for one-dimensional sequences. We prove in Section 4.2 that *any* multi-dimensional Sturmian sequence is $S$-adic. In Section 4.3, we prove that a multi-dimensional Sturmian sequence is substitutive if the vector of its parameters (which generalizes the *slope* of a Sturmian sequence) has an eventually periodic Brun expansion (Theorem 4). The Brun expansion is a multi-dimensional continued fraction expansion which is recalled in Section 4.1. Note that it is only a sufficient condition. We end the paper by discussing the difficulty of completing this characterization.

## 2. Multi-dimensional Sturmian Sequences

### *2.1. Aperiodic Sequences of Minimal Complexity*

In this section, we would like to extend, in the multi-dimensional case, the definition of Sturmian sequences as aperiodic sequences of minimal complexity.

The first step consists in defining *multi-dimensional sequences*. It is natural to define an *$n$-dimensional sequence* over the alphabet $\mathcal{A}$ as an infinite array in $\mathcal{A}^{\mathbb{Z}^n}$. For $n = 1$, we do not exactly obtain classic sequences, but rather a *two-sided* version of them, *i.e.* with letters indexed by $\mathbb{Z}$ instead of $\mathbb{N}$. This however does not matter since the results for one-sided sequences can be easily and similarly stated for two-sided sequences (see [17]).

The second step has to do with the notion of periodicity. Generalizing the one-dimensional case, a non-zero vector $t \in \mathbb{Z}^n$ is a vector of periodicity for the $n$-dimensional sequence $u$ if $u(x + t) = u(t)$ for all $x \in \mathbb{Z}^n$, where $u(y)$ denotes the letter of $u$ at position $y$. But now $u$ can have up to $n$ linearly independent vectors of periodicity- this leads to a notion of *$r$-periodicity*, with $r$ being the maximal number of linearly independent vectors of periodicity. Aperiodicity then corresponds to 0-periodicity.

The last step would be to define multi-dimensional Sturmian sequences as aperiodic multi-dimensional sequences of minimal complexity. It seems natural to define on $\mathbb{N}^n$ the complexity function $p_u$ of an $n$-dimensional sequence $u$ by the number $p_u(k_1, \ldots, k_n)$ of rectangular factors of size $k_1 \times \ldots \times k_n$ which occur in $u$. For $n = 1$, we retrieve the classic complexity function, and let us recall that, in this case, $u$ is aperiodic if and only if $p_u(k) \geq k+1$ for all $k$. For $n = 2$, let us recall a conjecture of M. Nivat ([30]): if the 2-dimensional sequence $u$ is aperiodic, then $p_u(k, k') \geq kk'+1$ for all $k, k'$. This conjecture has been proven in the particular case $k = 2$ or $k' = 2$ in [39], and for all $k, k'$ in the slightly weaker version $p_u(k, k') \geq \frac{kk'}{16} + 1$ in [34]. However, it is, conversely, not hard to find a *periodic* 2-dimensional sequence $u$ with complexity $p_u(k, k') \geq kk' + 1$. This conjecture thus will never be a sufficient part of a complete characterization of the aperiodicity of multi-dimensional sequences.

In order to get around this problem, a first way could be to define the complexity by replacing the *rectangular* shapes of factors with some other shapes. In this way, various shapes of factors are considered in [16]. In particular, those obtained by discretization of real convex sets seem promising. A second way could be to characterize the aperiodicity within only a *subset* of multi-dimensional sequences. Indeed, in the one-dimensional case, Sturmian sequences turn out to be all *uniformly recurrent*, *i.e.* such that a factor which occurs somewhere reoccurs at a bounded distance from every point. But it is worth noticing that aperiodic 2-dimensional sequences of rectangular complexity $kk' + 1$ are not uniformly recurrent, although they would be sequences of minimal complexity among aperiodic sequences, according to the conjecture of M. Nivat (see [15]). It thus seems reasonable to characterize aperiodicity within the set of uniformly recurrent multi-dimensional sequences. In this way, it is proven in [12] that among uniformly recurrent 2-dimensional sequences, those with rectangular complexity $kk' + k$ (or $kk' + k'$) are aperiodic, and no aperiodic ones of lower complexity are known. One thus can hope for a complete characterization, although additional restrictions on the considered set of sequences would possibly be required. It is also worth noticing that the aperiodic 2-dimensional sequences of [12] were obtained from a 2-dimensional version of the multi-dimensional sequences defined in the next section.

### *2.2. Hyperplane Sequences*

In the one-dimensional case, Sturmian sequences can be described as *cutting sequences*, *i.e.* as discretizations of real lines of the plane. Here we consider discretizations of $(n-1)$-dimensional real hyperplanes of $\mathbb{R}^n$ which yield $(n-1)$-dimensional sequences over $\{1, \ldots, n\}$.

The basic tools for our discretization are the faces of the unit hypercubes of $\mathbb{R}^n$:

**Definition 1** *The* face $(\vec{x}, i^*)$ *is the subset of* $\mathbb{R}^n$ *defined by:*

$$(\vec{x}, i^*) = \{\vec{x} + \vec{e}_i + \sum_{j \neq i} \lambda_j \vec{e}_j, \ 0 \leq \lambda_j \leq 1\}.$$

*The set of faces is denoted by* $\mathcal{F}$.

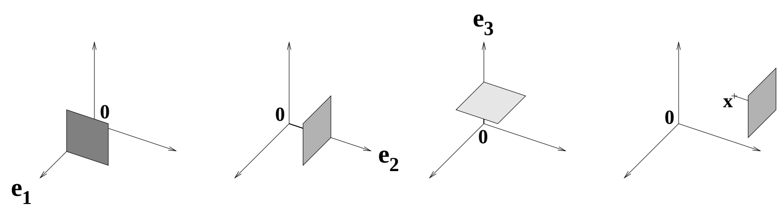


Figure 1: From left to right, the faces $(\vec{0}, 1^*)$, $(\vec{0}, 2^*)$, $(\vec{0}, 3^*)$ and $(\vec{x}, 2^*)$ (in $\mathbb{R}^3$).

We use these faces to discretize the real hyperplane:

$$\mathcal{P}_{\vec{\alpha},\rho} = \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x}, \vec{\alpha} \rangle + \rho = 0\},$$

where $\langle \vec{x}, \vec{y} \rangle$ denotes the canonical inner product of $\vec{x}$ and $\vec{y}$.

**Definition 2** *Let* $\vec{\alpha} \in \mathbb{R}^n_+ \backslash \{0\}$ *and* $\rho \in \mathbb{R}$. *The* stepped hyperplanes $\mathcal{S}^+_{\vec{\alpha},\rho}$ *and* $\mathcal{S}^-_{\vec{\alpha},\rho}$ *are the sets of faces defined by:*

$$\mathcal{S}^+_{\vec{\alpha},\rho} = \{(\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle + \rho < 0 \leq \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle + \rho\},$$

$$\mathcal{S}^-_{\vec{\alpha},\rho} = \{(\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle + \rho \leq 0 < \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle + \rho\}.$$

Note that $\mathcal{S}^+_{\vec{\alpha},\rho} = \mathcal{S}^-_{\vec{\alpha},\rho}$ for almost all $\rho \in \mathbb{R}$. In what follows, we write $\mathcal{S}_{\vec{\alpha},\rho}$ for both $\mathcal{S}^+_{\vec{\alpha},\rho}$ or $\mathcal{S}^-_{\vec{\alpha},\rho}$ when it does not matter.

Let us encode stepped hyperplanes into $(n-1)$-dimensional sequences over the alphabet $\{1, \ldots, n\}$. We first define a bijection between the faces of a given stepped

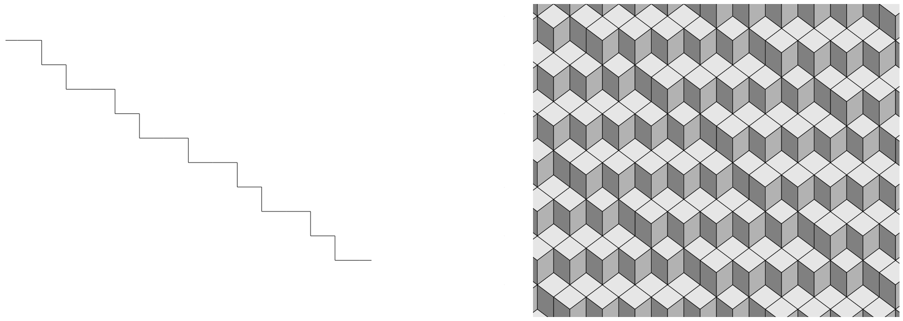

Figure 2: Stepped hyperplanes in the $n = 2$ (left) and $n = 3$ (right) cases.

hyperplane and the points of the $(n-1)$-dimensional lattice $\mathbb{Z}^{n-1}$. This bijection is already stated in [3, 5, 11] in the case of $n = 3$. We here provide a new proof for any $n$.

Informally, this bijection consists of two steps. First, a *proper* vertex is associated with each face of a stepped hyperplane, *i.e.* a point of $\mathbb{Z}^n$ which belongs to no other face of the stepped hyperplane. Then we use the orthogonal projection onto the real hyperplane normal to the vector $(1, \ldots, 1)$- the faces are mapped onto three types of lozenges, identical up to rotation, and it turns out that the proper vertices of the faces are mapped onto an $n-1$ dimensional lattice. Fig.3 illustrates this.

