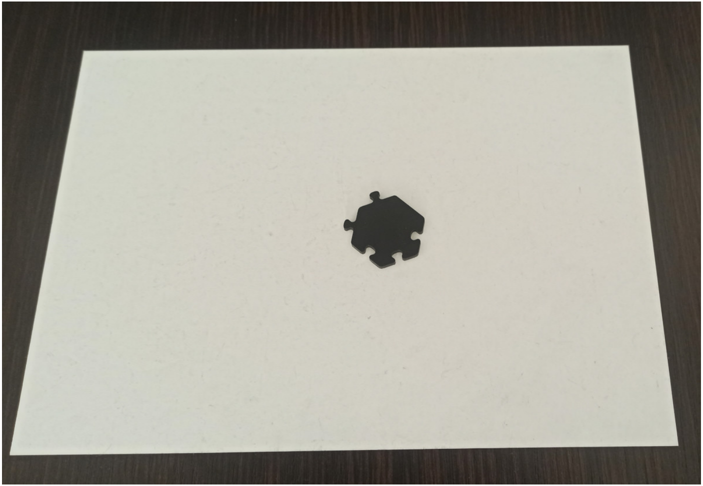
# Heesch Numbers & the Einstein Problem

Thomas Fernique

# Puzzle challenge (room S937)



Can an entire A4 sheet be covered by tiles all identical to this one?

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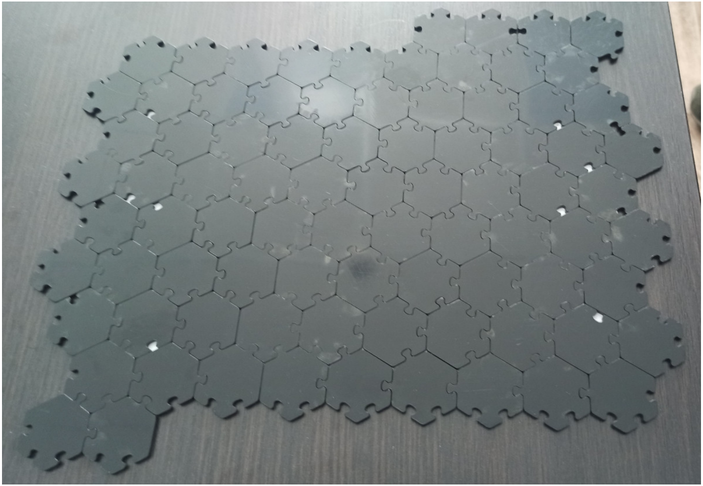

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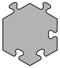
# Heesch numbers (1968)



Tile:
A topological disk $T$ (e.g., a polygon).

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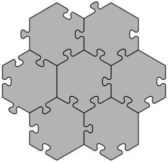

Patch:
A simply connected union of interior-disjoint isometric copies of $T$.

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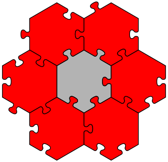

Corona $C$ of a patch $P$:
Tiles s.t. $P \cup C$ is a patch with no point of $P$ on its boundary.

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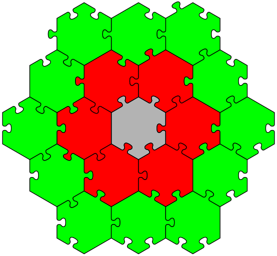

Heesch number of $T$:
Maximum number of coronas that can be successively added to $T$.

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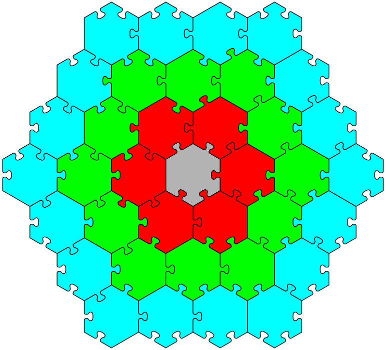

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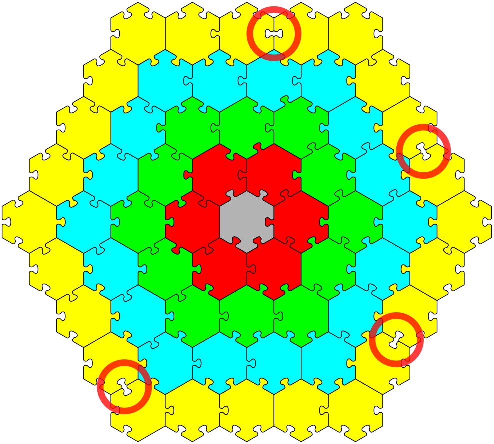

