Digital Surfaces in High Codimensions: When Periodicity Enforces Aperiodicity

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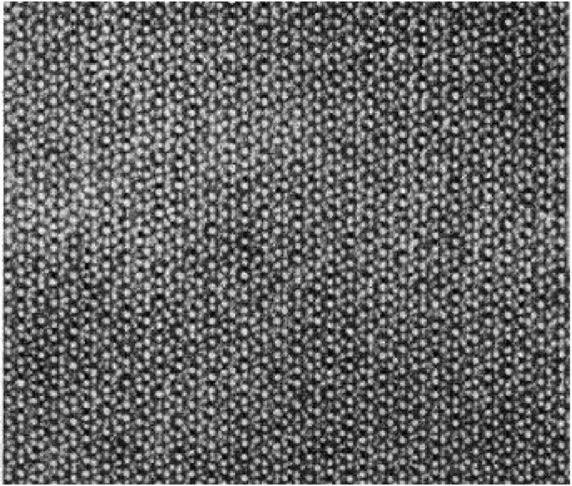
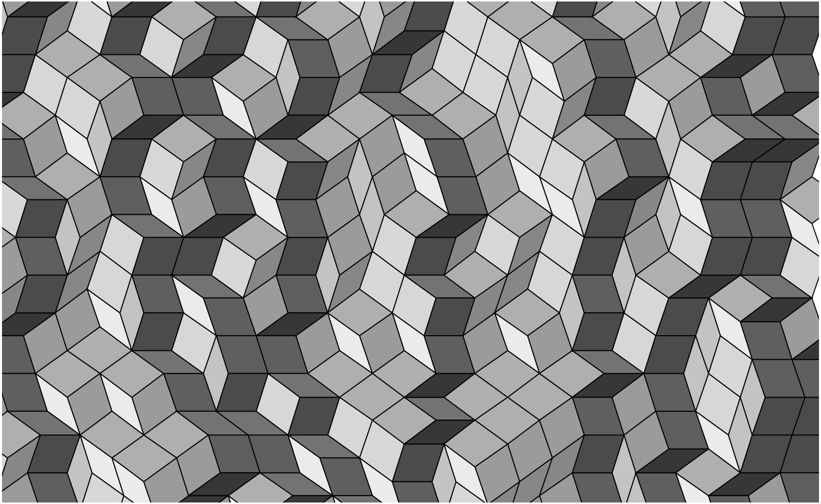
Quasicrystal



Matter with *non-periodic order* enforced by *finite range* interactions.

Rhombus tiling (Digital surface)



Tiling of the plane by $\binom{n}{2}$ rhombi defined over n non-colinear edges.

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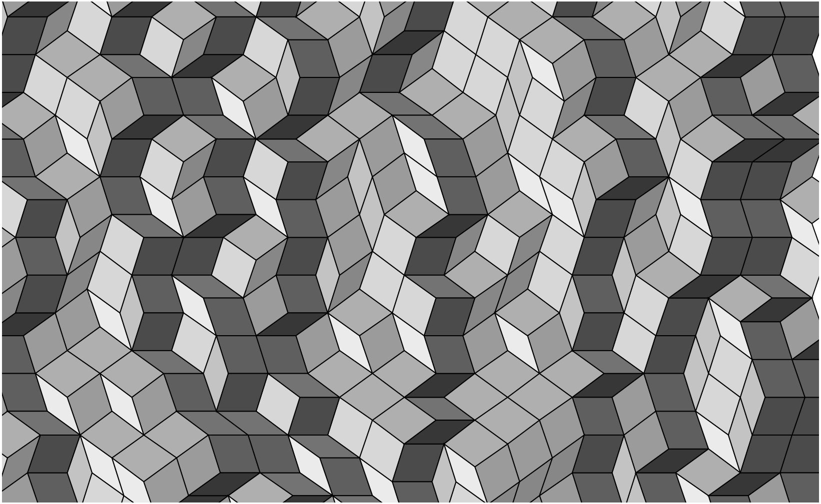
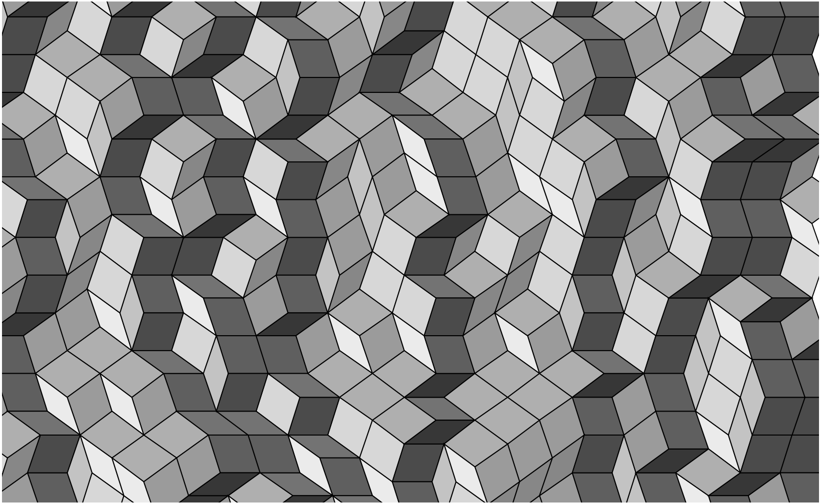


