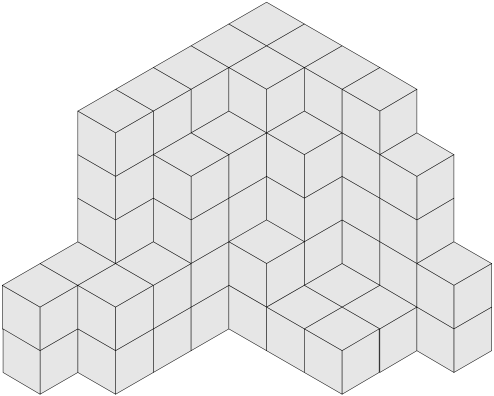
Lift & Flip to Sample Tilings

Thomas Fernique & Olga Sizova

Outline

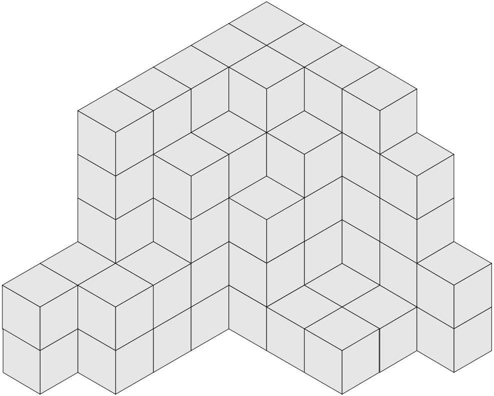
Outline

Lozenge tilings & flips



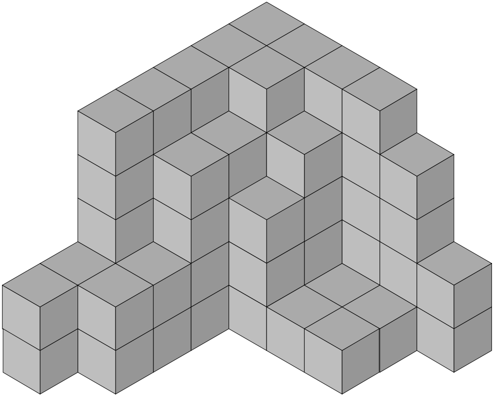
Tiling of a simply connected region by lozenges with a 30° angle.
Flip: rotation of three tiles sharing a vertex.

Lozenge tilings & flips



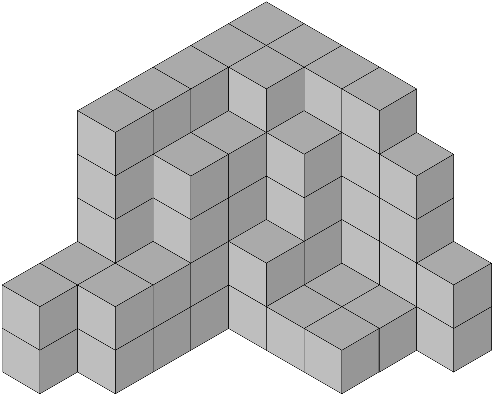
Tiling of a simply connected region by lozenges with a 30° angle.
Flip: rotation of three tiles sharing a vertex.

Lift & Flip-connexity



Lift: see tiles as 2D facets of $\mathbb{Z}^3$.
Flip: adding/removing a cube (in $\mathbb{R}^3$).

Lift & Flip-connexity



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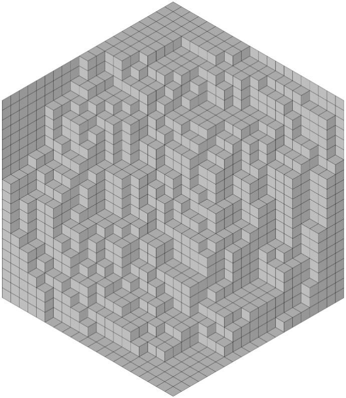
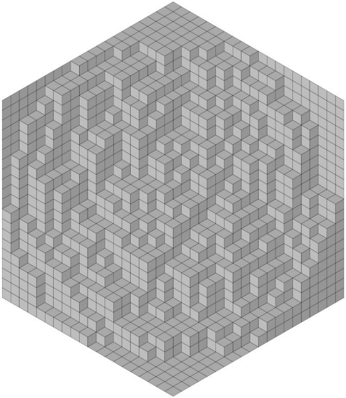
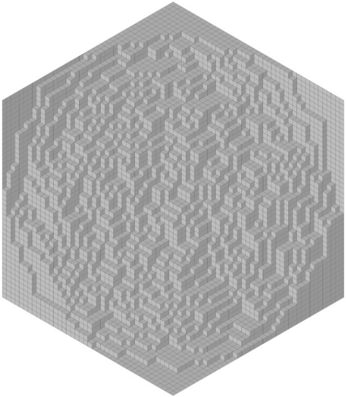
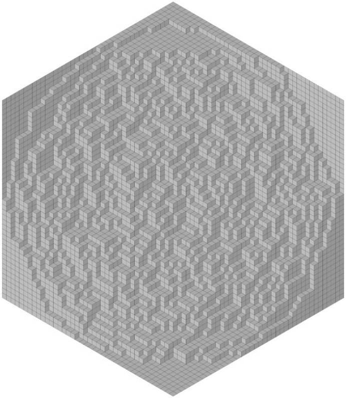
What for?





Random sampling (mixing time issues) to guess “typical shapes”.

What for?