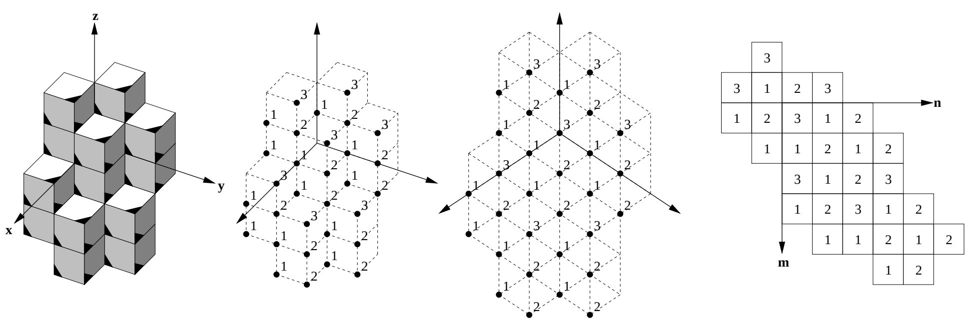


Figure 3: From left to right (for $n = 3$): a proper vertex corresponds to each face (at its black corner); the type $1, \ldots, n$ of a vertex depends on the type of its corresponding face; the projection onto the hyperplane normal to the vector ${}^t(1, \ldots, 1)$ maps the vertices to an $(n-1)$-dimensional lattice; we thus obtain an $(n-1)$-dimensional sequence over $\{1, \ldots, n\}$.

Formally:

**Proposition 1** *Let $v : \mathcal{F} \to \mathbb{Z}^n$ and $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ be the maps defined by:*

$$v(\vec{x}, i^*) = \vec{x} + \vec{e}_1 + \ldots + \vec{e}_i \quad \textit{and} \quad \pi(x_1, \ldots, x_n) = (x_1 - x_n, \ldots, x_{n-1} - x_n).$$

*Then, $\pi \circ v$ maps bijectively the faces of a given stepped hyperplane onto $\mathbb{Z}^{n-1}$.*

**Proof.** We prove this for $\mathcal{S}^+_{\vec{\alpha},\rho}$, with the proof for $\mathcal{S}^-_{\vec{\alpha},\rho}$ being similar. We denote by $\mathcal{V}^+_{\vec{\alpha},\rho} \in \mathbb{Z}^n$ the set of vertices of $\mathcal{S}^+_{\vec{\alpha},\rho}$, *i.e.* the points of $\mathbb{Z}^n$ which can be written $\vec{x} + \sum_{j \in I} \vec{e}_j$ for $(\vec{x}, i^*) \in \mathcal{S}^+_{\vec{\alpha},\rho}$ and $I \subset \{1, \ldots, n\}$ with $i \in I$.

We first prove that $v$ maps bijectively the faces of $\mathcal{S}^+_{\vec{\alpha},\rho}$ onto $\mathcal{V}^+_{\vec{\alpha},\rho}$. It is clear that $v(\mathcal{S}^+_{\vec{\alpha},\rho}) \subset \mathcal{V}^+_{\vec{\alpha},\rho}$. Let then $(\vec{x}, i^*)$ and $(\vec{y}, j^*)$ be two faces of $\mathcal{S}^+_{\vec{\alpha},\rho}$ such that $v(\vec{x}, i^*) = v(\vec{y}, j^*)$. If $i < j$, then $\vec{x} = \vec{y} + \vec{e}_{i+1} + \ldots + \vec{e}_j$. So $\langle \vec{x}, \vec{\alpha} \rangle = \langle \vec{y} + \vec{e}_{i+1} + \ldots + \vec{e}_j, \vec{\alpha} \rangle \geq \langle \vec{y} + \vec{e}_j, \vec{\alpha} \rangle$. Since $(\vec{y}, j^*) \in \mathcal{S}^+_{\vec{\alpha},\rho}$, $\langle \vec{y} + \vec{e}_j, \vec{\alpha} \rangle + \rho \geq 0$ and thus $\langle \vec{x}, \vec{\alpha} \rangle + \rho \geq 0$. But $(\vec{x}, i^*) \in \mathcal{S}^+_{\vec{\alpha},\rho}$ yields $\langle \vec{x}, \vec{\alpha} \rangle + \rho < 0$: $i < j$ is impossible. Similarly, $i > j$ is impossible. Hence $i = j$, and $\vec{x} = \vec{y}$ follows. Thus, $v$ is one-to-one from $\mathcal{S}^+_{\vec{\alpha},\rho}$ to $\mathcal{V}^+_{\vec{\alpha},\rho}$.
Conversely, let $\vec{y} \in \mathcal{V}^+_{\vec{\alpha},\rho}$. Let $(\vec{x}, i^*) \in \mathcal{S}^+_{\vec{\alpha},\rho}$ and $I \subset \{1, \ldots, n\}$, $i \in I$, such that $\vec{y} = \vec{x} + \sum_{j \in I} \vec{e}_j$. Let us define $f \; : \; k \mapsto \langle \vec{x} + \sum_{j \in I} \vec{e}_j - \vec{e}_1 - \ldots - \vec{e}_k, \vec{\alpha} \rangle + \rho$. One has:

$$f(0) \geq \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle + \rho \geq 0, \qquad f(n) \leq \langle \vec{x}, \vec{\alpha} \rangle + \rho < 0, \qquad f(k+1) \leq f(k).$$

So, let $k_0$ such that $f(k_0 - 1) \geq 0$ and $f(k_0) < 0$, and set $\vec{y}_0 = \vec{y} - \vec{e}_1 - \ldots - \vec{e}_{k_0}$. One has $\langle \vec{y}_0, \vec{\alpha} \rangle + \rho = f(k_0) < 0$ and $\langle \vec{y}_0 + \vec{e}_{k_0}, \vec{\alpha} \rangle + \rho = f(k_0 - 1) \geq 0$, *i.e.* $(\vec{y}_0, k_0^*) \in \mathcal{S}^+_{\vec{\alpha},\rho}$. Since $v(\vec{y}_0, k_0^*) = y$, this proves that $v$ is onto on $\mathcal{V}^+_{\vec{\alpha},\rho}$.

Let us prove now that $\pi$ bijectively maps the vertices of $\mathcal{V}^+_{\vec{\alpha},\rho}$ onto $\mathbb{Z}^{n-1}$. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$. It is clear that $\pi(\mathcal{V}^+_{\vec{\alpha},\rho}) \subset \mathbb{Z}^{n-1}$. Then let $\vec{y} = (y_1, \ldots, y_{n-1}) \in \mathbb{Z}^{n-1}$, and prove that there is a unique $\vec{x} = (x_1, \ldots, x_n) \in \mathcal{V}^+_{\vec{\alpha},\rho}$ such that $\pi(\vec{x}) = \vec{y}$. Suppose that $\vec{x}$ satisfies $\pi(\vec{x}) = \vec{y}$, and let $(\vec{x'}, i^*) \in \mathcal{S}^+_{\vec{\alpha},\rho}$ be the face such that $v(\vec{x'}, i^*) = x$, *i.e.* $\vec{x} = \vec{x'} - \vec{e}_1 - \ldots - \vec{e}_i$. One has $\langle \vec{x'}, \vec{\alpha} \rangle + \rho < 0 \leq \langle \vec{x'} + \vec{e}_i, \vec{\alpha} \rangle + \rho$, hence $0 \leq \langle \vec{x'}, \vec{\alpha} \rangle + \rho + \alpha_i < \alpha_i$. Thus:

$$0 \leq \sum_{j=1}^{n} x_j \alpha_j - \sum_{j=1}^{i} \alpha_j + \rho + \alpha_i < \alpha_i.$$

Then, $\pi(\vec{x}) = \vec{y}$, that is, $x_j = y_j + x_n$ for $j < n$, yields:

$$0 \leq \sum_{j=1}^{n-1} y_j \alpha_j + x_n \sum_{j=1}^{n} \alpha_j + \rho < \sum_{j=1}^{i} \alpha_j \leq \sum_{j=1}^{n} \alpha_j,$$

and performing the division by $\sum_{j=1}^{n} \alpha_j > 0$, we obtain:

$$0 \leq \frac{\sum_{j=1}^{n-1} y_j \alpha_j + \rho}{\sum_{j=1}^{n} \alpha_j} + x_n < 1.$$

Since $x_n \in \mathbb{Z}$, the previous inequalities completely characterize $x_n$. Hence $\vec{x}$ is unique, if it exists. And it does exist, since setting $x_n$ as above and $x_i = y_i + x_n$ for $i = 1 \ldots n-1$ yields $\vec{x} = (x_1, \ldots, x_n) \in \mathcal{V}^+_{\vec{\alpha},\rho}$ and $\pi(\vec{x}) = \vec{y}$. □

We then use this bijection to map a face orthogonal to $\vec{e}_i$ onto a letter $i$ indexed by $\mathbb{Z}^{n-1}$:

**Definition 3** *A* hyperplane sequence *is a $(n-1)$-dimensional sequence over the alphabet $\{1, \ldots, n\}$ which is image of a stepped hyperplane by the map:*

$$\phi \ : \ \begin{array}{ccc} \mathcal{F} & \rightarrow & \mathbb{Z}^{n-1} \times \{1, \ldots, n\} \\ (\vec{x}, i^*) & \mapsto & (\pi \circ v(\vec{x}, i^*), i). \end{array}$$

Note that not *all* the $(n-1)$-dimensional sequences over $\{1, \ldots, n\}$ are hyperplane sequences, some of them cannot be obtained as images by $\phi$ of a stepped hyperplane.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 2 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 3 | 1 | 3 | 
| 3 | 1 | 3 | 1 | 2 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 1 | 2 |
| 2 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 3 | 1 | 3 | 1 | 2 | 1 |
| 1 | 2 | 1 | 2 | 3 | 1 | 3 | 1 | 2 | 1 | 2 | 3 | 1 | 2 |
| 3 | 1 | 2 | 1 | 2 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 3 | 1 |
| 2 | 3 | 1 | 3 | 1 | 2 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 1 |
| 1 | 2 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 3 | 1 | 3 | 1 | 2 |
| 3 | 1 | 2 | 1 | 2 | 3 | 1 | 3 | 1 | 2 | 1 | 2 | 3 | 1 |

Figure 4: A 2-dimensional hyperplane sequence. Note that it seems to be strongly regular although not periodic.

