Local and Matching Rules for Canonical Tilings

Thomas Fernique

LIF – CNRS & Univ. de Provence

FRAC, 12/01/2008

This talk is mostly a survey of the two papers:

- L. Levitov, *Local rules for quasicrystals* (1988)
- J. Socolar, *Weak matching rules for quasicrystals* (1990)

With some guest contributions.

Interesting results, but proofs are probably not in the *Book*.

1. Some words about crystals
2. Canonical tilings
3. Strong Local Rules
4. Weak Local Rules
5. Matching Rules

1 Some words about crystals

2 Canonical tilings

3 Strong Local Rules

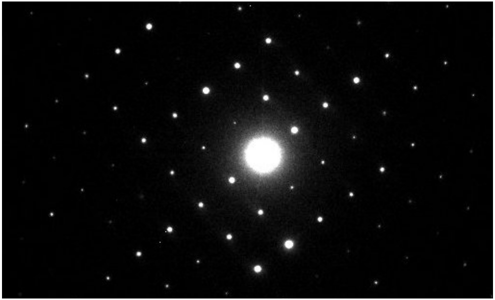
4 Weak Local Rules

5 Matching Rules

Definition

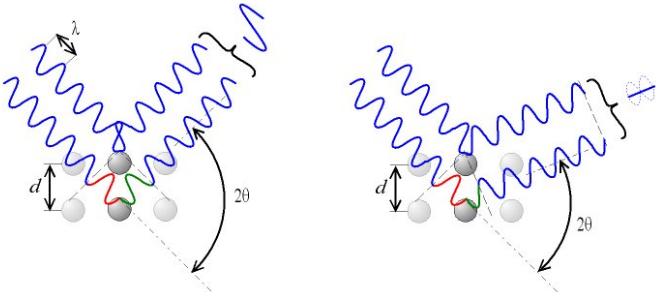
Crystal: lattice decorated with an atomic pattern.

i.e., periodic structure. Experimentally, crystals "diffract":



Principle

XRay diffract on atoms; if those show some long range order, then phases can sum up or cancel out (according to the outgoing angle).



A material "diffracts" if one get sharp bright spots (Bragg peaks).
$\rightsquigarrow$ Diffraction reveals long-range order.

$A \subset \mathbb{R}^2$ has D-fold symmetry if $R_{\frac{2\pi}{D}}(A) = A$.
A is *periodic* if there is a rank 2 lattice Λ s.t. $A + \Lambda = A$.

Crystallographic restriction

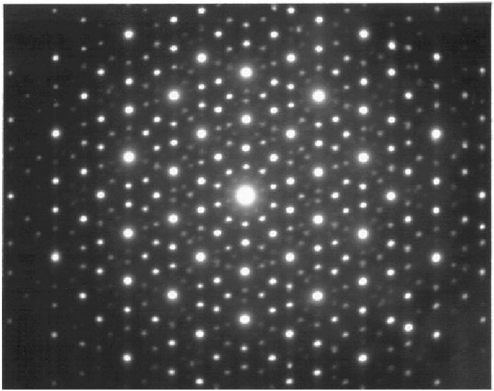
Only $2, 3, 4, 6$-fold symmetries can be periodicity-compatible.

Proof:

- A-preserving rotation in a base of Λ: integer matrix;
- trace of R_θ in any base: $2\cos(\theta)$.

$$2\cos(\theta) \in \mathbb{Z} \Leftrightarrow \cos(\theta) \in \{0, \pm\frac{1}{2}, \pm 1\} \Leftrightarrow \theta \in \{0, \frac{2\pi}{2}, \frac{2\pi}{3}, \frac{2\pi}{4}, \frac{2\pi}{6}\}.$$

In 1984, the following diffraction pattern was observed:



10-fold symmetry $\rightsquigarrow$ not a crystal.
Other non-crystallographic symmetries observed: $5, 8, 10, 12$-fold.

