Maximally Dense Sphere Packings

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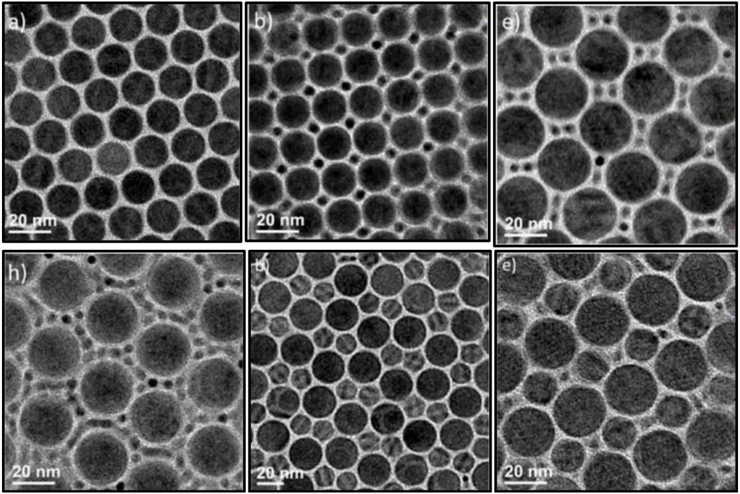
Considered Problem



What is the maximal density of a packing of the Euclidean space by spheres with finitely many prescribed sizes $r_1, \dots, r_k$?

Outline

The localization problem

The m-localization

Beyond the m-localization?

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One circle

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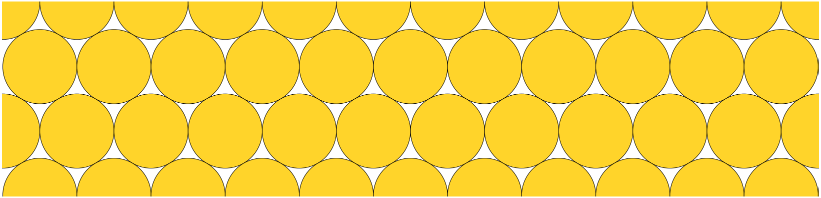

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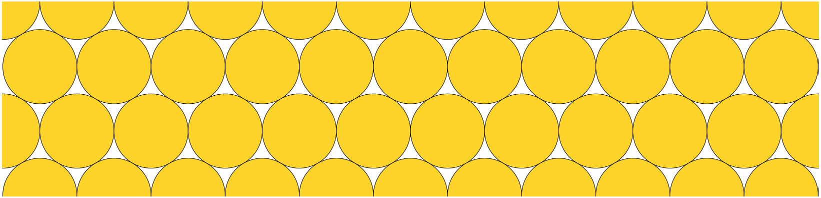

The maximal density is derived quite locally!

Two circles

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- and would alternate around ;
- the angle of the triangle in is larger than $\frac{\pi}{3}$;
- would thus be surrounded by , but this is impossible.

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Localization (following Lagarias, Discrete Comput. Geom. **27**, 2002)

Weighting rule:

- packing $\Omega \rightsquigarrow$ partition $\mathcal{P}$ of the space into bounded polytopes;
- each cell has a weight related to the empty volume it contains;
- each cell distributes its weight among nearby spheres.

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Weighting rule:

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Proof sketch:

$$\sum_{R \in \mathcal{P}} \alpha \mathrm{vol}(R) - \mathrm{cov}(R) \geq \sum_{v \in \Omega} \beta$$

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$$\frac{\kappa|\Omega|}{\mathrm{vol}(\mathcal{P})} \leq \frac{\alpha\kappa}{\kappa + \beta}.$$

Example: one circle

Weighting rule:

- circle packing $\Omega \rightsquigarrow$ Delaunay triangulation of the plane;
- the weight of a triangle R is equal to $\frac{\pi}{2\sqrt{3}}\mathrm{vol}(R) - \mathrm{cov}(R)$;
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Upper bound on the density:

$$\delta \leq \frac{\pi}{2\sqrt{3}}.$$

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Weighting rule (Hales, 1992):

- sphere packing $\Omega \rightsquigarrow$ Delaunay tetrahedrization of the space;
- the weight of a tetrahedron R is equal to $\delta_{\mathrm{oct}}\mathrm{vol}(R) - \mathrm{cov}(R)$, where $\delta_{\mathrm{oct}} \approx 0.720903$ is the density in a regular octahedron;
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Upper bound on the density:

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Proving a local density inequality

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check(K):
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Problem 2

Optimal local densities may not exist (aperiodic densest packings?)

Outline

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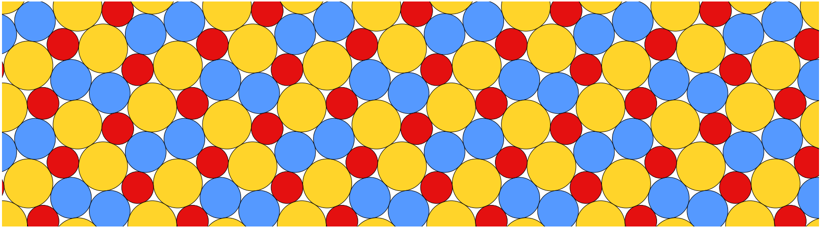
The *m*-localization: teaser

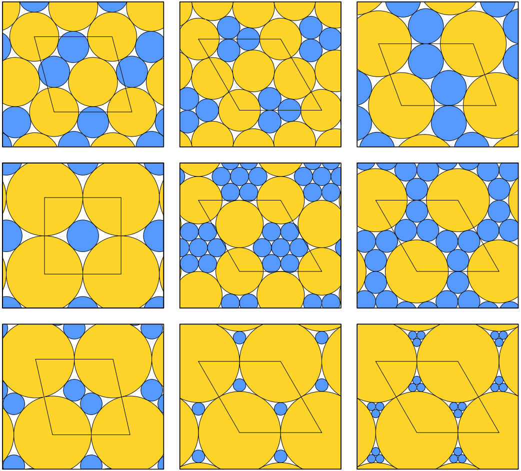
This is a weighting rule:

- introduced under that name by T. Kennedy in 2004;
- relying on ideas introduced by A. Heppes in 2000-02;
- improved by N. Bédaride and Th. Fernique in 2020.