We finally define *multi-dimensional Sturmian sequences.* Let us recall that a classic Sturmian sequence is a discretization of a line of direction $(1, \alpha)$, with $1, \alpha$ linearly independent over $\mathbb{Q}$. Thus we set:

**Definition 4** *An $n$-*dimensional Sturmian sequence *is the image by $\phi$ of a stepped hyperplane $\mathcal{S}_{{}^t(1,\alpha_1,\ldots,\alpha_n),\rho}$, with $1, \alpha_1, \ldots, \alpha_n$ linearly independent over $\mathbb{Q}$.*

Then, according to this definition, 1-dimensional Sturmian sequences correspond to classic two-sided Sturmian sequences.

### *2.3. Rotation Sequences*

In this section, we define *rotation sequences.* They are $(n-1)$-dimensional sequences over $\{1, \ldots, n\}$ obtained as the coding, relative to a partition into $n$ interval of $[0, 1)$, of the action of $n-1$ rotations on an element $\rho \in [0, 1)$. We prove that we exactly retrieve the hyperplane sequences of the previous section. Rotation sequences have been defined in [29] for $n = 1$ and in [11, 12] for $n = 2$, where the correspondance with the hyperplane sequences is also proven. Here we state and prove these results for any $n$.

Let us first summarize basic definitions from Chap. 2 of [27]. For $0 < \alpha < 1$, the rotation of angle $\alpha$ is the mapping $R_\alpha$ from $[0,1)$ into itself defined by:

$$R_\alpha(z) = \{z + \alpha\},$$

where $\{z\}$ denotes the fractional part of $z$. It is convenient to identify $[0,1)$ with the torus $\mathbb{R}/\mathbb{Z}$, *i.e.* the unit circle. For $0 \leq b < a < 1$, the set $[a,1] \cup [0,b)$ is then considered as an interval and denoted by $[a,b)$. Thus, for any subinterval $I$ of $[0,1)$, the sets $R_\alpha(I)$ and $R_\alpha^{-1}(I)$ are always intervals (even when overlapping the point 0). We now define *rotation sequences*:

**Definition 5** *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$ such that $\sum_j \alpha_j = 1$ and $\rho \in [0,1)$. The* rotation sequences *$\mathcal{U}_{\vec{\alpha},\rho}^+$ and $\mathcal{U}_{\vec{\alpha},\rho}^-$ are the $(n-1)$-dimensional sequences over $\{1, \ldots, n\}$ defined by:*

$$\mathcal{U}_{\vec{\alpha},\rho}^+(y_1, \ldots, y_{n-1}) = i \;\Leftrightarrow\; R_{\alpha_1}^{y_1} \circ \cdots \circ R_{\alpha_{n-1}}^{y_{n-1}}(\rho) \in I_i^+,$$

$$\mathcal{U}_{\vec{\alpha},\rho}^-(y_1, \ldots, y_{n-1}) = i \;\Leftrightarrow\; R_{\alpha_1}^{y_1} \circ \cdots \circ R_{\alpha_{n-1}}^{y_{n-1}}(\rho) \in I_i^-,$$

*where $I_i^+ = \sum_{j<i} \alpha_j + [0, \alpha_i)$ and $I_i^- = \sum_{j<i} \alpha_j + (0, \alpha_i]$.*

The following proposition shows that these rotation sequences and the hyperplane sequences of Definition 3 are in fact the same:

**Proposition 2** *For $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$ such that $\sum_j \alpha_j = 1$ and $\rho \in [0,1)$:*

$$\mathcal{U}_{\vec{\alpha},\rho}^+ = \phi(\mathcal{S}_{\vec{\alpha},\rho}^+) \qquad \text{and} \qquad \mathcal{U}_{\vec{\alpha},\rho}^- = \phi(\mathcal{S}_{\vec{\alpha},\rho}^-).$$

**Proof.** We prove this for $\mathcal{S}_{\vec{\alpha},\rho}^+$, with the proof for $\mathcal{S}_{\vec{\alpha},\rho}^-$ being similar. Let $\vec{y} \in \mathbb{Z}^{n-1}$, denoted by $\vec{y} = (y_1, \ldots, y_{n-1})$, and $i = \phi(\mathcal{S}_{\vec{\alpha},\rho}^+)(\vec{y})$. By Proposition 1, there is a unique face $(\vec{x}, i^*) \in \mathcal{S}_{\vec{\alpha},\rho}^+$ such that $\pi \circ v(\vec{x}, i^*) = \vec{y}$. Writing $\vec{x} = (x_1, \ldots, x_n)$, this yields $y_j = x_j + 1 - x_n$ for $j \leq i$ and $y_j = x_j - x_n$ for $i < j < n$, with $x_n \in \mathbb{Z}$. One thus computes:

$$\langle \vec{x}, \vec{\alpha} \rangle = \sum_{j=1}^{n-1} y_j \alpha_j - \sum_{j \leq i} \alpha_j + x_n \sum_{j=1}^{n} \alpha_j.$$

Hence one has, modulo 1:

$$\sum_{j \leq i} \alpha_j + \langle x, \vec{\alpha} \rangle = \sum_{j=1}^{n-1} y_j \alpha_j = \sum_{j<i} \alpha_j + \langle x + \vec{e}_i, \vec{\alpha} \rangle,$$

and $\langle \vec{x}, \vec{\alpha} \rangle + \rho < 0 \leq \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle + \rho$ then yields, modulo 1:

$$\sum_{j<i} \alpha_j \leq \sum_{j=1}^{n-1} y_j \alpha_j + \rho < \sum_{j \leq i} \alpha_j,$$

that is, $\mathcal{U}^+_{\vec{\alpha},\rho}(\vec{y}) = i$. □

Such a definition by rotations is, for example, very convenient for studying frequencies of factors (see [1, 11] in one- or two-dimensional cases).

## 3. Multi-dimensional Substitutions

### 3.1. Generalized Substitutions

We here briefly recall the definition of *generalized substitutions*. The reader is referred to [6] or Chap.8 of [32] for a less succinct presentation.

Let us recall that the *incidence matrix* of a substitution $\sigma$ over $\{1, \ldots, n\}$ is the $n \times n$ integer matrix whose coefficient at row $i$ and column $j$ is the number of occurences of the letter $i$ in the word $\sigma(j)$. A substitution $\sigma$ is then said to be *unimodular* if $\det M_\sigma = \pm 1$.

To a unimodular substitution is associated a map acting on the faces introduced in Section 2.2 (Definition 1):

**Definition 6 ([6])** *Let $\sigma$ be a unimodular substitution over $\{1, \ldots, n\}$. The* generalized substitution $\Theta^*_\sigma$ *is defined on the face $(\vec{x}, i^*)$ of $\mathbb{R}^n$ by:*

$$\Theta^*_\sigma(\vec{x}, i^*) = \bigcup_{\substack{p \mid \sigma(j) = p \cdot i \cdot s \\ 1 \le j \le n}} \left( M_\sigma^{-1}(\vec{x} - \vec{f}(p)), j^* \right),$$

*where $\vec{f}(u)$ is the vector of $\mathbb{Z}^n$ whose $i$-th coordinate is the number of occurrences of the letter $i$ in the word $u$ (in particular, $\vec{f}(\sigma(i)) = M_\sigma \vec{e}_i$). Note that since $\sigma$ is unimodular, $M_\sigma^{-1}$ also has* integer *coefficients, and thus $\Theta^*_\sigma(\vec{x}, i^*) \subset \mathcal{F}$. This definition is then extended to sets of faces by:*

$$\forall E \subset \mathcal{F}, \qquad \Theta^*_\sigma(E) = \bigcup_{(\vec{x}, i^*) \in E} \Theta^*_\sigma(\vec{x}, i^*).$$

It is convenient to define an action of $\mathbb{Z}^n$ on the faces by:

$$\vec{x} + (\vec{y}, i^*) = (\vec{y} + \vec{x}, i^*).$$

Indeed, one then checks that $\Theta^*_\sigma(\vec{x}, i^*) = M_\sigma^{-1}\vec{x} + \Theta^*_\sigma(\vec{0}, i^*)$, so that the images by $\Theta^*_\sigma$ of the faces $(\vec{0}, i^*)$ for $1 \le i \le n$ suffice to define $\Theta^*_\sigma$ on the whole $\mathcal{F}$.

**Example 1** *Let us consider the substitution $\sigma : 1 \mapsto 12, 2 \mapsto 1$. One has:*

$$M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad M_\sigma^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

*and one computes (see also Fig.5):*

$$\begin{aligned} \Theta^*_\sigma(\vec{0}, 1^*) &= \left\{ (\vec{0}, 1^*), (\vec{0}, 2^*) \right\}, \\ \Theta^*_\sigma(\vec{0}, 2^*) &= \{(-\vec{e}_2, 1^*)\}. \end{aligned}$$

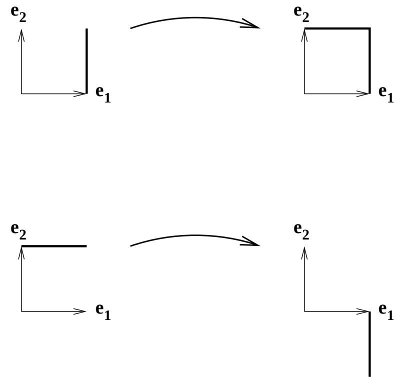


Figure 5: The action of the generalized substitution associated to $\sigma : 1 \mapsto 12, 2 \mapsto 1$ on the faces $(\vec{0}, 1^*)$ and $(\vec{0}, 2^*)$.