1 Some words about crystals

2 Canonical tilings

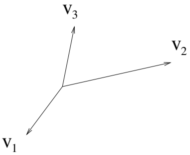
3 Strong Local Rules

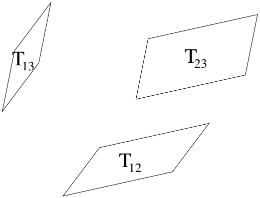
4 Weak Local Rules

5 Matching Rules

$\vec{v}_1, \ldots, \vec{v}_D$ non-colinear vectors of $\mathbb{R}^d \rightsquigarrow \binom{D}{d}$ parallelepipedal tiles:

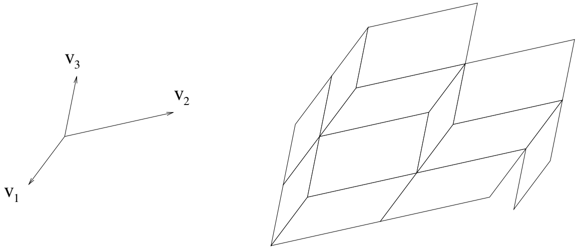
$$T_{i_1,\ldots,i_d} = \{\lambda_{i_1}\vec{v}_{i_1} + \ldots + \lambda_{i_d}\vec{v}_{i_d} \mid 0 \leq \lambda_{i_k} \leq 1\}.$$





$\vec{v}_1, \ldots, \vec{v}_D$ non-colinear vectors of $\mathbb{R}^d \rightsquigarrow \binom{D}{d}$ parallelepipedal tiles:

$$T_{i_1,\ldots,i_d} = \{\lambda_{i_1}\vec{v}_{i_1} + \ldots + \lambda_{i_d}\vec{v}_{i_d} \mid 0 \leq \lambda_{i_k} \leq 1\}.$$



Definition

A $D \to d$ canonical tiling is a tiling of $\mathbb{R}^d$ with these tiles.

Let $(\vec{e}_1, \ldots, \vec{e}_D)$ be the canonical basis of $\mathbb{R}^D$.
Map ϕ on vertices and edges of a $D \to d$ canonical tiling $\mathcal{T}$:

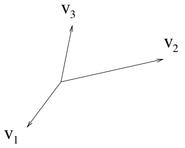
$$\phi(\vec{x}_0) = \vec{y}_0 \in \mathbb{Z}^D, \quad \phi([\vec{x}, \vec{x} + \vec{v}_i]) = [\phi(\vec{x}), \phi(\vec{x}) + \vec{e}_i].$$

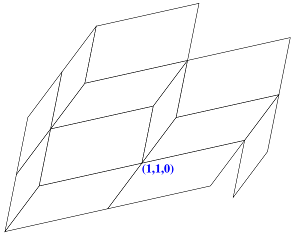
Linearity over tiles $\rightsquigarrow$ map ϕ from $\mathcal{T}$ to $\mathbb{R}^D$, called lift.

Let $(\vec{e}_1,\ldots,\vec{e}_D)$ be the canonical basis of $\mathbb{R}^D$.
Map ϕ on vertices and edges of a $D \to d$ canonical tiling $\mathcal{T}$:

$$\phi(\vec{x}_0)=\vec{y}_0\in\mathbb{Z}^D,\quad \phi([\vec{x},\vec{x}+\vec{v}_i])=[\phi(\vec{x}),\phi(\vec{x})+\vec{e}_i].$$

Linearity over tiles $\rightsquigarrow$ map ϕ from $\mathcal{T}$ to $\mathbb{R}^D$, called lift.



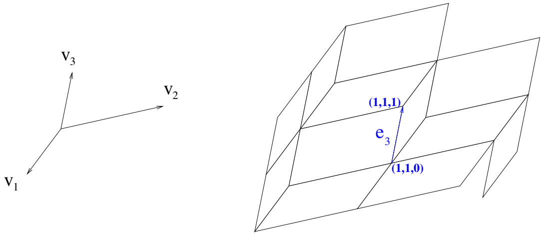


$\phi(\vec{x}_0)=(1,1,0)$.

Let $(\vec{e}_1, \ldots, \vec{e}_D)$ be the canonical basis of $\mathbb{R}^D$.
Map ϕ on vertices and edges of a $D \to d$ canonical tiling $\mathcal{T}$:

$$\phi(\vec{x}_0) = \vec{y}_0 \in \mathbb{Z}^D, \quad \phi([\vec{x}, \vec{x} + \vec{v}_i]) = [\phi(\vec{x}), \phi(\vec{x}) + \vec{e}_i].$$

Linearity over tiles $\rightsquigarrow$ map ϕ from $\mathcal{T}$ to $\mathbb{R}^D$, called lift.