Lift: homeomorphism which maps rhombi on 2-faces of unit n-cubes.

Shadow



Orthogonal projection of the lift over 3 vectors of the canonical basis.

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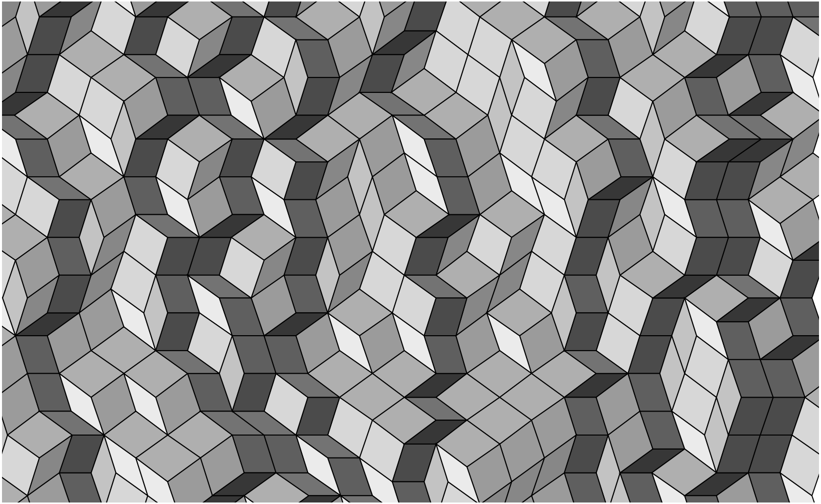

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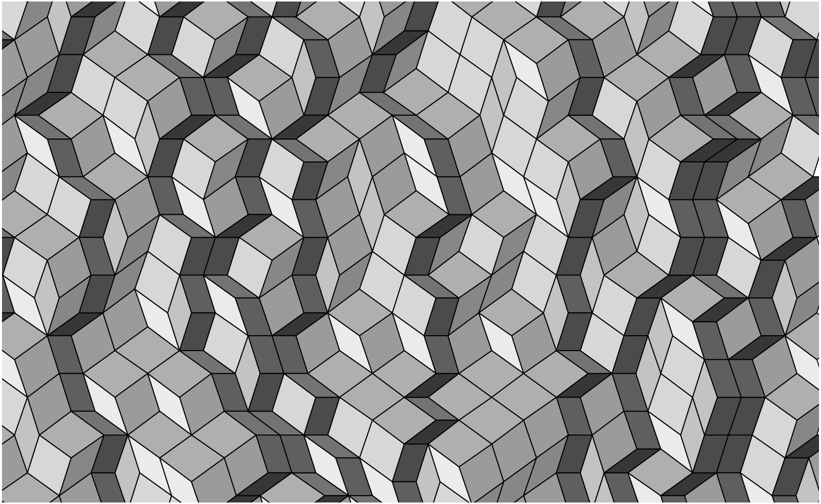