**Example 2** *Let us consider the substitution* $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$. *One has:*

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \textit{and} \quad M_\sigma^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix},$$

*and one computes (see also Fig.6):*

$$\begin{aligned} \Theta_\sigma^*(\vec{0}, 1^*) &= \left\{(\vec{0}, 1^*), (\vec{0}, 2^*), (\vec{0}, 3^*)\right\}, \\ \Theta_\sigma^*(\vec{0}, 2^*) &= \{(-\vec{e}_3, 1^*)\}, \\ \Theta_\sigma^*(\vec{0}, 3^*) &= \{(-\vec{e}_3, 2^*)\}. \end{aligned}$$

Note that two distinct faces can be mapped by a generalized substitution onto the *same* face. It is certainly possible according to Definition 6, but we would like to avoid this, in order to define a suitable notion of multi-dimensional substitution. To this aim, in the following section we consider particular sets of faces, namely stepped hyperplanes (Definition 2). We prove that distinct faces of such sets are mapped by a generalized substitution onto distinct sets of faces.

*3.2. Action on Stepped Hyperplanes*

In [6], it is proven that if $\vec{\alpha}$ is the left eigenvector of the incidence matrix $M_\sigma$ of a unimodular substitution $\sigma$, then the generalized substitution $\Theta_\sigma^*$ maps distinct faces of the stepped hyperplane $\mathcal{S}_{\vec{\alpha},0}$ onto disjoint sets of faces of $\mathcal{S}_{\vec{\alpha},0}$. We here extend this result, proving that *any* stepped hyperplane $\mathcal{S}_{\vec{\alpha},\rho}$ is mapped *without overlaps* onto the *whole* stepped hyperplane $\mathcal{S}_{{}^tM_\sigma\vec{\alpha},\rho}$. In particular, distinct faces of $\mathcal{S}_{\vec{\alpha},\rho}$ are mapped onto distinct faces of $\mathcal{S}_{{}^tM_\sigma\vec{\alpha},\rho}$.

**Theorem 1** *Let* $\sigma$ *be a unimodular substitution on* $\{1, \ldots, n\}$, $M_\sigma$ *denote its incidence matrix and* $\Theta_\sigma^*$ *be the associated generalized substitution. Then, for* $\vec{\alpha} \in \mathbb{R}_+^n \backslash \{0\}$ *and* $\rho \in \mathbb{R}$, *one has:*

$$\Theta_\sigma^*(\mathcal{S}_{\vec{\alpha},\rho}) = \mathcal{S}_{{}^tM_\sigma\vec{\alpha},\rho}.$$

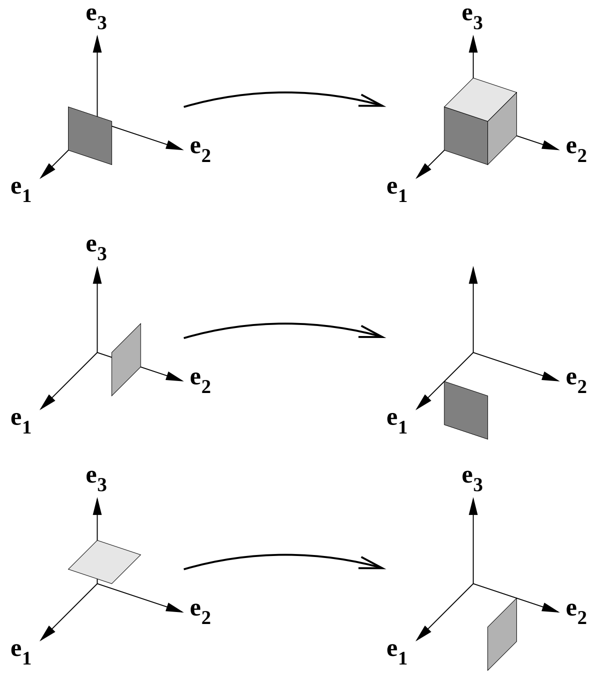


Figure 6: The action of the generalized substitution associated with $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ on the faces $(\vec{0}, 1^*)$, $(\vec{0}, 2^*)$ and $(\vec{0}, 3^*)$.

**Proof.** We prove this for $\mathcal{S}^+_{\vec{\alpha},\rho}$, with the proof for $\mathcal{S}^-_{\vec{\alpha},\rho}$ being similar. The proof is in two steps: we first prove that a face of $\mathcal{S}^+_{\vec{\alpha},\rho}$ is mapped by $\Theta^*_\sigma$ onto faces of $\mathcal{S}^+_{{}^tM_\sigma\vec{\alpha},\rho}$, and then that a face of $\mathcal{S}^+_{{}^tM_\sigma\vec{\alpha},\rho}$ belongs to the image of exactly one face of $\mathcal{S}^+_{\vec{\alpha},\rho}$.

Let $(\vec{x}, i^*) \in \mathcal{S}^+_{\vec{\alpha},\rho}$ and $(\vec{y}, j^*) \in \Theta^*_\sigma(x, i^*)$. One can write $\vec{y} = M_\sigma^{-1}(\vec{x} - \vec{f}(p))$ with $\sigma(j) = p \cdot i \cdot s$. One then has:

$$\begin{aligned}\langle \vec{y}, {}^tM_\sigma\vec{\alpha}\rangle &= \langle M_\sigma^{-1}(\vec{x} - \vec{f}(p)), {}^tM_\sigma\vec{\alpha}\rangle \\ &= \langle M_\sigma M_\sigma^{-1}(\vec{x} - \vec{f}(p)), \vec{\alpha}\rangle \\ &= \langle \vec{x} - \vec{f}(p), \vec{\alpha}\rangle \\ &\leq \langle \vec{x}, \vec{\alpha}\rangle,\end{aligned}$$

and:

$$\begin{aligned}\langle \vec{y} + \vec{e}_j, {}^tM_\sigma\vec{\alpha}\rangle &= \langle M_\sigma^{-1}(\vec{x} - \vec{f}(p) + M_\sigma\vec{e}_j), {}^tM_\sigma\vec{\alpha}\rangle \\ &= \langle \vec{x} - \vec{f}(p) + M_\sigma\vec{e}_j, \vec{\alpha}\rangle \\ &= \langle \vec{x} - \vec{f}(p) + \vec{f}(p \cdot i \cdot s), \vec{\alpha}\rangle \\ &= \langle \vec{x} + \vec{e}_i + \vec{f}(s), \vec{\alpha}\rangle \\ &\geq \langle \vec{x} + \vec{e}_i, \vec{\alpha}\rangle.\end{aligned}$$

Thus, $(\vec{x}, i^*) \in \mathcal{S}^+_{\vec{\alpha},\rho}$ yields $\langle \vec{y}, {}^t M_\sigma \vec{\alpha} \rangle + \rho < 0$ and $\langle \vec{y} + \vec{e}_j, {}^t M_\sigma \vec{\alpha} \rangle + \rho \geq 0$, that is, $(\vec{y}, j^*) \in \mathcal{S}^+_{{}^t M_\sigma \vec{\alpha},\rho}$. This yields $\Theta^*_\sigma(\mathcal{S}^+_{\vec{\alpha},\rho}) \subset \mathcal{S}^+_{{}^t M_\sigma \vec{\alpha},\rho}$.

Now, let $(\vec{y}, j^*) \in \mathcal{S}^+_{{}^t M_\sigma \vec{\alpha},\rho}$. Note that $(\vec{x}, i^*)$ is such that $(\vec{y}, j^*) \in \Theta^*_\sigma(\vec{x}, i^*)$ if and only if one can write $\sigma(j) = p \cdot i \cdot s$ so that $\vec{y} = M_\sigma^{-1}(\vec{x} - \vec{f}(p))$. Thus, for $\sigma(j) = i_1 \cdots i_q$, the preimages by $\Theta^*_\sigma$ of $(\vec{y}, j^*)$ are exactly the faces $(\vec{x}_k, i^*_k)_{k=0\ldots q}$ where $\vec{x}_k = M_\sigma \vec{y} + \vec{f}(i_1 \cdots i_{k-1})$. Let us prove that exactly one of these preimages belongs to $\mathcal{S}_{\vec{\alpha},\rho}$. Let $h$ be the function defined for $k = 0 \ldots q$ by $h(k) = \langle \vec{x}_k, \vec{\alpha} \rangle + \rho$. Since $(\vec{y}, j^*) \in \mathcal{S}_{{}^t M_\sigma \vec{\alpha},\rho}$, one has:

$$h(0) = \langle M_\sigma \vec{y}, \vec{\alpha} \rangle + \rho = \langle \vec{y}, {}^t M_\sigma \vec{\alpha} \rangle + \rho < 0,$$

$$h(q) = \langle M_\sigma \vec{y} + \vec{f}(\sigma(j)), \vec{\alpha} \rangle + \rho = \langle M_\sigma \vec{y} + M_\sigma \vec{e}_j, \vec{\alpha} \rangle + \rho = \langle \vec{y} + \vec{e}_j, {}^t M_\sigma \vec{\alpha} \rangle + \rho \geq 0.$$

Moreover, for $k \geq 0$:

$$h(k+1) = \langle \vec{x}_{k+1}, \vec{\alpha} \rangle + \rho = \langle \vec{x}_k + \vec{f}(i_k), \vec{\alpha} \rangle + \rho \geq h(k).$$

Thus, there is a unique $k_0$, $0 \leq k_0 < q$, such that $h(k_0) < 0 \leq h(k_0 + 1)$, which exactly means that $(\vec{x}_{k_0}, i^*_{k_0})$ is the unique preimage by $\Theta^*_\sigma$ of $(\vec{y}, j^*)$ which belongs to $\mathcal{S}^+_{\vec{\alpha},\rho}$. □

Figure 7 illustrates Theorem 1.