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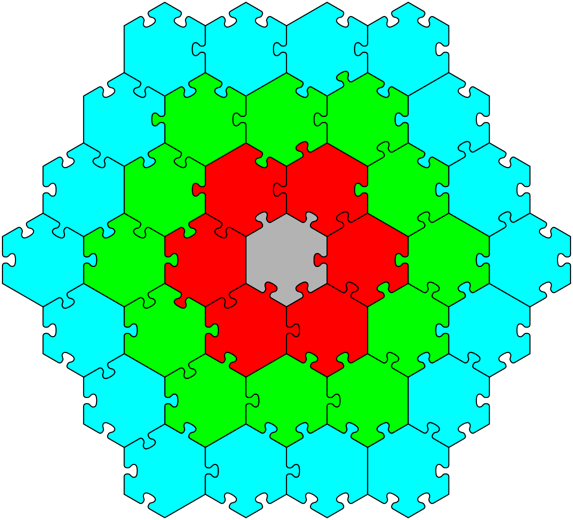
# Records



The previous tile has Heesch number 3 (Robert Ammann, 1990s).

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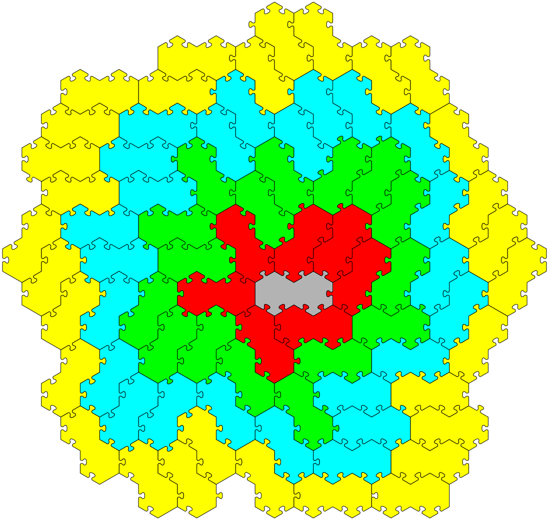

This generalization has Heesch number 4 (Casey Mann, 2001).

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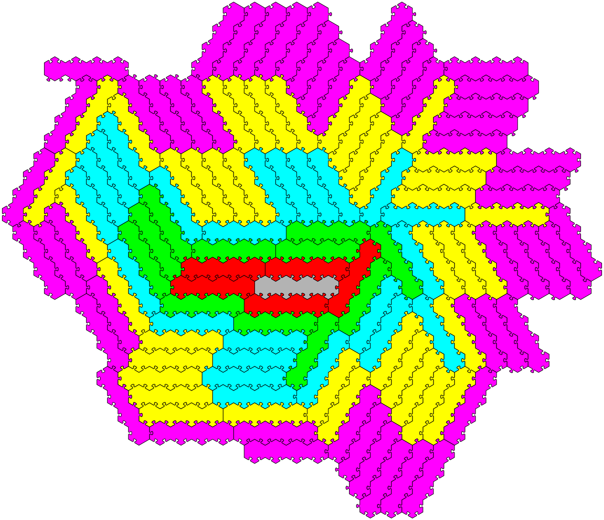

It can be continued up to Heesch number 5 (Casey Mann, 2001).

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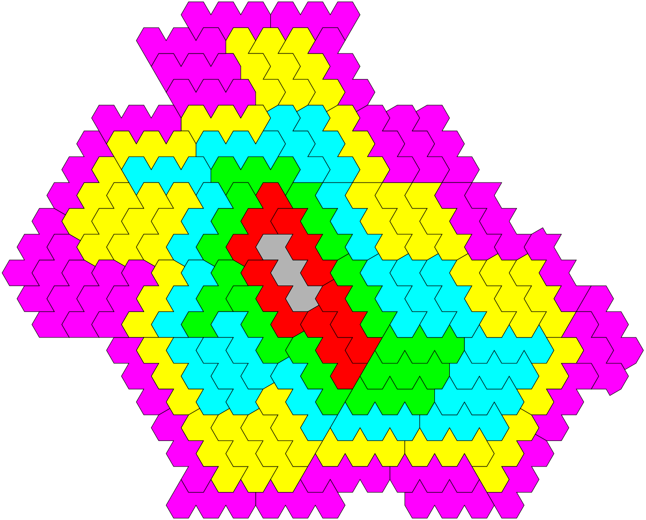

Another similar idea yields Heesch number 5 (David Smith, 2019).