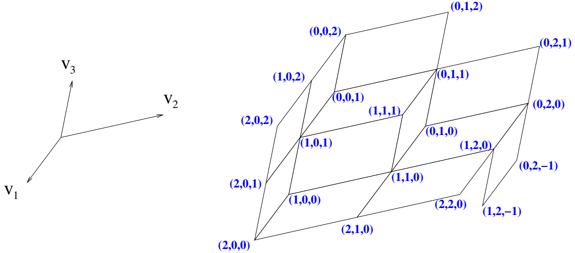


$$\phi([\vec{x}_0, \vec{x}_0 + \vec{v}_3]) = [\phi(\vec{x}_0), \phi(\vec{x}_0) + \vec{e}_3]) = [(1,1,0),(1,1,1)].$$

Let $(\vec{e}_1, \ldots, \vec{e}_D)$ be the canonical basis of $\mathbb{R}^D$.
Map ϕ on vertices and edges of a $D \to d$ canonical tiling $\mathcal{T}$:

$$\phi(\vec{x}_0) = \vec{y}_0 \in \mathbb{Z}^D, \quad \phi([\vec{x}, \vec{x} + \vec{v}_i]) = [\phi(\vec{x}), \phi(\vec{x}) + \vec{e}_i].$$

Linearity over tiles $\rightsquigarrow$ map ϕ from $\mathcal{T}$ to $\mathbb{R}^D$, called lift.

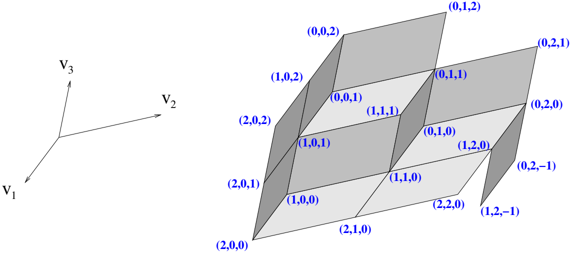


$\phi(\mathcal{T})$: surface made of d-dim. facets of D-dim. unit cubes.

Let $(\vec{e}_1, \ldots, \vec{e}_D)$ be the canonical basis of $\mathbb{R}^D$.
Map ϕ on vertices and edges of a $D \to d$ canonical tiling $\mathcal{T}$:

$$\phi(\vec{x}_0) = \vec{y}_0 \in \mathbb{Z}^D, \quad \phi([\vec{x}, \vec{x} + \vec{v}_i]) = [\phi(\vec{x}), \phi(\vec{x}) + \vec{e}_i].$$

Linearity over tiles $\rightsquigarrow$ map ϕ from $\mathcal{T}$ to $\mathbb{R}^D$, called lift.

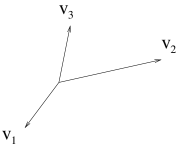


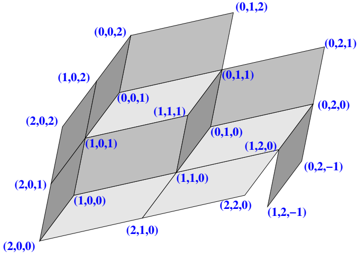
Easily seen in the particular $3 \to 2$ case, by shadowing faces.

Let $(\vec{e}_1, \ldots, \vec{e}_D)$ be the canonical basis of $\mathbb{R}^D$.
Map ϕ on vertices and edges of a $D \to d$ canonical tiling $\mathcal{T}$:

$$\phi(\vec{x}_0) = \vec{y}_0 \in \mathbb{Z}^D, \quad \phi([\vec{x}, \vec{x} + \vec{v}_i]) = [\phi(\vec{x}), \phi(\vec{x}) + \vec{e}_i].$$

Linearity over tiles $\rightsquigarrow$ map ϕ from $\mathcal{T}$ to $\mathbb{R}^D$, called lift.





Note: the lift is unique up to a translation in $\mathbb{Z}^D$.

Let C be a d-dim. affine subspace of $\mathbb{R}^D$.

Definition

A C-tiling is a canonical tiling with a lift in the slice $C + [0,1)^D$.

One shows that, given C, there is a unique C-tiling.

Such tilings diffract $\rightsquigarrow$ model for long-range order of quasi-crystals.

Let C be a d-dim. affine subspace of $\mathbb{R}^D$.

Definition

A C-tiling is a canonical tiling with a lift in the slice $C + [0,1)^D$.

One shows that, given C, there is a unique C-tiling.
Such tilings diffract $\rightsquigarrow$ model for long-range order of quasi-crystals.