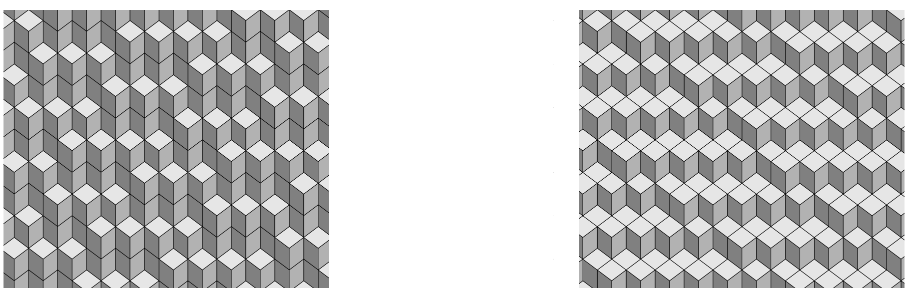

Figure 7: A stepped hyperplane $\mathcal{S}_{\vec{\alpha},\rho}$ (left) is mapped by the generalized substitution $\Theta^*_\sigma$ onto the stepped hyperplane $\mathcal{S}_{{}^t M_\sigma \vec{\alpha},\rho}$ (right).

Note that although the proof Theorem 1 is rather technical, it is on the contrary easy to check that:

$$M_\sigma^{-1} \mathcal{P}_{\vec{\alpha},\rho} = \mathcal{P}_{{}^t M_\sigma \vec{\alpha},\rho},$$

where $\mathcal{P}_{\vec{\alpha},\rho}$ is the hyperplane introduced in Section 2.2 whose discretization is the stepped hyperplane $\mathcal{S}_{\vec{\alpha},\rho}$. In other words, the generalized substitution $\Theta^*_\sigma$ acts as a "discretization" of the linear map $M_\sigma^{-1}$. This viewpoint can help to more intuitively understand the way generalized substitutions act.

Note also that two *different* substitutions can have the *same* incidence matrices. In this case, the associated generalized substitutions are different, but they act similarly on stepped hyperplanes according to Theorem 1. Indeed, they act in the same way *globally*, *i.e.* on whole stepped hyperplanes, but not *locally*, *i.e.* on subsets of faces (see Fig.8).

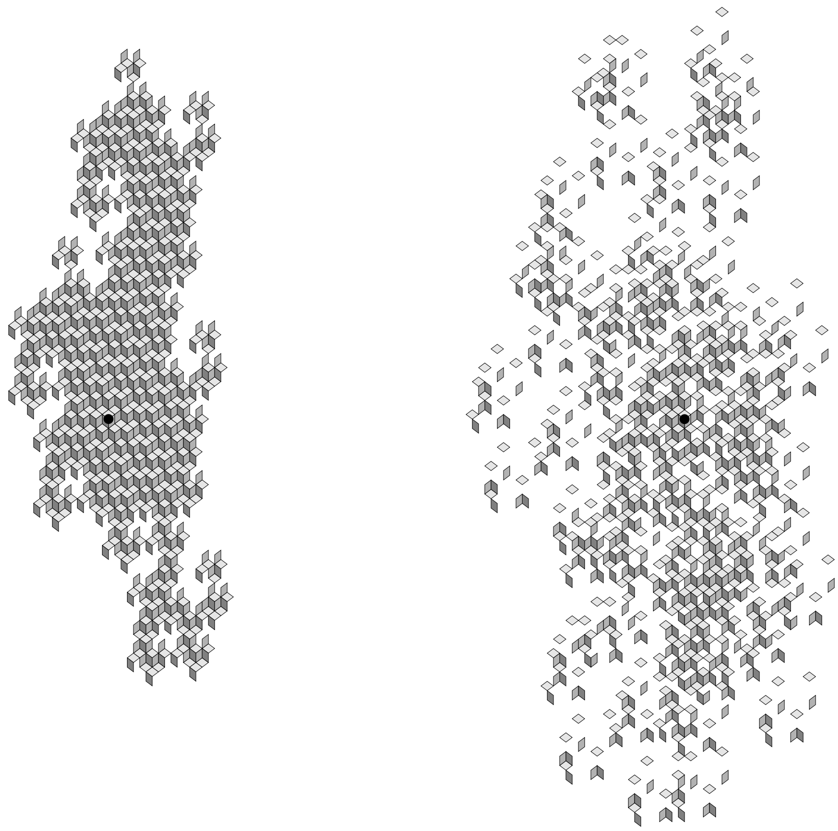

Figure 8: Let $\sigma : 1 \to 12, 2 \to 23, 3 \to 123$ and $\sigma' : 1 \to 12, 2 \to 32, 3 \to 231$. Although $M_\sigma = M_{\sigma'}$, one can see that $\Theta^*_\sigma$ (left) and $\Theta^*_{\sigma'}$ (right) act differently on the same initial set of faces. However, it is worth noting that both obtained sets of faces are included in the *same* stepped hyperplane (the origin is highlighted by the black circle).

Finally, it is straightforward to restate Theorem 1 in terms of multi-dimensional sequences, thanks to the projection onto hyperplane sequences of Section 2.2 (Proposition 1 and Definition 3). Hence, generalized substitutions map hyperplane sequences (resp. Sturmian sequences) onto hyperplane sequences (resp. Sturmian sequences). Note that, among classic substitutions on words, only *Sturmian morphisms* map Sturmian words onto Sturmian words (see [9] or Chap.2 of [27]). It is therefore natural to consider generalized substitutions as a multi-dimensional extension of Sturmian morphisms rather than a general notion of multi-dimensional substitution. Last, note that Sturmian morphisms do not act only over Sturmian words but over all the two-letter words. It is thus natural to extend the action of generalized substitutions beyond stepped hyperplanes. In this direction, the way explored by [4] seems promising.

### *3.3. Effective Generation of Stepped Hyperplanes*

Theorem 1 shows the way generalized substitutions act on stepped hyperplanes. But these substitutions also allow one to generate *effectively* sequences. So we discuss here the possibility of obtaining a stepped hyperplane as the limit of successive applications of a generalized substitution on a finite set of faces (see Fig.9).

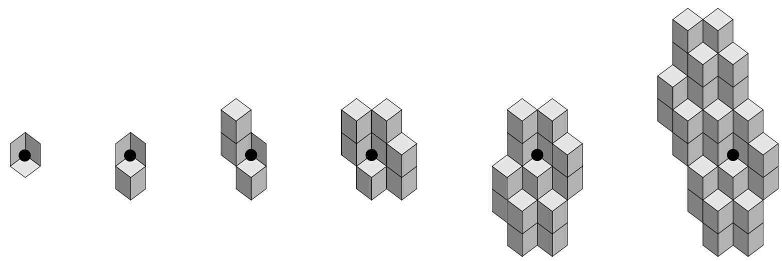

Figure 9: From left to right, some iterations on a finite set of faces of the generalized substitution associated with $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ (described Fig.6). The origin is highlighted by the black circle.

In what follows, we consider only *characteristic* stepped hyperplanes:

**Definition 7** *The stepped hyperplane $\mathcal{S}_{\vec{\alpha},\rho}$ is said to be* characteristic *if $\rho = 0$. For the sake of simplicity, a characteristic stepped hyperplane $\mathcal{S}_{\vec{\alpha},0}$ is denoted by $\mathcal{S}_{\vec{\alpha}}$.*

Theorem 1 yields that generalized substitutions are permutations onto the set of characteristic stepped hyperplanes. Moreover, we also consider here only *Pisot* substitutions:

**Definition 8** *A substitution $\sigma$ on $\{1, \dots, n\}$ with incidence matrix $M_\sigma$ is* Pisot *if $M_\sigma$ has a spectrum $\{\lambda, \mu_1, \dots, \mu_k\}$ such that $0 < |\mu_i| < 1 < \lambda$, for $i = 1, \dots, k$.*

In [14], it is proven that if $\sigma$ is a Pisot substitution, then $M_\sigma$ (and hence ${}^tM_\sigma$) is irreducible and diagonalizable with simple eigenvalues. Thus, according to the Perron-Frobenius theorem, ${}^tM_\sigma$ has a positive eigenvector $\vec{\alpha}$ associated with its dominant eigenvalue $\lambda$:

$$ {}^tM_\sigma \vec{\alpha} = \lambda \vec{\alpha}. $$

We will use the following lemma:

**Lemma 1** *Let $\sigma$ be a Pisot substitution. Let $0 < |\mu_i|_i < 1 < \lambda$ be the eigenvalues of $M_\sigma$ and $\vec{\alpha}$ be a positive eigenvector of ${}^tM_\sigma$ associated with $\lambda$. Then:*

1. *the eigenspace of $M_\sigma^{-1}$ associated with the $(\mu_i^{-1})_i$ is the real hyperplane:*

$$ \mathcal{P}_{\vec{\alpha}} = \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x}, \vec{\alpha} \rangle = 0\}; $$

2. *if* $\mu = \max_i |\mu_i| < 1$, *then there is a positive real number* $c$ *such that:*

$$\forall n \in \mathbb{N},\ \forall \vec{x} \in \mathcal{P}_{\vec{\alpha}},\ \|M_\sigma^{-n}\vec{x}\| \geq \frac{c}{\mu^n}\|\vec{x}\|.$$

**Proof.** Note that $M_\sigma^{-1}$ has eigenvalues $\lambda^{-1}$ and $\mu_i^{-1}$ for $i = 1, \ldots, n-1$.

1. Let $\vec{x}$ be an eigenvector of $M_\sigma^{-1}$ associated with $\mu_i^{-1}$. Since $\mu_i M_\sigma^{-1}\vec{x} = \vec{x}$ and $\lambda\vec{\alpha} = {}^t M_\sigma \vec{\alpha}$, one has:

$$\langle \vec{x}, \vec{\alpha} \rangle = \langle \mu_i M_\sigma^{-1}\vec{x}, \frac{1}{\lambda}{}^t M_\sigma \vec{\alpha} \rangle = \frac{\mu_i}{\lambda}\langle \vec{x}, \vec{\alpha} \rangle,$$

and since $|\frac{\mu_i}{\lambda}| < 1$, this implies $\langle \vec{x}, \vec{\alpha} \rangle = 0$, *i.e.* $\vec{x} \in \mathcal{P}_{\vec{\alpha}}$. Thus, the eigenspace of $M_\sigma^{-1}$ associated with the $(\mu_i^{-1})_i$ is included in $\mathcal{P}_{\vec{\alpha}}$. With these spaces having both dimension $n-1$, this inclusion turns out to be an equality.