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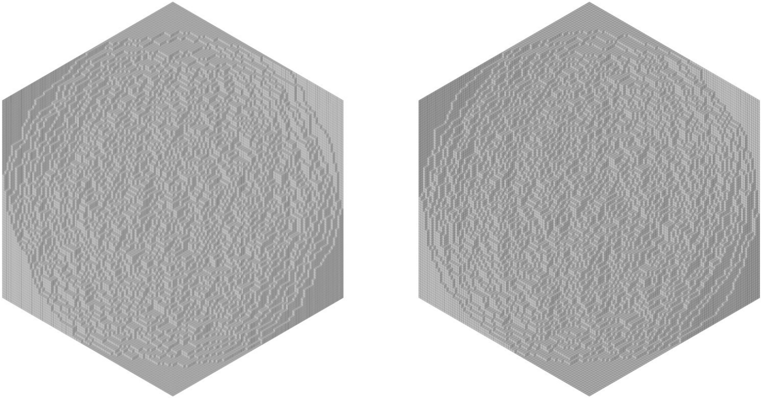

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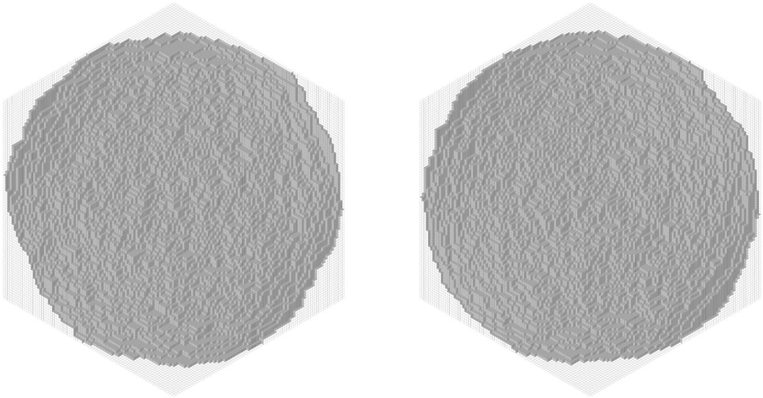
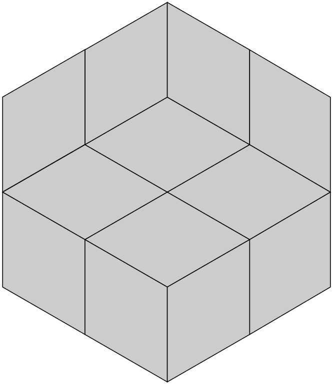


Random sampling (mixing time issues) to guess “typical shapes”.

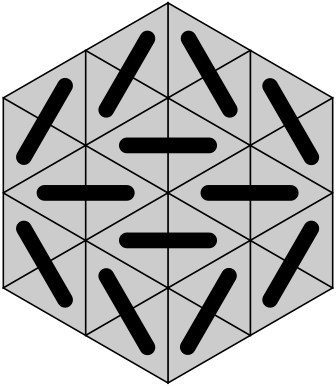
Outline

On the triangular grid



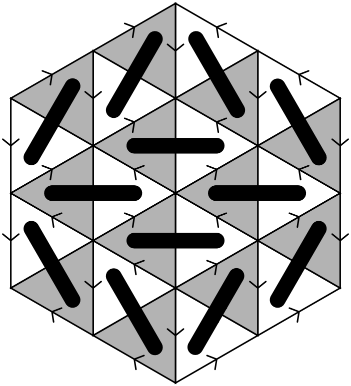
Consider again a lozenge tiling.

On the triangular grid



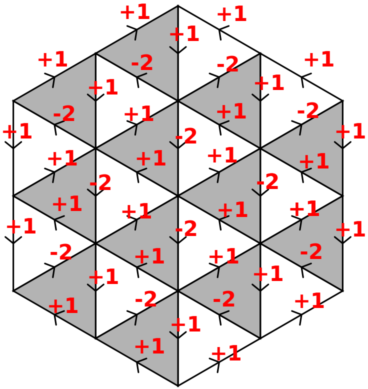
It can be seen as a perfect matching on the triangular grid.

Height function (Thurston'89)



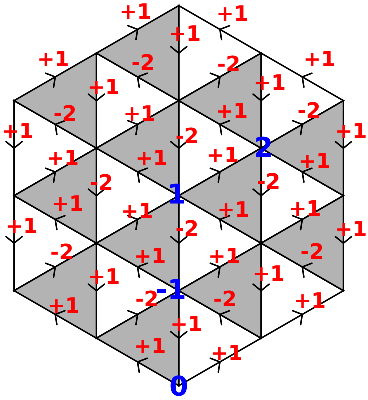
Color triangles in black & white. Orient black ones clockwise.

Height function (Thurston'89)



Give weight -2 to edges which cut a tile, $+1$ to all the other ones.

Height function (Thurston'89)



Give height 0 to a vertex v_0. Height of v: weight of a path $v_0 \rightsquigarrow v$.

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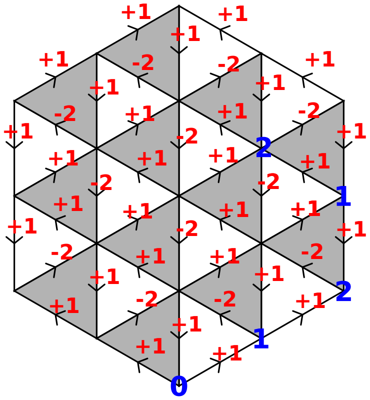

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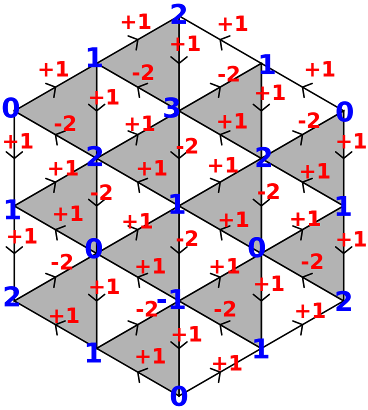