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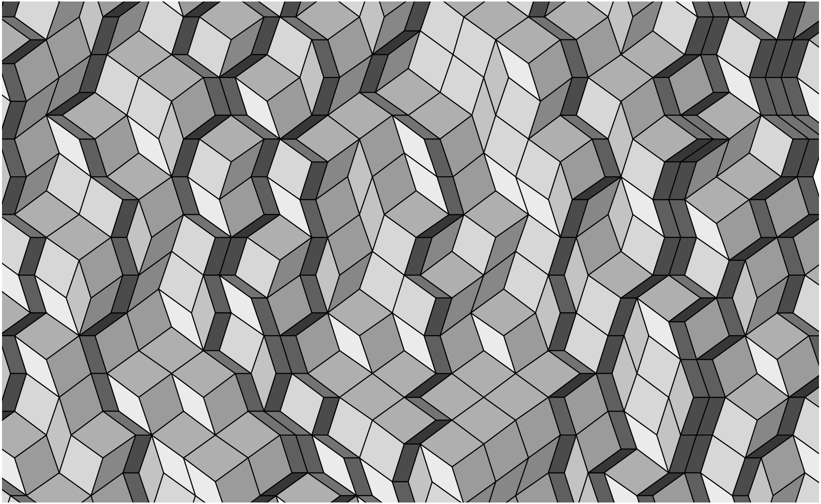

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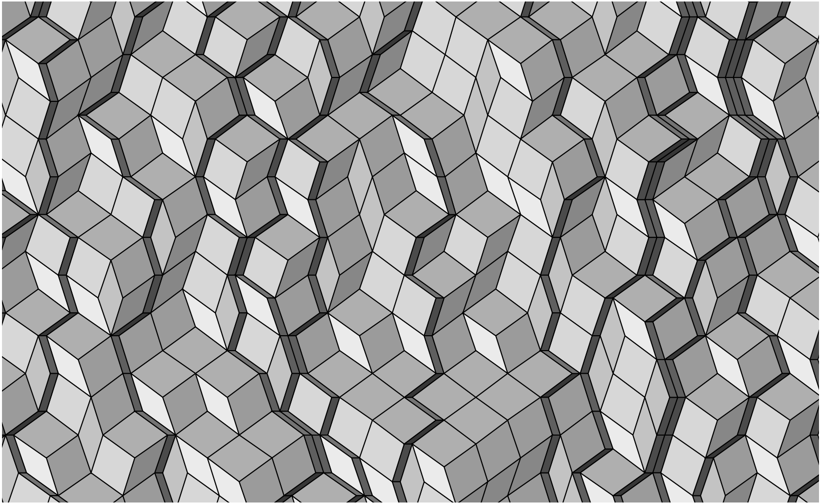

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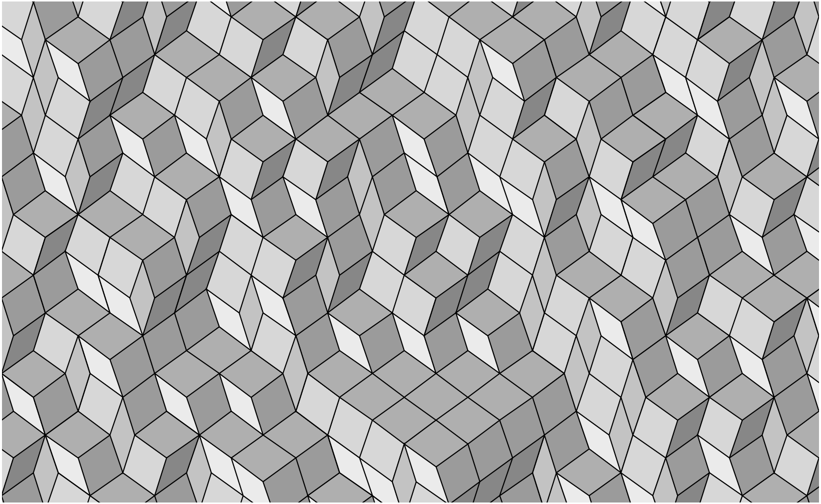