2. For $i = 1, \ldots, n-1$, let $\vec{e_i'}$ be an eigenvector of $M_\sigma^{-1}$ associated with $\mu_i^{-1}$. A vector $x \in \mathcal{P}_{\vec{\alpha}}$ can be written $\vec{x} = \sum_i x_i \vec{e_i'}$. We set:

$$N(\vec{x}) = \sum_i |x_i|.$$

It defines a norm $N$ over $\mathcal{P}_{\vec{\alpha}}$. One has:

$$N(M_\sigma^{-n}\vec{x}) = N\left(\sum_i x_i M_\sigma^{-n}\vec{e_i'}\right) = N\left(\sum_i \frac{x_i}{\mu_i^n}\vec{e_i'}\right) = \sum_i \left|\frac{x_i}{\mu_i^n}\right| \leq \frac{1}{\mu^n}N(\vec{x}).$$

Then, the equivalence of the norms $N(.)$ and $\|.\|$ yields $a > 0$ such that $\frac{1}{a}\|\vec{x}\| \leq N(\vec{x}) \leq a\|\vec{x}\|$ for all $\vec{x}$. One thus computes:

$$\|M_\sigma^{-n}\vec{x}\| \leq aN(M_\sigma^{-n}\vec{x}) \leq \frac{a}{\mu^n}N(\vec{x}) \leq \frac{a^2}{\mu^n}\|\vec{x}\|,$$

and this yields the result with $c = a^2$.

□

The previous Lemma tells us that the real hyperplane $\mathcal{P}_{\vec{\alpha}}$ is *invariant* under the linear map $M_\sigma^{-1}$, and that this map is *expansive* on this real hyperplane. We have already mentioned that it is worth considering the generalized substitution $\Theta_\sigma^*$ and the stepped hyperplane $\mathcal{S}_{\vec{\alpha}}$ as discretizations of, respectively, the linear map $M_\sigma^{-1}$ and the real hyperplane $\mathcal{P}_{\vec{\alpha}}$. According to this viewpoint, the following theorem is just a "discrete version" of Lemma 1:

**Theorem 2** *Let* $\sigma$ *be a Pisot unimodular substitution and* $\vec{\alpha}$ *be a positive left eigenvector of* $M_\sigma$ *associated with its dominant eigenvalue. Then, there is a finite set of faces* $P \subset \mathcal{S}_{\vec{\alpha}}$ *such that:*

$$\lim_{n\to\infty} \left(\Theta_\sigma^*\right)^n (P) = \mathcal{S}_{\vec{\alpha}}.$$

**Proof.** Let $(\vec{y}_0, j_0^*) \in \mathcal{S}_{\vec{\alpha}}$. It follows from Theorem 1 that there are a unique sequence $(\vec{y}_n, j_n^*)_{n\geq 0}$ of faces of $\mathcal{S}_{\vec{\alpha}}$ and a unique sequence $(p_n)_{n\geq 0}$ of words, with $p_n$ being a prefix of $\sigma(j_n)$, such that:

$$\forall n \geq 0, \quad \vec{y}_n = M_\sigma^{-1}(\vec{y}_{n+1} - \vec{f}(p_{n+1})).$$

In particular:

$$\vec{y}_0 = M_\sigma^{-n}\vec{y}_n - \sum_{k=1}^{n} M_\sigma^{-k}\vec{f}(p_k).$$

Let us prove that there is $C > 0$ such that if $\|\vec{y}_0\| \geq C$, then $\|\vec{y}_n\| < \|\vec{y}_0\|$ for some $n$.

Let us first bound the quantity $\sum_k M_\sigma^{-k}\vec{f}(p_k)$. Note that the sequence $(p_n)_{n\geq 0}$ ranges through a finite number of words. Thus, there is $F > 0$ which bounds $(\|\vec{f}(p_n)\|)_{n\geq 0}$. Moreover, a classic result of linear algebra yields:

$$\forall \vec{x} \in \mathbb{R}^n, \quad \|M_\sigma^{-1}\vec{x}\| \leq \varrho\|x\|,$$

where $\varrho = \varrho(M_\sigma^{-1}) > 1$ is the spectral radius of $M_\sigma^{-1}$. One computes:

$$\left\|\sum_{k=1}^{n} M_\sigma^{-k}\vec{f}(p_k)\right\| \leq \sum_{k=1}^{n} \varrho^k F = F\frac{\varrho^{n+1}-1}{\varrho - 1}.$$

Let us now examine $\|M_\sigma^{-n}\vec{y}_n\|$. One can write $\vec{y}_n = \vec{y_n'} + \vec{y_n''}$ with $\vec{y_n'} \in \mathcal{P}_{\vec{\alpha}}$ and $\vec{y_n''} \in \mathbb{R}\vec{\alpha}$- $\|y_n''\|$ is the distance from $\vec{y}_n$ to $\mathcal{P}_{\vec{\alpha}}$, and $(\vec{y}_n, j_n^*) \in \mathcal{S}_{\vec{\alpha}}$ yields that this distance is bounded by $\max_i \alpha_i$. Thus, using Lemma 1 with $\vec{y_n'} \in \mathcal{P}_{\vec{\alpha}}$:

$$\begin{aligned}
\|M_\sigma^{-n}\vec{y}_n\| &\geq \|M_\sigma^{-n}\vec{y_n'}\| - \|M_\sigma^{-n}\vec{y_n''}\| \\
&\geq \frac{c}{\mu^n}\|\vec{y_n'}\| - \left\|\frac{1}{\lambda^n}\vec{y_n''}\right\| \\
&\geq \frac{c}{\mu^n}(\|\vec{y}_n\| - \|\vec{y_n''}\|) - \left\|\frac{1}{\lambda^n}\vec{y_n''}\right\| \\
&\geq \frac{c}{\mu^n}\|\vec{y}_n\| - \left(\frac{c}{\mu^n} + \frac{1}{\lambda^n}\right)\max_i \alpha_i.
\end{aligned}$$

So finally, if we set

$$T(n) = F\frac{\varrho^{n+1}-1}{\varrho-1} + \left(\frac{c}{\mu^n} + \frac{1}{\lambda^n}\right)\max_i \alpha_i,$$

one has:

$$\|\vec{y}_0\| \geq \frac{c}{\mu^n}\|\vec{y}_n\| - T(n).$$

Then let $n_0$ be such that $\frac{c}{\mu^{n_0}} > 2$, and set $C = T(n_0)$. This yields $\|\vec{y}_{n_0}\| < \|\vec{y}_0\|$ for $\|\vec{y}_0\| \geq C$. In other words, any face $(\vec{y}_0, j_0^*)$ of $\mathcal{S}_{\vec{\alpha}}$ such that $\|\vec{y}_0\| \geq C$ is the image, by iterated applications of $\Theta_\sigma^*$, of a face $(\vec{y}_{n_0}, j_{n_0}^*)$ with $\|\vec{y}_{n_0}\| < \|\vec{y}_0\|$. Hence, the finite set of faces $P = \{(\vec{y}, j^*) \in \mathcal{S}_{\vec{\alpha}} \mid \|\vec{y}\| \leq C\}$ suffices to generate the whole stepped hyperplane $\mathcal{S}_{\vec{\alpha}}$. □

Note that the previous proof even provides a suitable set of faces which generates the stepped hyperplane. However, it is not so easy to compute this set, especially because of the constant $c$ coming from Lemma 1. Figure 10 illustrates Theorem 2.

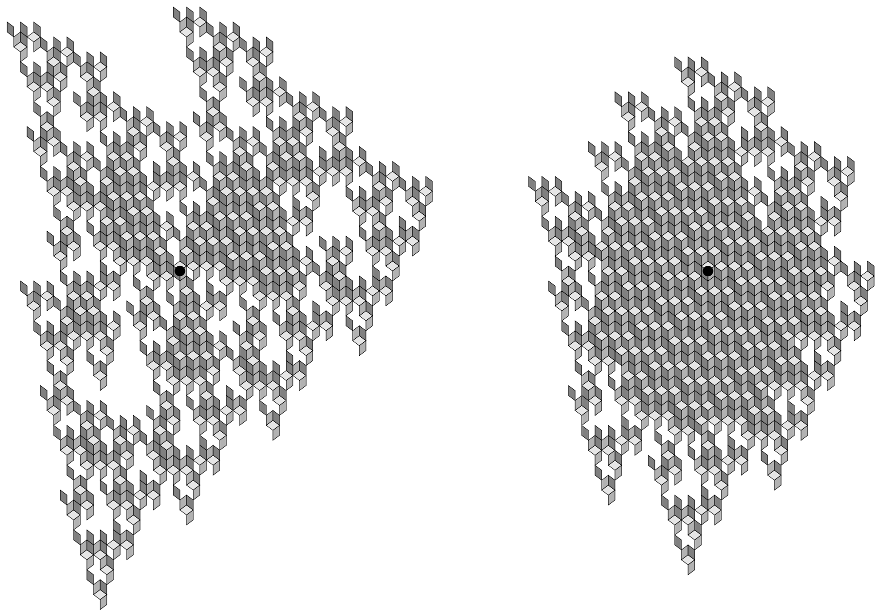

Figure 10: Let us consider the unimodular Pisot substitution $\sigma : 1 \mapsto 21, 2 \mapsto 13, 3 \mapsto 1$. Left, some iterations of $\Theta_\sigma^*$ on a finite set of faces. Some faces near the origin (black circle) turn out to be never obtained (gaps). However, according to Theorem 2, there is a finite set of faces which suffices to generate the whole stepped hyperplane. Right, the set of faces obtained after some iterations of $\Theta_\sigma^*$ on a sufficiently large initial set.