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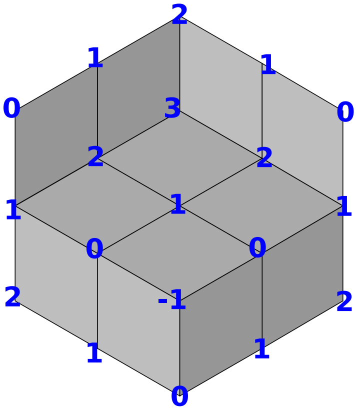
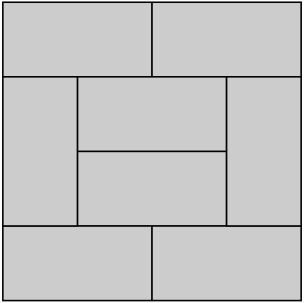


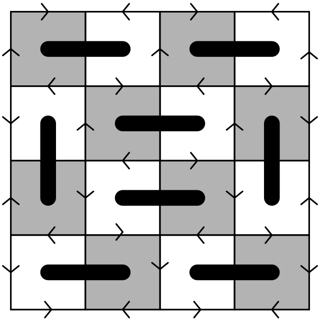
Height: scalar product of the lift with the cube diagonal.

On the square grid



This easily extends to domino tilings on the square grid!

On the square grid



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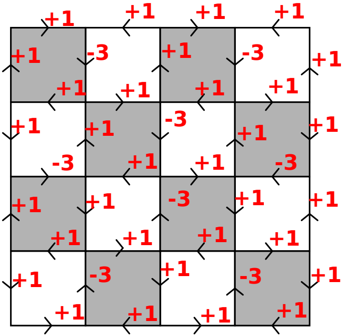

This easily extends to domino tilings on the square grid!

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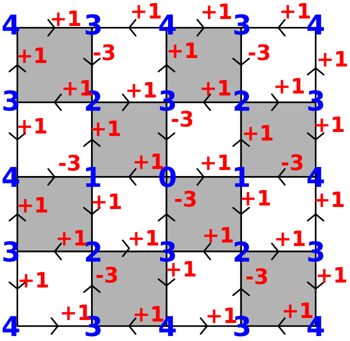

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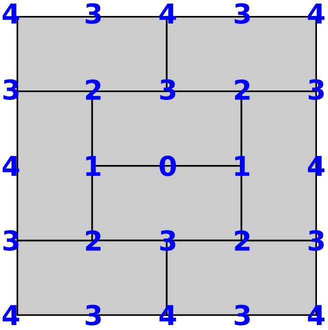
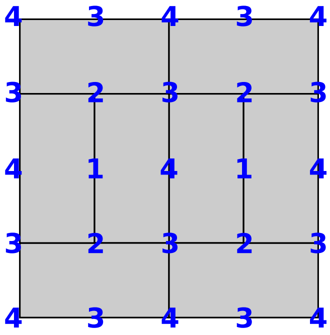


Lift: not that easy to visualize. . .

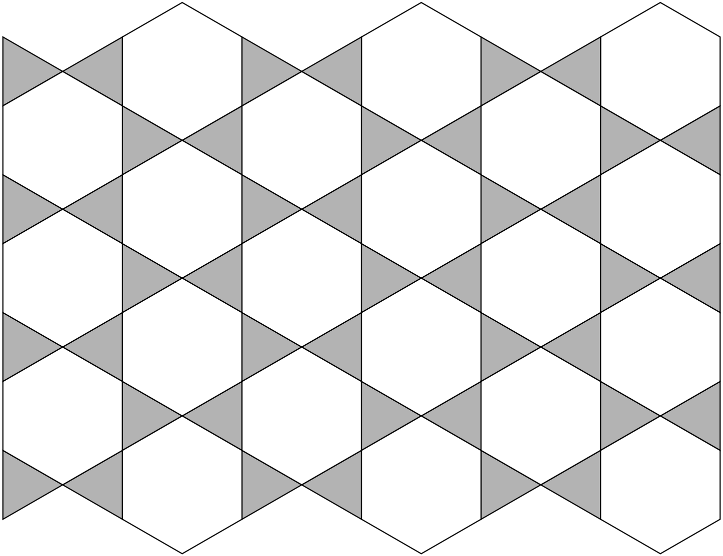
On the square grid



Lift: not that easy to visualize...
Flip: adding/removing a sort of bumpy square...

Outline

On the Kagome grid (Bodini'06)



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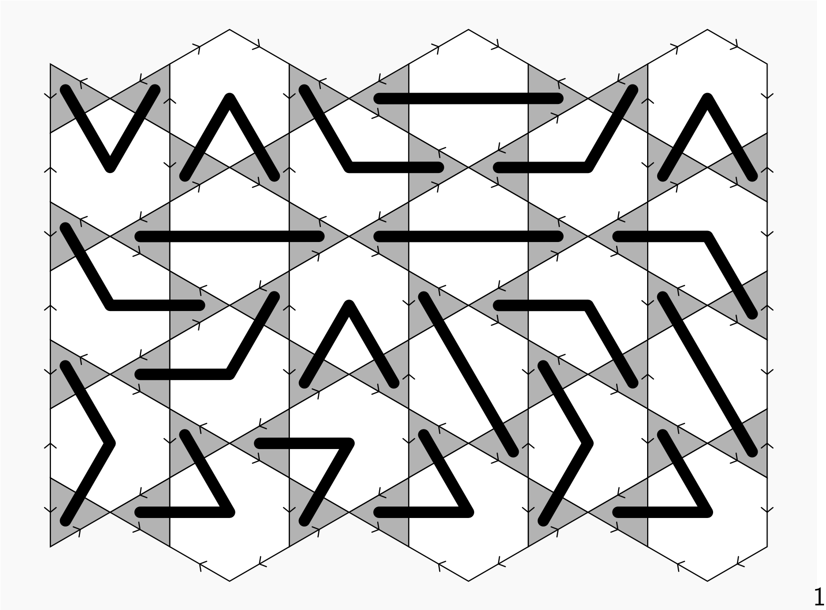