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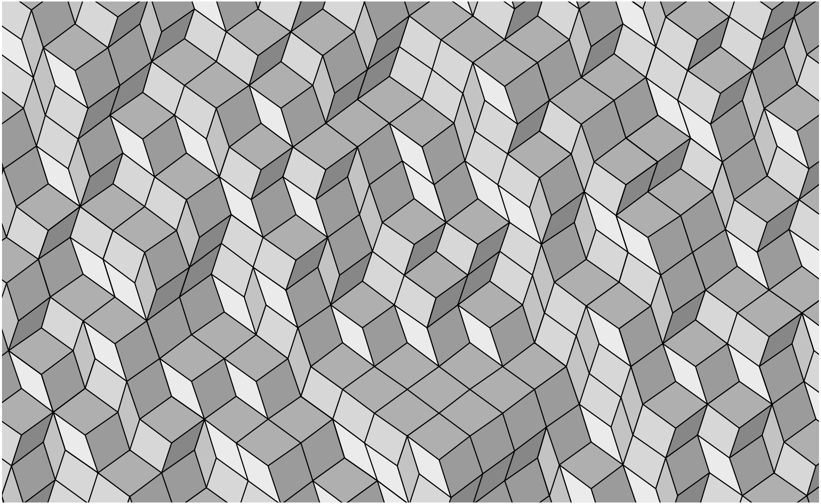

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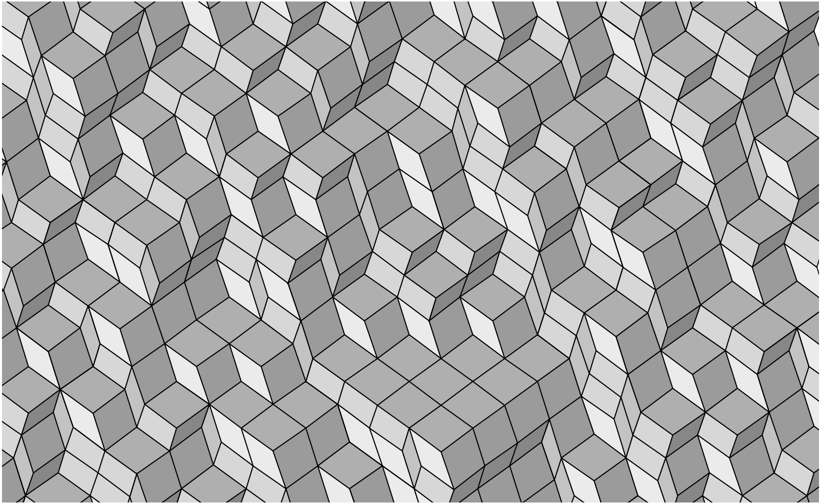

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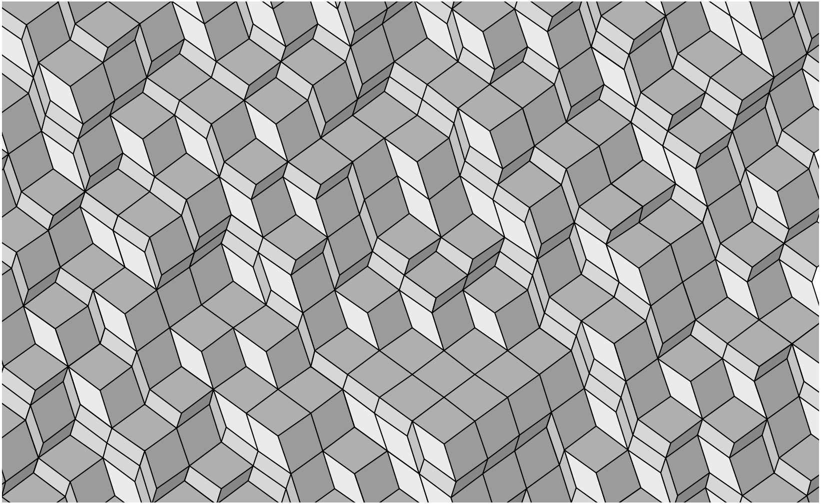