## 4. Multi-dimensional Continued Fractions

### *4.1. Brun Expansions*

There are various multi-dimensional generalizations of continued fractions (see *e.g.* [13] for an overview). Here we describe the Brun algorithm, also called the modified Jacobi-Perron algorithm (see [26]).

Let $X = [0,1)^n$ and denote $X \backslash \{0\}$ by $X^*$. Let $T : X^* \to X$ be defined by:

$$T(\alpha_1, \ldots, \alpha_n) = \left( \frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_n}{\alpha_i} \right),$$

where $i$ is the smallest index such that $\alpha_i = \max \alpha_j$. Let also $a : X^* \to \mathbb{N}^*$ and

$\varepsilon : X \to \{1, \ldots, n\}$ be defined by:

$$a(\alpha_1, \ldots, \alpha_n) = \left\lfloor \frac{1}{\max \alpha_j} \right\rfloor \qquad \text{and} \qquad \varepsilon(\alpha_1, \ldots, \alpha_n) = \min\{i \mid \alpha_i = \max_j \alpha_j\}.$$

The *Brun expansion* of $\vec{\alpha} \in X^*$ is the (possibly finite) sequence:

$$(a_k, \varepsilon_k)_{k \geq 0} = (a(T^k(\vec{\alpha})), \varepsilon(T^k(\vec{\alpha})))_{k \geq 0},$$

and one writes:

$$[\vec{\alpha}] = [(a_0, \varepsilon_0), (a_1, \varepsilon_1), \ldots].$$

Let us give a matrix viewpoint. For $a \in \mathbb{N}$ and $\varepsilon \in \{1, \ldots, n\}$, one defines the following $(n+1) \times (n+1)$ matrix:

$$A_{a,\varepsilon} = \begin{pmatrix} a & & 1 & \\ & I_{\varepsilon-1} & & \\ 1 & & 0 & \\ & & & I_{n-\varepsilon} \end{pmatrix},$$

where $I_p$ is the $p \times p$ identity matrix and all the unspecified coefficients are zeroes. One checks for $\vec{\alpha} \in X^*$:

$$\alpha_{\varepsilon(\vec{\alpha})} A_{a(\vec{\alpha}),\varepsilon(\vec{\alpha})} {}^t(1, T(\vec{\alpha})) = {}^t(1, \vec{\alpha}), \tag{1}$$

where $(1, \vec{u}) = (1, u_1, \ldots, u_n)$ for $\vec{u} = (u_1, \ldots, u_n)$.
Finally, if the Brun expansion of $\vec{\alpha}$ is greater than $k$ in length, then the *$k$-th convergent* of $\vec{\alpha}$ is defined by:

$$[(a_0, \varepsilon_0), \ldots, (a_k, \varepsilon_k)] = \left( \frac{p_1}{q}, \ldots, \frac{p_n}{q} \right),$$

where:

$${}^t(q, p_1, \ldots, p_n) = A_{a_0,\varepsilon_0} \cdots A_{a_k,\varepsilon_k} {}^t(1, 0, \ldots, 0).$$

Note that for $n = 1$, $T$ is exactly the Gauss map. Sequence $(a_n)_n$ then turns out to be the continued fraction expansion of $\alpha$, and one has $\varepsilon_n = 1$ for all $n$. Eq.(1) becomes:

$$\alpha \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ T(\alpha) \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix},$$

and one checks, for example:

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix} \Leftrightarrow \frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}.$$

For $n > 1$, map $T$ applies the classic Gauss map on the greatest coordinate of $\vec{\alpha}$. The result of the operation is then $a(\vec{\alpha})$, while $\varepsilon(\vec{\alpha})$ indicates the coordinates the operation was performed on.

There are several results concerning the convergence of this algorithm. For $n = 1$, it is known that the sequence of the convergents of $\alpha$ tends to $\alpha$ with an exponential rate. For $n = 2$, the exponential convergence is proven for *almost every* $\vec{\alpha}$ (see [24, 26, 28]). For $n > 2$, [25] provides a general method to prove the convergence almost everywhere, which however has been used only for $n \leq 3$.

### *4.2. S-adicity*

We here combine Theorem 1 and the Brun expansions to write any stepped hyperplane as a continued composition of generalized substitutions. As in Section 3.3, we consider here only characteristic stepped hyperplanes (see Definition 7).

Let $a \in \mathbb{N}^*$ and $\varepsilon \in \{1, \ldots, n\}$. The substitution $\sigma_{a,\varepsilon}$ is defined over $\{1, \ldots, n+1\}$ by:

$$\sigma_{a,\varepsilon} \ : \ \begin{cases} 1 & \mapsto \ 1^a \cdot (\varepsilon+1) \\ (\varepsilon+1) & \mapsto \ 1 \\ i & \mapsto \ i \end{cases}$$

One checks that the incidence matrix of $\sigma_{a,\varepsilon}$ is $A_{a,\varepsilon}$. Moreover $\sigma_{a,\varepsilon}$ is unimodular since one computes $\det(A_{a,\varepsilon}) = -1$. One thus can combine Eq.(1) and Theorem 1 to obtain:

$$\Theta^*_{\sigma_{a(\vec{\alpha}),\varepsilon(\vec{\alpha})}} \left(\mathcal{S}_{{}^t(1,T(\vec{\alpha}))}\right) = \mathcal{S}_{{}^t(1,\vec{\alpha})}.$$

In particular, if $\vec{\alpha}$ has a Brun expansion at least $k$ in length, this yields:

$$\Theta^*_{\sigma_{a_0,\varepsilon_0}} \circ \ldots \circ \Theta^*_{\sigma_{a_k,\varepsilon_k}} \left(\mathcal{S}_{{}^t(1,T^{k+1}(\vec{\alpha}))}\right) = \mathcal{S}_{{}^t(1,\vec{\alpha})}. \tag{2}$$

Note that the number of different substitutions in Eq.(2) is unbounded. Since it is not very suitable for effective computations, let us rewrite this equation using a *finite* number of substitutions. For $\varepsilon = 1 \ldots n$, the substitutions $\tau_\varepsilon$ and $\sigma_\varepsilon$ are defined over $\{1, \ldots, n+1\}$ by:

$$\tau_\varepsilon \ : \ i \mapsto \begin{cases} 1 & \text{if } i = \varepsilon+1, \\ \varepsilon+1 & \text{if } i = 1, \\ i & \text{otherwise} \end{cases} \qquad \text{and} \qquad \sigma_\varepsilon \ : \ i \mapsto \begin{cases} 1 \cdot i & \text{if } i = \varepsilon+1, \\ i & \text{otherwise,} \end{cases}$$

and an induction easily proves:

$$\sigma_{a,\varepsilon} = \sigma_\varepsilon^a \circ \tau_\varepsilon. \tag{3}$$

Moreover, $\sigma_\varepsilon$ and $\tau_\varepsilon$ are also unimodular, so the generalized substitutions $\Theta^*_{\tau_\varepsilon}$ and $\Theta^*_{\sigma_\varepsilon}$ are defined, and one has:

$$\Theta^*_{\sigma_{a,\varepsilon}} = \Theta^*_{\sigma_\varepsilon^a \circ \tau_\varepsilon} = \Theta^*_{\tau_\varepsilon} \circ \left(\Theta^*_{\sigma_\varepsilon}\right)^a.$$

Thus, Eq.(2) can be rewritten using a finite number of non-trivial generalized substitutions as follows:

$$\mathcal{S}_{{}^t(1,\vec{\alpha})} = \Theta^*_{\tau_{\varepsilon_0}} \circ \left(\Theta^*_{\sigma_{\varepsilon_0}}\right)^{a_0} \circ \ldots \circ \Theta^*_{\tau_{\varepsilon_k}} \circ \left(\Theta^*_{\sigma_{\varepsilon_k}}\right)^{a_k} \left(\mathcal{S}_{{}^t(1,T^{k+1}(\vec{\alpha}))}\right). \tag{4}$$

Generalizing the terminology for classic substitutions on sequences (see *e.g.* [20, 21, 32, 42]), Eq.(4) is called an *$S$-adic expansion at order $k$* of the stepped hyperplane $\mathcal{S}_{^t(1,\vec{\alpha})}$. Then, $\mathcal{S}_{^t(1,\vec{\alpha})}$ is said to be *$S$-adic* if it has $S$-adic expansions at any order. This is particularly the case if $\vec{\alpha}$ has irrational coordinates linearly independent over $\mathbb{Q}$, since it then turns out that $\vec{\alpha}$ has an infinite Brun expansion. Note that this case corresponds to multi-dimensional Sturmian sequences (see Definition 4).