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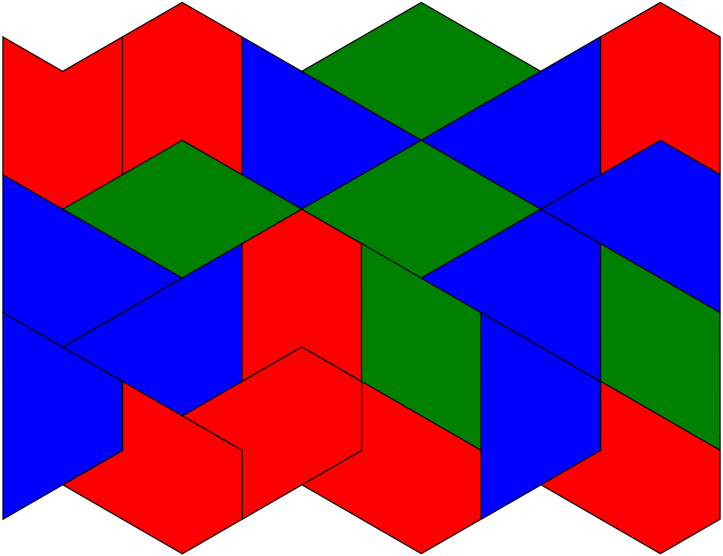

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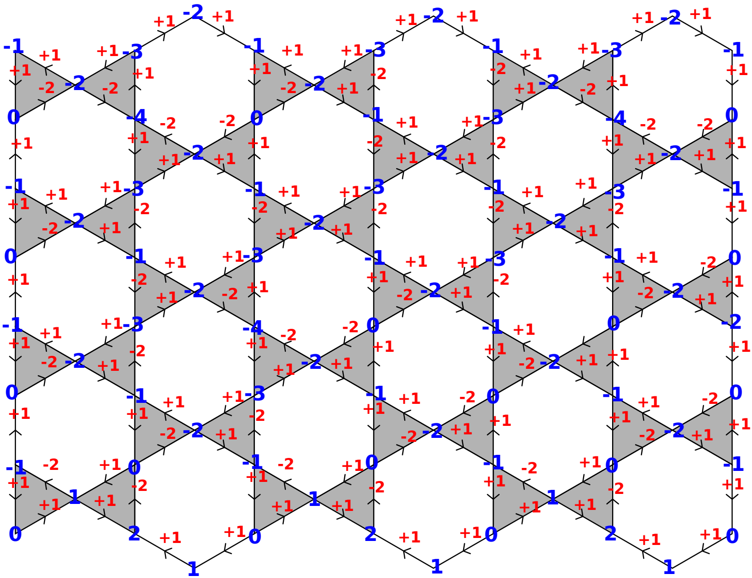

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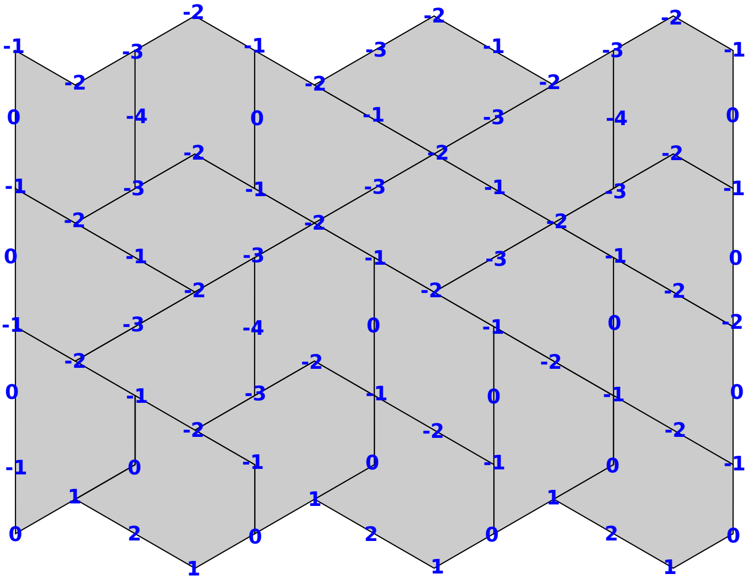

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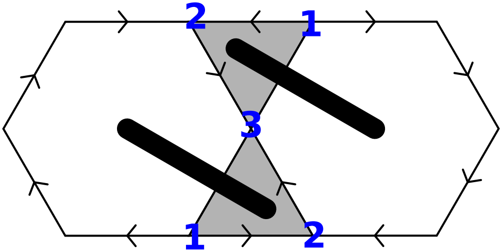

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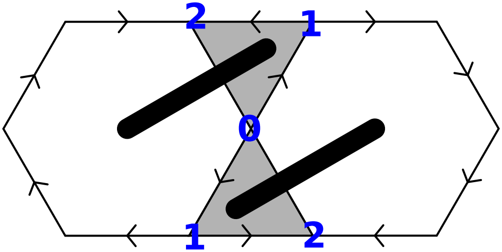

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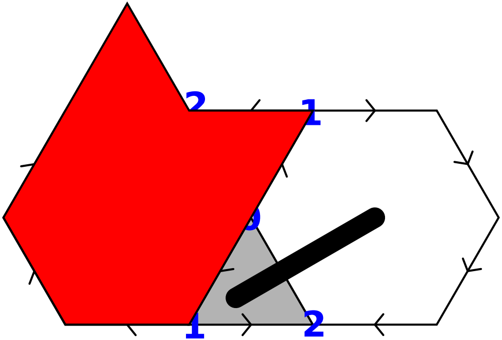