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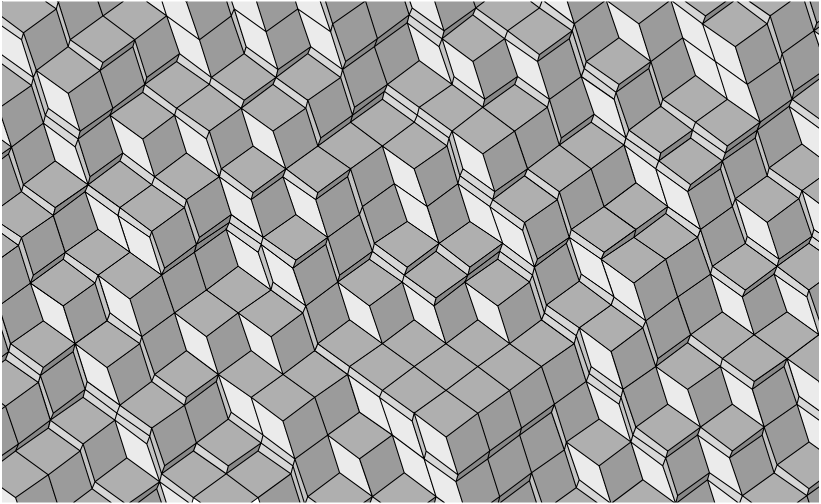

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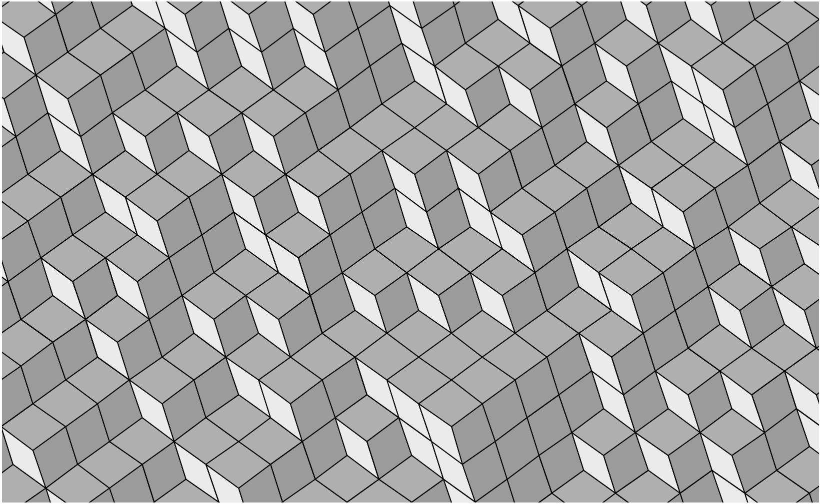

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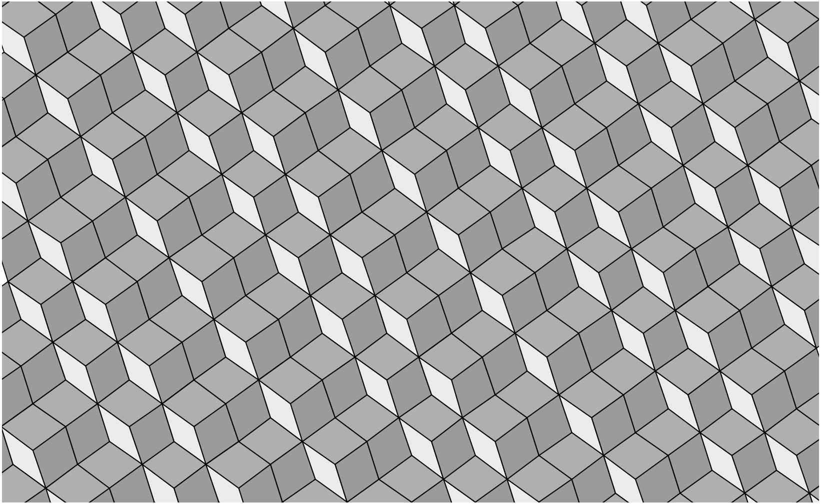
Planar rhombus tiling (Digital plane)



Its lift lies in $P + [0, t)^n$, where P is a plane and $t \geq 1$ the *thickness*.