**Example 3** *Let us consider the $n = 1$ case. One has:*

$$\tau_1 \ : \ \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \end{cases} \qquad \textit{and} \qquad \sigma_1 \ : \ \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 12 \end{cases}.$$

*One can check Eq.(3), for example, for $a = 3$ and $\varepsilon = 1$:*

$$\sigma_1^3 \circ \tau_1(1) = \sigma_1^3(2) = \sigma_1^2(12) = \sigma(112) = 1112 = \sigma_{3,1}(1),$$

$$\sigma_1^3 \circ \tau_1(2) = \sigma_1^3(1) = 1 = \sigma_{3,1}(2).$$

*Then, the associated generalized substitutions $\Theta^*_{\tau_1}$ and $\Theta^*_{\sigma_1}$ are defined by:*

$$\Theta^*_{\tau_1} \ : \ \begin{cases} (\vec{0}, 1^*) \mapsto \{(\vec{0}, 2^*)\} \\ (\vec{0}, 2^*) \mapsto \{(\vec{0}, 1^*)\} \end{cases} \quad \textit{and} \quad \Theta^*_{\sigma_1} \ : \ \begin{cases} (\vec{0}, 1^*) \mapsto \{(\vec{0}, 1^*), (\vec{0}, 2^*)\} \\ (\vec{0}, 2^*) \mapsto \{(-\vec{e}_1, 2^*)\} \end{cases}.$$

*Figure 11 illustrates some applications of these basic substitutions,* i.e. *the beginning of an expansion such as the one of Eq.(4).*

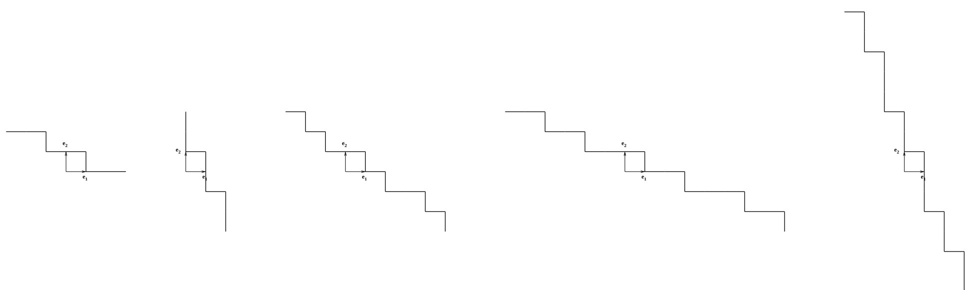

Figure 11: Starting from a finite set of faces of a stepped hyperplane, the generalized substitutions $\Theta^*_{\tau_1}$, $\Theta^*_{\sigma_1}$, $\Theta^*_{\sigma_1}$ and $\Theta^*_{\tau_1}$ are successively applied (from left to right). All of these sets of faces belong to stepped hyperplanes.

**Example 4** *Let us consider the case $n = 2$. One has:*

$$\tau_1 \ : \ \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{cases}, \quad \tau_2 \ : \ \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 2 \\ 3 \mapsto 1 \end{cases}, \quad \sigma_1 \ : \ \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 12 \\ 3 \mapsto 3 \end{cases}, \quad \sigma_2 \ : \ \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 13 \end{cases}.$$

*Then, the associated generalized substitutions $\Theta^*_{\tau_1}$, $\Theta^*_{\tau_2}$, $\Theta^*_{\sigma_1}$ and $\Theta^*_{\sigma_2}$ are defined by:*

$$\Theta^*_{\tau_1} \ : \ \begin{cases} (\vec{0}, 1^*) \mapsto \{(\vec{0}, 2^*)\} \\ (\vec{0}, 2^*) \mapsto \{(\vec{0}, 1^*)\} \\ (\vec{0}, 3^*) \mapsto \{(\vec{0}, 3^*)\} \end{cases}, \quad \Theta^*_{\tau_2} \ : \ \begin{cases} (\vec{0}, 1^*) \mapsto \{(\vec{0}, 3^*)\} \\ (\vec{0}, 2^*) \mapsto \{(\vec{0}, 2^*)\} \\ (\vec{0}, 3^*) \mapsto \{(\vec{0}, 1^*)\} \end{cases},$$

$$\Theta^*_{\sigma_1} : \begin{cases} (\vec{0},1^*) \mapsto \{(\vec{0},1^*),(\vec{0},2^*)\} \\ (\vec{0},2^*) \mapsto \{(-\vec{e}_1,2^*)\} \\ (\vec{0},3^*) \mapsto \{(\vec{0},3^*)\} \end{cases}, \quad \Theta^*_{\sigma_2} : \begin{cases} (\vec{0},1^*) \mapsto \{(\vec{0},1^*),(\vec{0},3^*)\} \\ (\vec{0},2^*) \mapsto \{(\vec{0},2^*)\} \\ (\vec{0},3^*) \mapsto \{(-\vec{e}_1,3^*)\} \end{cases}.$$

*Figure 11 illustrates some applications of these basic substitutions,* i.e. *the beginning of an expansion such as the one of Eq.(4).*

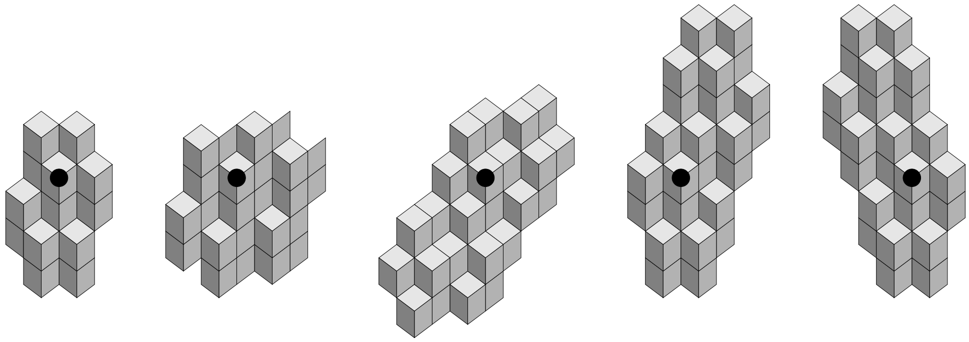

Figure 12: Starting from a finite set of faces of a stepped hyperplane, the generalized substitutions $\Theta^*_{\sigma_1}$, $\Theta^*_{\sigma_2}$, $\Theta^*_{\tau_2}$ and $\Theta^*_{\tau_1}$ are successively applied (from left to right). All of these sets of faces belong to stepped hyperplanes. Note that these four steps are a decomposition of the last step of Fig.9.

*4.3. Substitutivity*

In this section, we are interested in characterizations of the following stepped hyperplanes:

**Definition 9** *A stepped hyperplane is said to be* invariant *if it is invariant under a non-trivial generalized substitution. It is said to be* substitutive *if it is the image by a generalized substitution of an invariant stepped hyperplane.*

This generalizes the notions of invariant and substitutive sequences in the one-dimensional case. Such characterizations are particularly interesting since, according to Theorem 1, one can *effectively* generate stepped hyperplanes which are invariant under a *Pisot* generalized substitution or image of such invariant stepped hyperplanes.

In order to characterize substitutive characteristic stepped hyperplanes, let us state the following propostion:

**Proposition 3** *Let $\vec{\alpha} \in \mathbb{R}^n_+$. If the characteristic stepped hyperplane $\mathcal{S}_{^t(1,\vec{\alpha})}$ is substitutive, then the coordinates of $\vec{\alpha}$ belong to a field extension $\mathbb{Q}(\lambda)$, with $\lambda$ being an algebraic number of degree $n+1$ at most.*

**Proof.** Suppose that $\mathcal{S}_{^t(1,\vec{\alpha})}$ is the image by $\Theta^*_{\sigma'}$ of a stepped hyperplane $\mathcal{S}_{^t(1,\vec{\beta})}$ which is invariant under $\Theta^*_\sigma$. Theorem 1 implies that ${}^tM_\sigma^{-1}{}^t(1,\vec{\beta}) = \lambda^t(1,\vec{\beta})$, with $\lambda$ being an eigenvalue of $M_\sigma^{-1}$. Since $M_\sigma^{-1}$ is a matrix of size $n+1$ with integer coefficients, $\lambda$ is an algebraic number of degree $n+1$ at most and the coordinates of $\vec{\beta}$ thus belong to $\mathbb{Q}(\lambda)$. But Theorem 1 also yields ${}^tM_{\sigma'}{}^t(1,\vec{\beta}) = {}^t(1,\vec{\alpha})$, so the coordinates of $\vec{\alpha}$ also belong to $\mathbb{Q}(\lambda)$. $\square$

Let us also recall this theorem put forward by Lagrange:

**Theorem 3 (Lagrange)** *The continued fraction expansion of $\alpha \in \mathbb{R}$ is ultimately periodic if and only if $\alpha$ belong to a field extension $\mathbb{Q}(\lambda)$, $\lambda$ being an algebraic number of degree 2 at most.*

We are now in a position to prove the following theorem:

**Theorem 4** *If $\vec{\alpha} \in [0,1)^n$ has an ultimately periodic Brun expansion, then the characteristic stepped hyperplane $\mathcal{S}_{{}^t(1,\vec{\alpha})}$ is substitutive. Moreover, the converse holds for $n = 1$.*

**Proof.** The sufficient condition results from Eq.(2) or Eq.(4). For $n = 1$, the converse follows from Proposition 3 and Theorem 3 (recall that the Brun expansions and continued fraction expansions are identical for $n = 1$). $\square$

Note that Theorem 4 yields a complete characterization only for $n = 1$. This case corresponds to Sturmian sequences (the stepped hyperplanes are "stepped lines"), and we get an already known result (see Introduction). For $n > 1$, a multi-dimensional extension of Theorem 3 would similarly yield a complete characterization. However, such an extension is thought to be a very hard problem.

To get around this problem, we would like to prove, without relying on Proposition 3, that substitutivity implies the periodicity of Brun expansions. A promising way seems to be the following characterization which relies on *derived sequences* (roughly speaking, a derived sequence encodes the way the prefix of a sequence reoccurs in this sequence):

**Theorem 5 ([19])** *A sequence is substitutive if and only if it has a finite number of derived sequences.*

Of course, we would here need a multi-dimensional generalization of this characterization. It seems, however, easier than generalizing the theorem of Langrange (Theorem 3). In this direction, one can mention [31, 37], where *derived Voronoï tilings* - a sort of generalization of derived sequences - are defined. In particular, the *pseudo-self-similar tilings* are characterized in a way which looks like a multi-dimensional extension of Theorem 5 (the reader is referred to [31, 36, 37, 38] for more details). In our case, we would need to link derived Voronoï tilings (or some adaptation to multi-dimensional sequences) with Brun expansions. In this direction, it is worth noticing that links between derived sequences and classic continued fractions have already been studied in [2, 10, 40].


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