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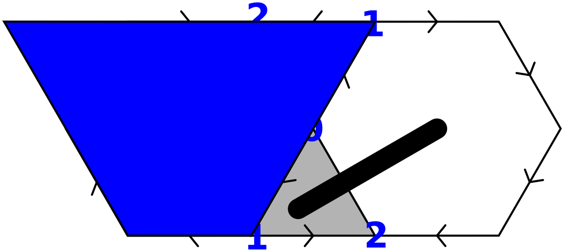

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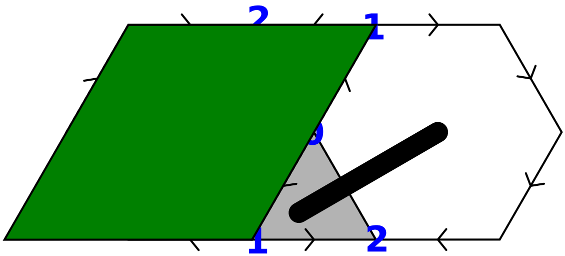

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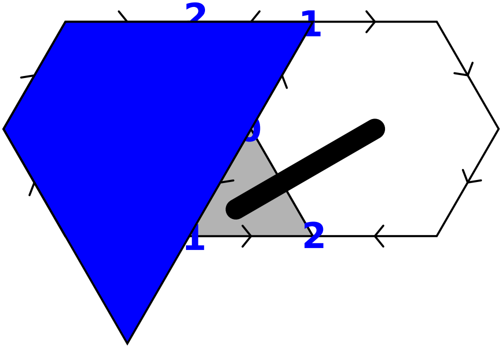

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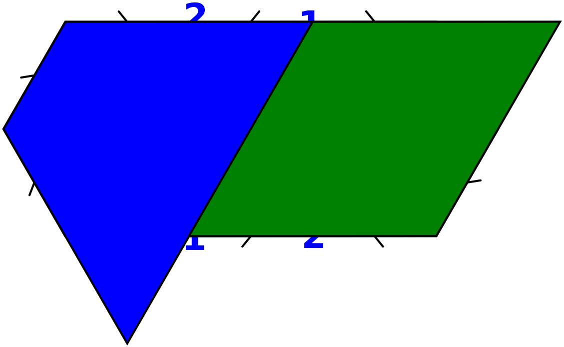

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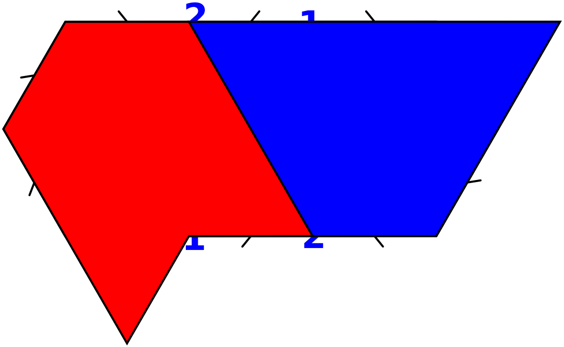

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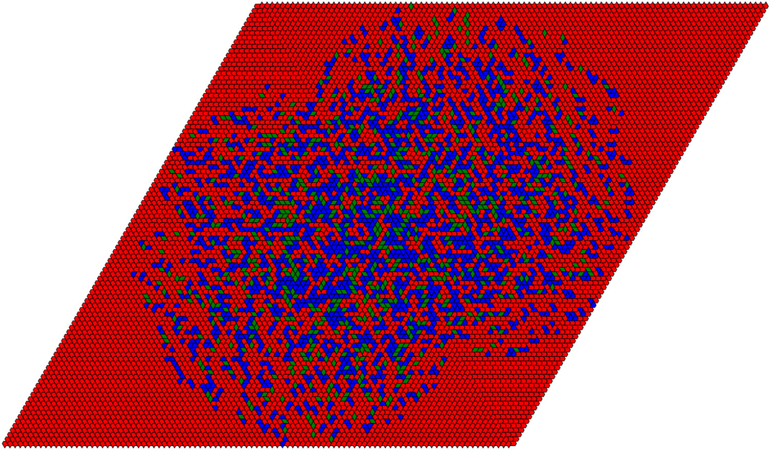
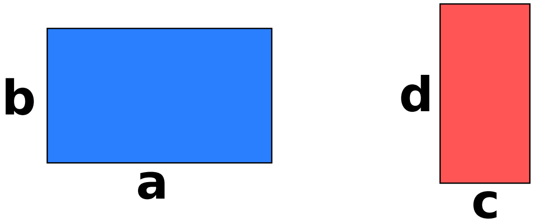
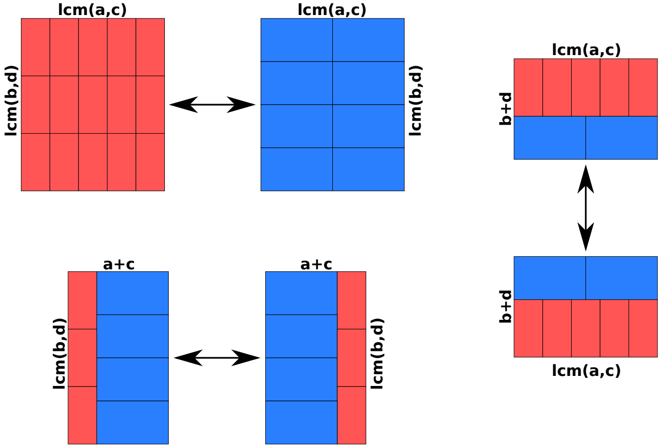


Two rectangles (Kenyon-Kenyon'92, Rémila'04)