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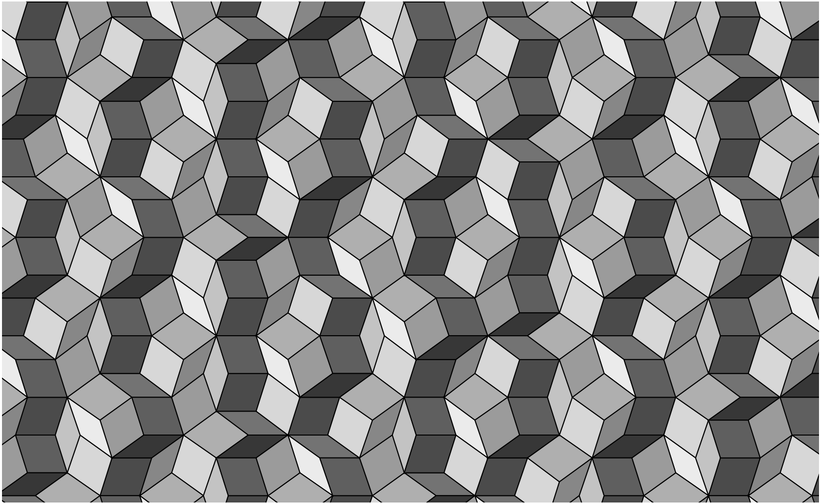

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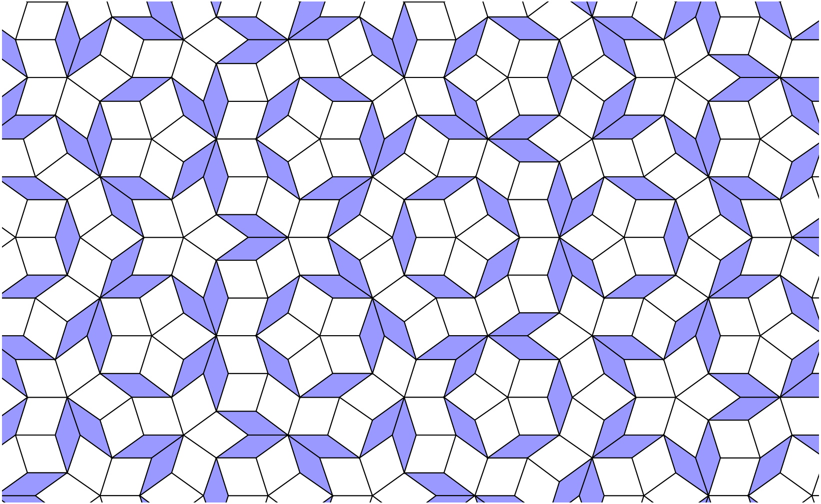

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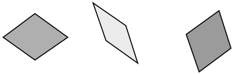
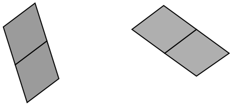
Local rules

Tiles





Forbidden patterns

Finite set of finite forbidden patterns. Enforcing non-periodic tilings?

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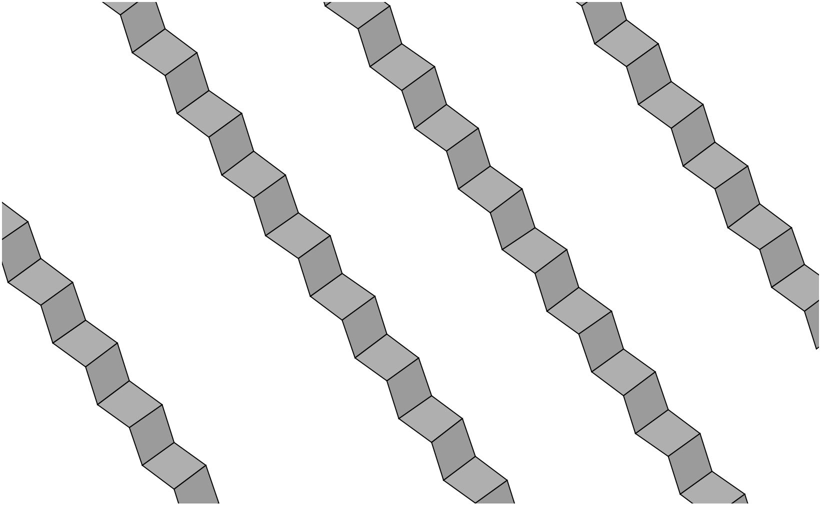