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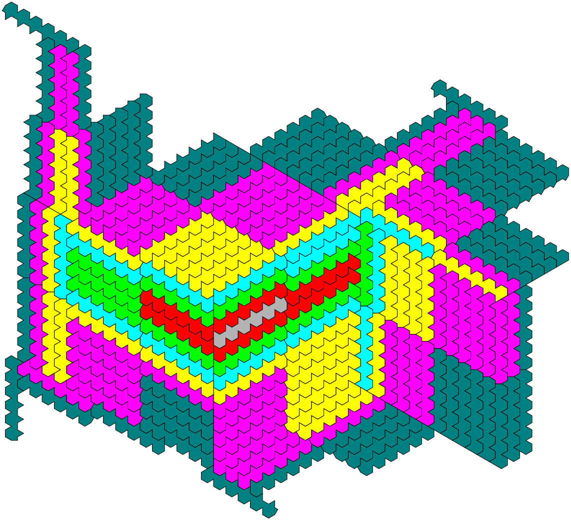
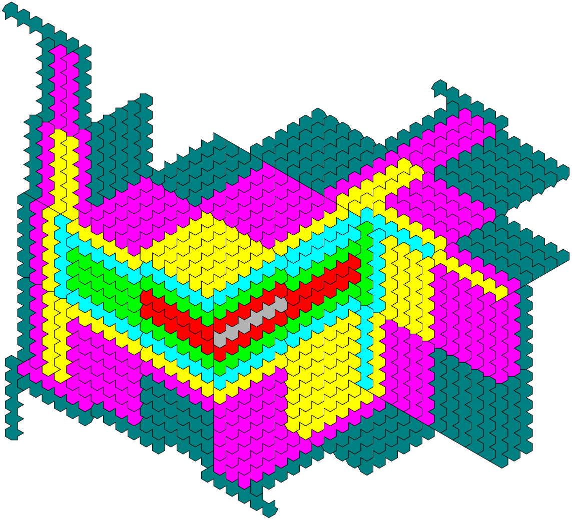


It has be continued to yield Heesch number 6 (Bojan Bašić, 2021)

# Records



This is the current record. . . among tiles with finite Heesch number!

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No such bound (yet) known. Personal guess: there are none.

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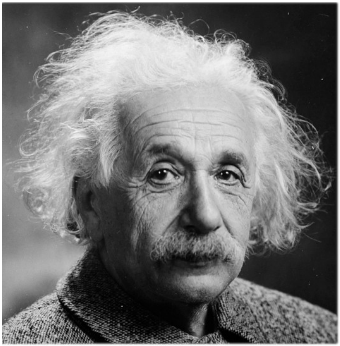
Theorem (David Smith *et al.*, 2023)
*Such a tile actually does exist!*

# Interlude



Einstein $\neq$ ein Stein

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But Heinrich Heesch defined the Heesch number.

(He also "almost" proved the 4-color theorem)

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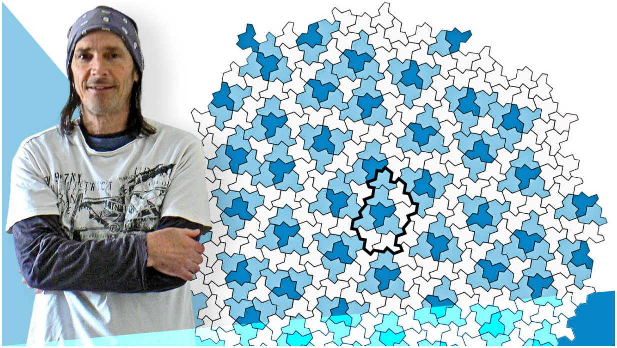

And David Smith is an english retired print technician!

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This tile was discovered while searching for large Heesch numbers.

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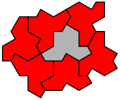

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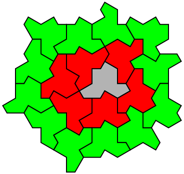

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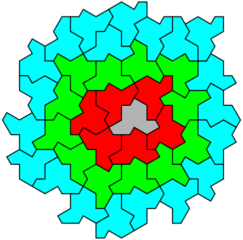

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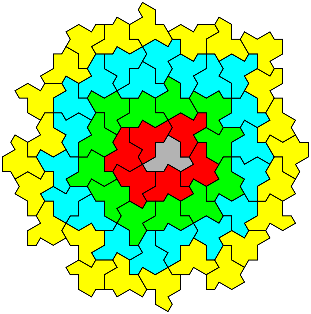