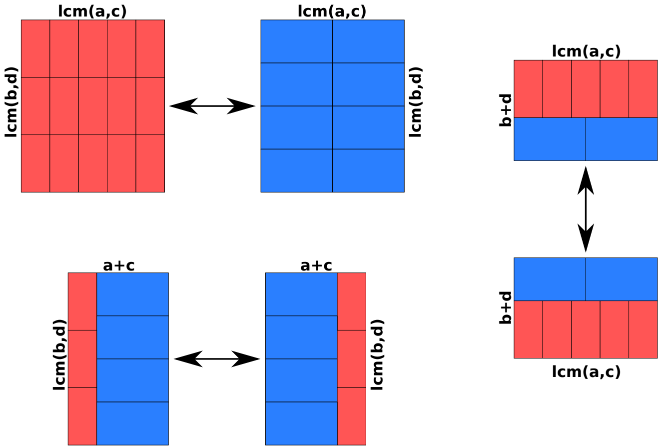
1×2 and 2×1 dominoes $\longrightarrow$ $a \times b$ and $c \times d$ rectangles.

Two rectangles (Kenyon-Kenyon'92, Rémila'04)



Flips: strong (rectangles are changed) or weak (they are moved).

Two rectangles (Kenyon-Kenyon'92, Rémila'04)



Lift: Tiling group theory (Conway-Lagarias'90) to lift in $\mathbb{R}^4$!

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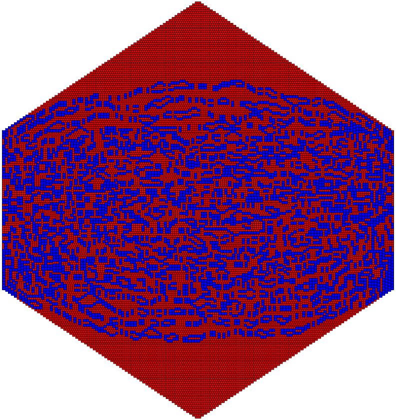

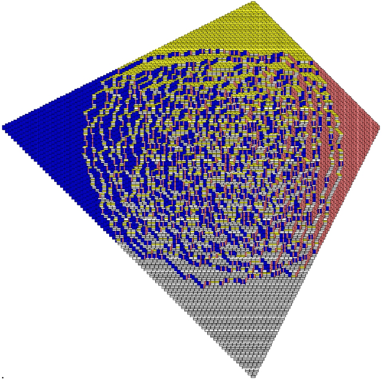

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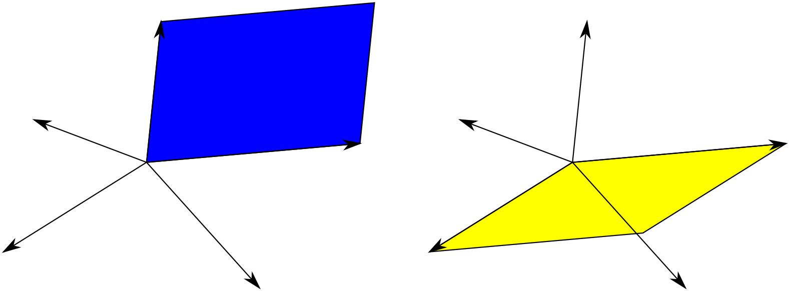
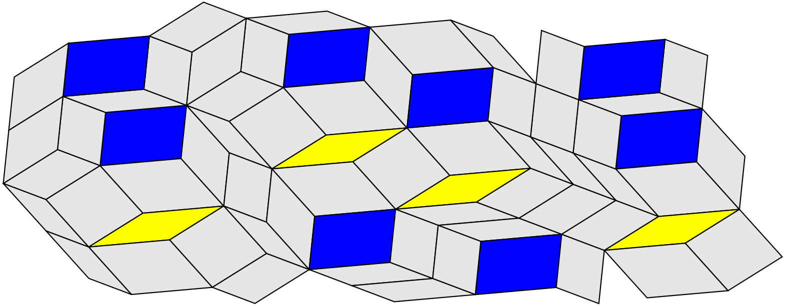
Rhombus tilings (Bodini-F.-Rémila'08)



What if we allow $n \geq 3$ edge directions to define rhombi?

Rhombus tilings (Bodini-F.-Rémila'08)



We get $\binom{n}{2}$ tiles and so-called $n \to 2$ tilings.

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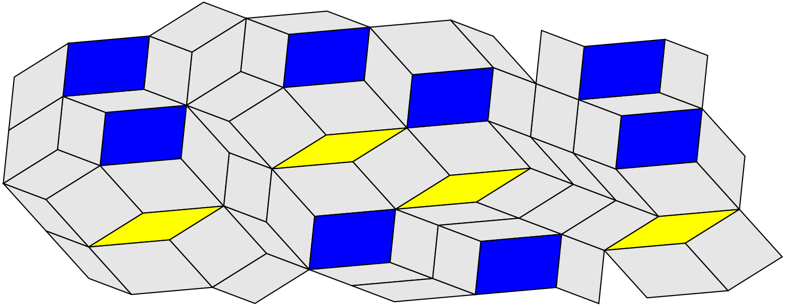

Lift: map the n edge directions onto the standard basis of $\mathbb{R}^n$.

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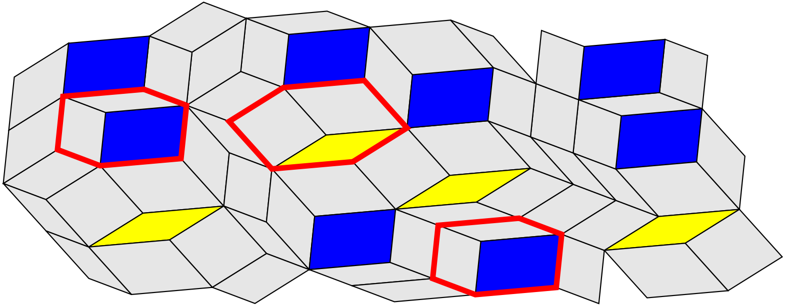

Flip: rotation of three lozenges sharing a vertex.