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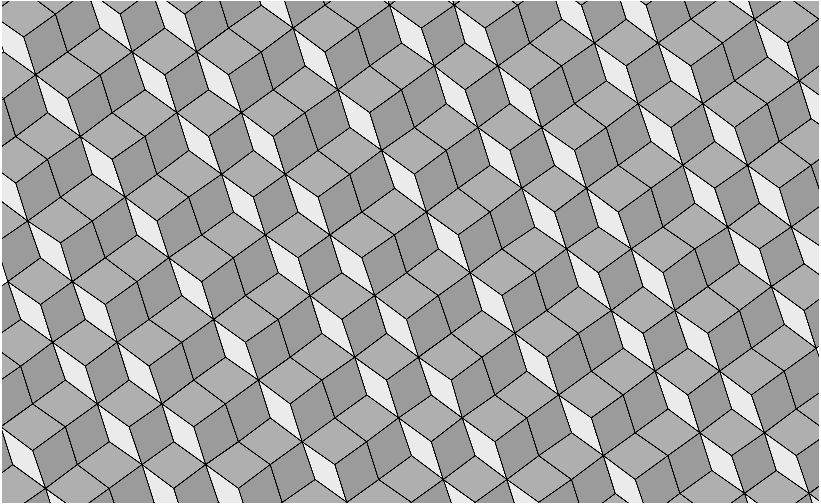

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Only a $\vec{p}$-periodic shadow can satisfy such local rules.

There are however local rules on shadows, not on tilings.
But they can be easily extended to tilings which are *planar*:

Proposition (BF)

Periodicities of planar tiling shadows are enforceable by local rules.

Grassmann coordinates

Grassmann coordinates $(G_{ij})_{i<j} \in \mathbb{RP}^{\binom{n}{2}-1}$ of $P = \mathbb{R}\vec{u} + \mathbb{R}\vec{v} \subset \mathbb{R}^n$

$$G_{ij} := u_i v_j - u_j v_i.$$

Theorem (Good ol' algebraic geometry)

Planes are uniquely characterized by their Grassmann coordinates.

Grassmann coordinates & Plücker relations

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Plücker relations on $(G_{ij})_{i<j} \in \mathbb{RP}^{\binom{n}{2}-1}$: the $\binom{n}{4}$ quadratic relations

$$G_{ij}G_{kl} = G_{ik}G_{jl} - G_{il}G_{jk}$$

Theorem (Good ol' algebraic geometry)

Only Grassmann coordinates satisfy all the Plücker relations.

Periodicity relations

Proposition (BF)

If the (i, j, k)-shadow of a planar tiling is (p, q, r)-periodic, then the Grassmann coordinates of the digitalized plane satisfy

$$pG_{jk} - qG_{ik} + rG_{ij} = 0$$

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Can this be achieved? What about non-planar tilings?

Example for $n = 5$ (generalized Penrose tilings)

Enforce by local rules the 10 periodicity relations $G_{i,j} = G_{2i-j,i}$, *i.e.*,

$$\begin{aligned} G_{12} = G_{51} = G_{45} = G_{34} = G_{23} &\quad =: \quad x, \\ G_{13} = G_{41} = G_{23} = G_{52} = G_{35} &\quad =: \quad y. \end{aligned}$$

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They are the Grassmann coordinates of $\mathbb{R}\cos(\frac{2i\pi}{5})_i + \mathbb{R}\sin(\frac{2i\pi}{5})_i$.

Generalized Levitov theorem

Theorem (BF)

If Plücker and periodicity relations characterize a unique plane, then the tilings satisfying the associated local rules are planar.

The thickness is moreover uniformly (but not precisely) bounded.

The plane has algebraic Grassmann coordinate of degree $d \leq 2^{\binom{n}{4}}$.

Tedious proof, but much easier analogue “continuous” statement.

Thank you for your attention