Example

A $D \to 2$ C-tiling has D-fold symmetry when $\vec{C}$ is spanned by:

$$(\cos(2k\pi/D))_{0\leq k<D} \quad \text{and} \quad (\sin(2k\pi/D))_{0\leq k<D}.$$

For $D = 5$ and $C = \vec{C}$, we get the celebrated Penrose tiling.

1 Some words about crystals

2 Canonical tilings

3 Strong Local Rules

4 Weak Local Rules

5 Matching Rules

For $r > 0$:

Definition (r-atlas)

A r-pattern of a tiling $\mathcal{T}$ is a union of tiles of $\mathcal{T}$ appearing in an open ball of radius r. The r-atlas of $\mathcal{T}$ is the set of its r-patterns.

For $r > 0$:

Definition (r-atlas)

A r-pattern of a tiling $\mathcal{T}$ is a union of tiles of $\mathcal{T}$ appearing in an open ball of radius r. The r-atlas of $\mathcal{T}$ is the set of its r-patterns.

Definition (Strong Local Rules (SLR))

A C-tiling has SLR if, for some $r > 0$, its r-atlas characterizes $\vec{C}$.

SLR fix the “slope” of a canonical tiling.

A necessary condition for the existence of Strong Local Rules:

Theorem (Levitov)

If a 2-dim. C-tiling has SLR, then $\vec{C}^{\perp}$ is spanned by vectors with entries in $\mathbb{Q}(\sqrt{n})$, for some $n \in \mathbb{N}$.

One speaks about *quadratic-based* slopes.

A necessary condition for the existence of Strong Local Rules:

Theorem (Levitov)

If a 2-dim. C-tiling has SLR, then $\vec{C}^{\perp}$ *is spanned by vectors with entries in* $\mathbb{Q}(\sqrt{n})$*, for some* $n \in \mathbb{N}$*.*

One speaks about *quadratic-based* slopes.

Corollary

Only $3, 4, 5, 6, 8, 10, 12$*-fold symmetries can be SLR-compatible.*

Proof: $\cos(\frac{2\pi}{D}) \in \mathbb{Q}(\sqrt{n})$ yields $n = 1|2|3|5$, $D = 3, 4, 6|8|12|5, 10$.

Can we effectively find SLR for all these symmetries?

- $D \in \{3, 4, 6\}$: yes, with periodic tilings;
- $D \in \{5, 10\}$: yes (Thang);
- $D = 8$: no (Beenker);
- $D = 12$: ?

Recall: 8-fold symmetry experimentally observed $\rightsquigarrow$ gap.

1 Some words about crystals

2 Canonical tilings

3 Strong Local Rules

4 Weak Local Rules

5 Matching Rules

Definition (Weak Local Rules (WLR))

A C-tiling $\mathcal{T}$ has WLR if, for some $r > 0$, the lift of any tiling with the same r-atlas as $\mathcal{T}$ stays at bounded distance from C.

WLR fix the “global slope” of a canonical tiling.

Definition (Weak Local Rules (WLR))

A C-tiling $\mathcal{T}$ has WLR if, for some $r > 0$, the lift of any tiling with the same r-atlas as $\mathcal{T}$ stays at bounded distance from C.

WLR fix the "global slope" of a canonical tiling.

Note: if $\mathcal{T}$ has D-fold symmetry, $D \notin \{2, 3, 4, 6\}$, then any tiling whose lift stays at bounded distance from C is aperiodic.

$\rightsquigarrow$ aperiodic sets of tiles (à la Kari)

A sufficient condition for the existence of Weak Matching Rules:

Theorem (Levitov)

If a 2-dim. C-tiling is such that $\vec{C}^{\perp}$ *is spanned by vectors with entries in* $\mathbb{Q}(\sqrt{n})$, *for some* $n \in \mathbb{N}$, *then it has WLR.*

Again, quadratic-base slopes.

A sufficient condition for the existence of Weak Matching Rules:

Theorem (Levitov)

If a 2-dim. C-tiling is such that $\vec{C}^{\perp}$ is spanned by vectors with entries in $\mathbb{Q}(\sqrt{n})$, for some $n \in \mathbb{N}$, then it has WLR.

Again, quadratic-base slopes.