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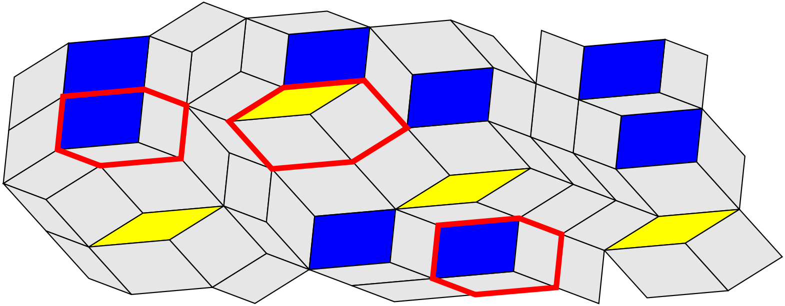

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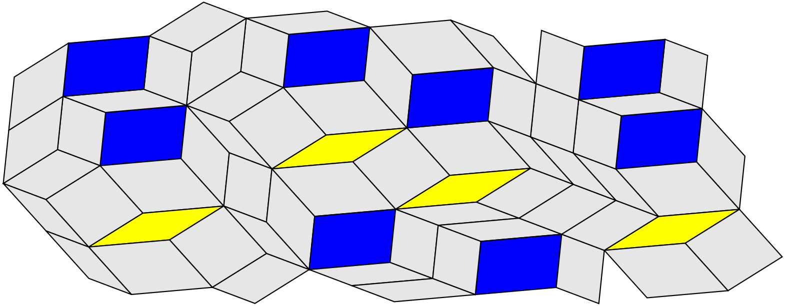
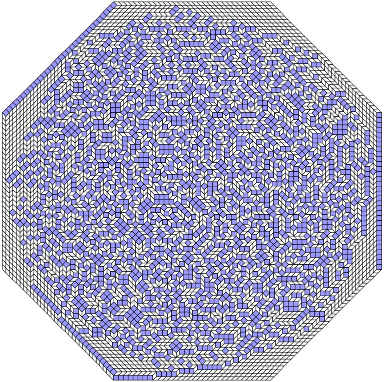
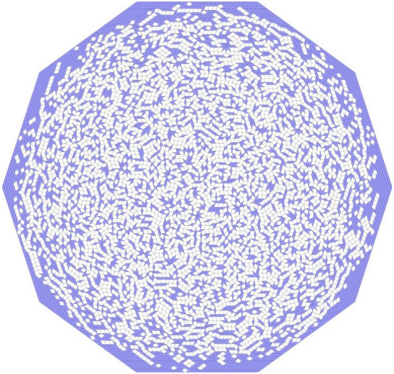


Each tile can be moved step by step towards its final position without moving tiles already at their final position $\rightsquigarrow$ connexity.

Rhombus tilings (Bodini-F.-Rémila'08)

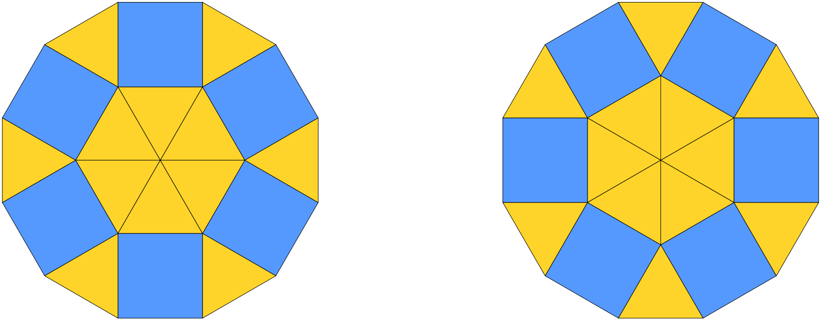




This allows to sample random tilings. And get arctic-circle?

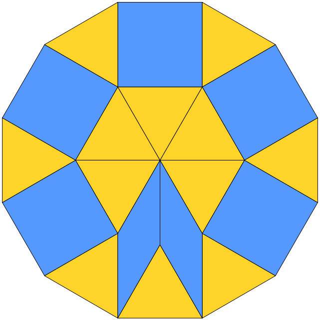
Outline

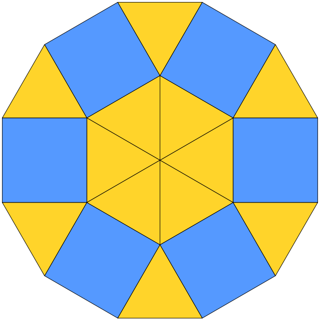
Tilings & flips



How to connect tilings by squares and triangles of a given region?

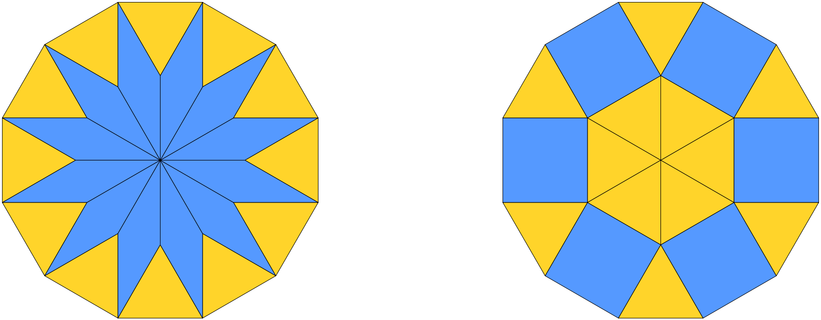
Tilings & flips





Physicists suggest “defect” (lozenges) to mediate between tilings.