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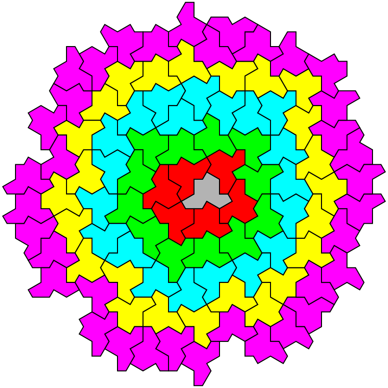

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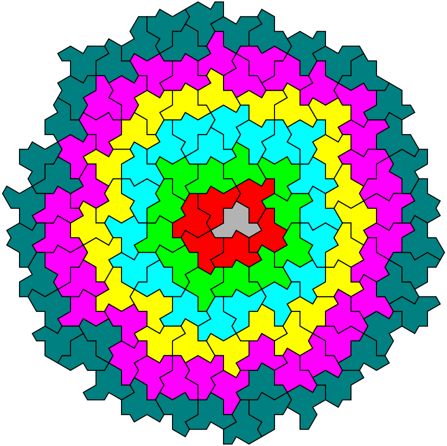

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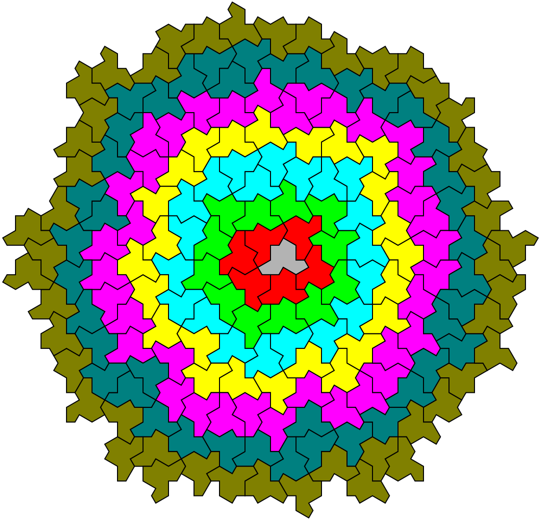

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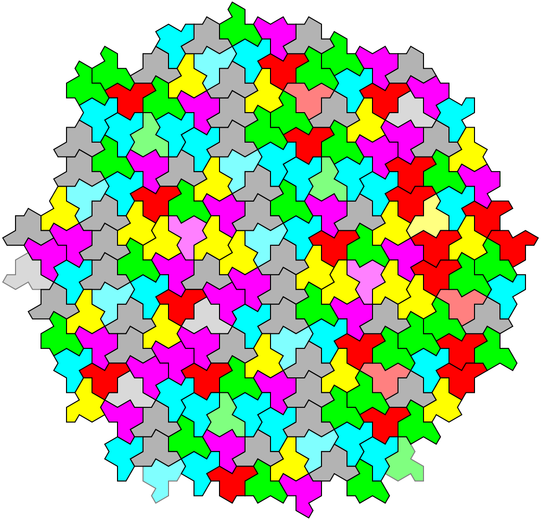

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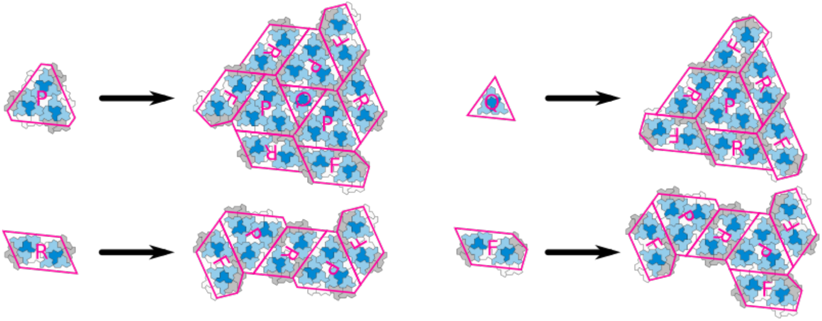
# Proof (Smith, Myers, Kaplan, Goodman-Strauss)



Sketch:

- Show that every tile can be replaced by a combinatorially equivalent patch: iterating yields a hierarchical tiling;
- conversely, prove that tiles can always be grouped into such patches: only such hierarchical tilings are possible.

Scale-invariance ensures non-periodicity.

# Conclusion

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Generalized Heesch problem:
Decide whether $k$ given tiles can cover the entire plane.

NO for $k \geq 5$
(Nicolas Ollinger, 2009)