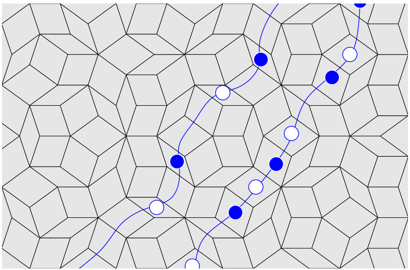
Another existence result:

Theorem (Socolar)

For $D \notin 4\mathbb{N}$, $D \geq 3$, D-fold symmetries are WLR-compatible.

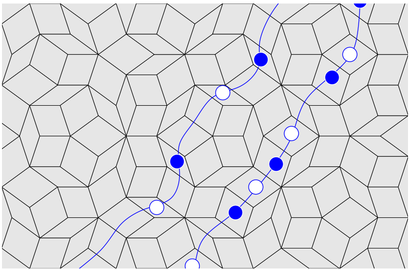
Here, the above sufficient condition does not always hold (hint: $\cos(2\pi/D)$ is quadratic only for $D \in \{3, 4, 5, 6, 8, 10, 12\}$).

Proof of Socolar's result: relies on an *Alternation Condition* (AC):



AC can be enforced by Weak Local Rules (not trivial).

Proof of Socolar's result: relies on an *Alternation Condition* (AC):



AC can be enforced by Weak Local Rules (not trivial).

AC could be easily enforced if tile's coloring would be allowed. . .

1 Some words about crystals

2 Canonical tilings

3 Strong Local Rules

4 Weak Local Rules

5 Matching Rules

Colored tiling: each tile has a color (finite set of colors).
$\rightsquigarrow$ sort of "colored Local Rules", called Matching Rules:

Definition (Strong Matching Rules (SMR))

A C-tiling has SMR if, for some $r > 0$, there is a colored C-tiling whose r-atlas characterizes $\vec{C}$.

Definition (Weak Matching Rules (WMR))

A C-tiling $\mathcal{T}$ has WMR if, for some $r > 0$, there is a colored C-tiling $\mathcal{T}_c$ such that the lift of any tiling with the same r-atlas as $\mathcal{T}_c$ stays at bounded distance from C.

Note: MR $\simeq$ Sofic subshift, whereas LR $\simeq$ Subshift of Finite Type.

Strong Matching Rules: maybe the most classic problem.

Widespread approach: for fixed-points of a substitution.
Principle: enforce the hierarchical structure by Matching Rules.

Strong Matching Rules: maybe the most classic problem.

Widespread approach: for fixed-points of a substitution.
Principle: enforce the hierarchical structure by Matching Rules.

Probably the most general result in this direction:

ToBeProven

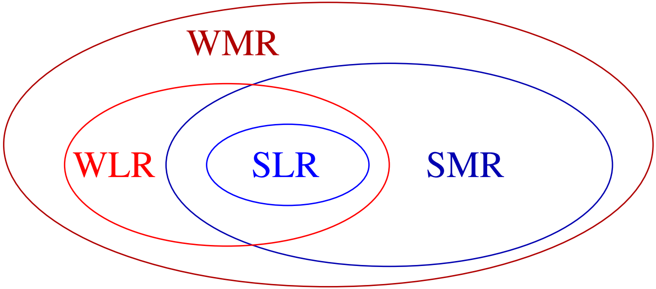
If a C-tiling is such that $\vec{C}$ is an eigenspace of a non-negative integer matrix, then it has SMR.

Proof would rely on *generalized substitutions* (Arnoux-Ito) to show the *pseudo-self-similarity* of this tiling, then obtain an equivalent *self-similar* tiling (Solomyak) and, last, derive matching rules (Goodmann-Strauss).

Other approaches for obtaining Strong Matching Rules?

And for Weak Matching Rules?

At least, there is the ones enforcing the Socolar's AC.
(in the $D = 5$ case, they enforce a Penrose tiling $\rightsquigarrow$ SMR).



Strong Local Rules (SLR): $\simeq$ quadratic-based slopes (*i.e.*, $\vec{C}$).
Weak Local Rules (WLR): strictly more than SLR. Less than SMR?
Strong Matching Rules (SMR): what?
Weak Matching Rules (WMR): strictly more than SMR?

1. Do weak rules still ensure diffraction?
2. Directed flip-acc: asynchronous flips on a tiling with matching errors, with probability depending on the number of corrected matching rules. Convergence towards a C-tiling?

A different definition of Matching Rules (Senechal):

Perfect MR	:	non-periodic and repetitive tilings, same LI-class;
Strong MR	:	non-periodic and repetitive tilings;
Weak MR	:	non-periodic tilings (Kari's rules)