Tilings & flips



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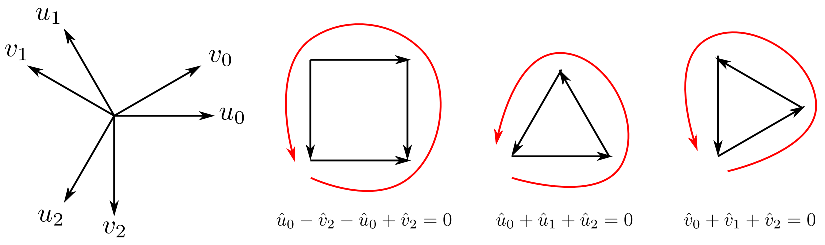
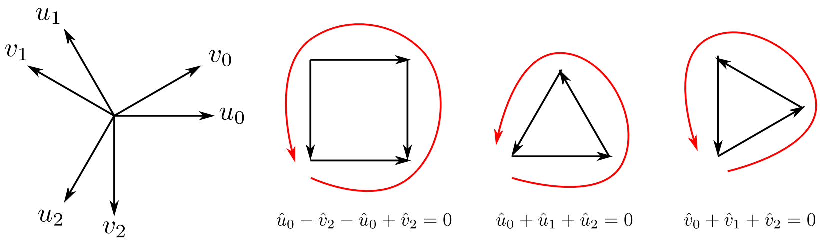


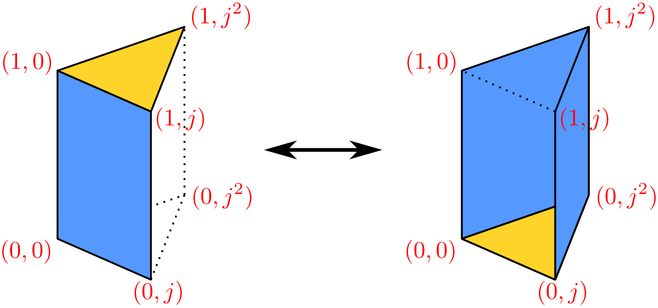
Lifts $\hat{u}_k/\hat{v}_k$ of edges u_k/v_k have to sum up to zero around each tile.

Lift



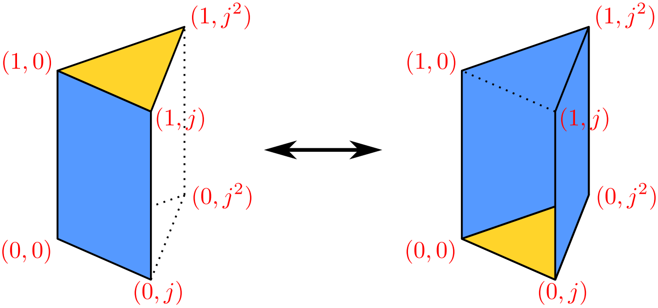
Set e.g. $\hat{u}_k = (j^k, 0)$ and $\hat{v}_k = (0, j^k)$, both in $\mathbb{C} \times \mathbb{C}$, with $j = e^{\frac{2i\pi}{3}}$.

Lift



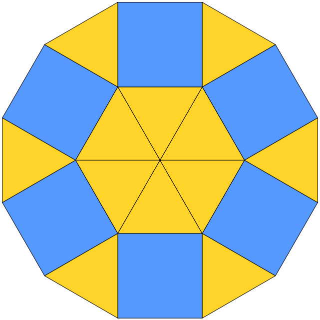
Flip: adding/removing a triangular prism (in $\mathbb{C} \times \mathbb{C}$).

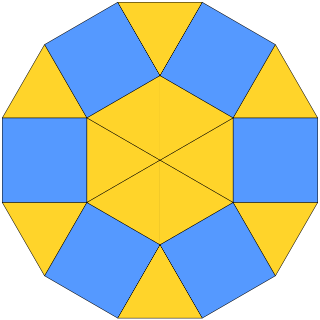
Lift



Flip: adding/removing a triangular prism (in $\mathbb{C} \times \mathbb{C}$).
Lozenges are squares like any other!

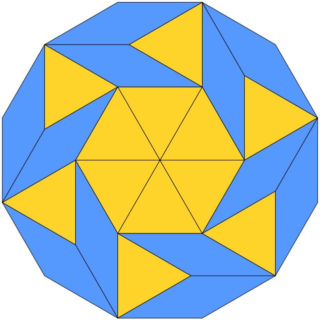
Flip-connexity

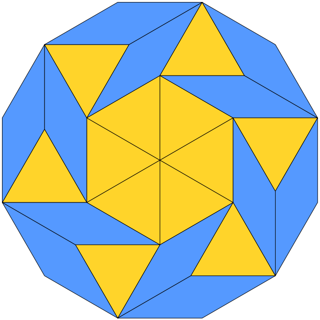




Physicists claim that any two square-triangle tilings are connected.

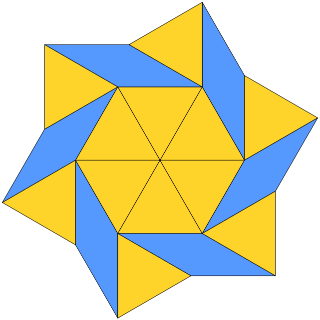
Flip-connexity

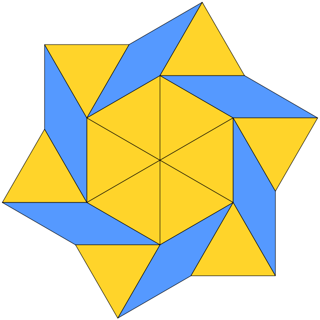




Physicists claim that any two square-triangle tilings are connected.
Does it holds for the superset of square-triangle-lozenges tilings?

Flip-connexity





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