Designed to prove maximal density of triangulated packings:



Triangulated binary packings

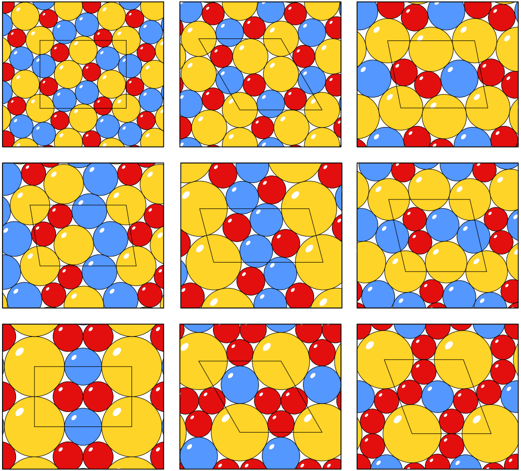


r_2/r_1	δ
0.6376	0.9106
0.5452	0.9116
0.5333	0.9141
0.4142	0.9201
0.3861	0.9200
0.3492	0.9246
0.2808	0.9319
0.1547	0.9503
0.1010	0.9624

Theorem (Kennedy, 2006)

There are exactly 9 sizes allowing a triangulated binary packing.

Triangulated ternary packings



r_2/r_1	r_3/r_1	δ
0.83	0.65	0.9093
0.79	0.63	0.9098
0.84	0.61	0.9101
0.88	0.62	0.9094
0.59	0.51	0.9124
0.79	0.57	0.9100
0.61	0.50	0.9135
0.66	0.46	0.9153
0.55	0.43	0.9178

Theorem (Fernique-Hashemi-Sizova, 2019)

There are exactly 164 *sizes allowing a triangulated ternary packing.*

Density of triangulated packings

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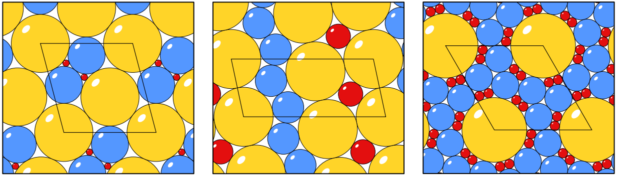

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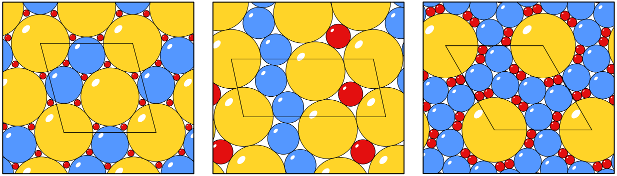

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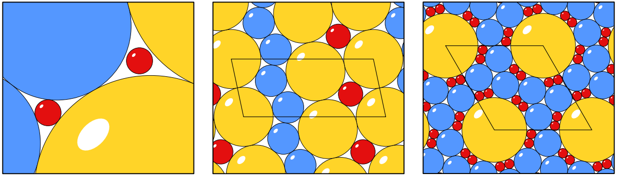

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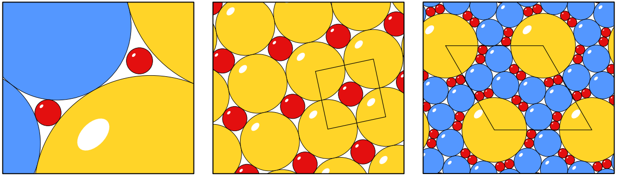

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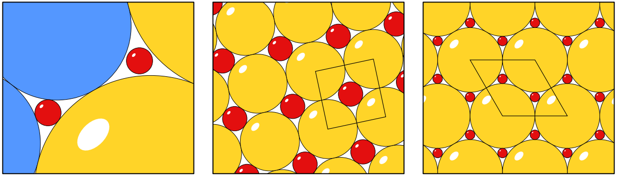

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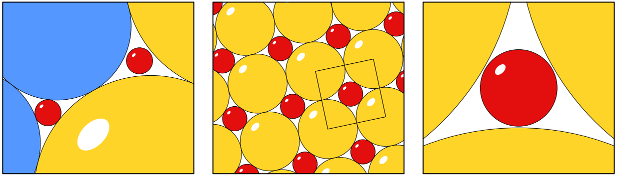

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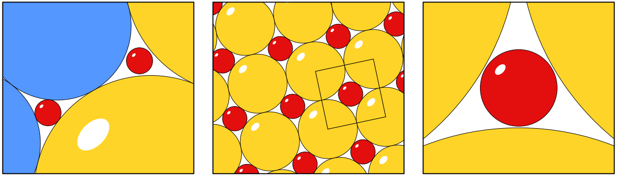

For 116 *sizes, the m-localization does not allow to conclude.*

The *m*-localization: definition

Weighting rule:

- packing $\Omega \rightsquigarrow$ *FM-triangulation* of the plane;
- each triangle R has weight at most $\delta^*\mathrm{vol}(R) - \mathrm{cov}(R)$;
- each triangle *cleverly* distributes its weight among its 3 circles.

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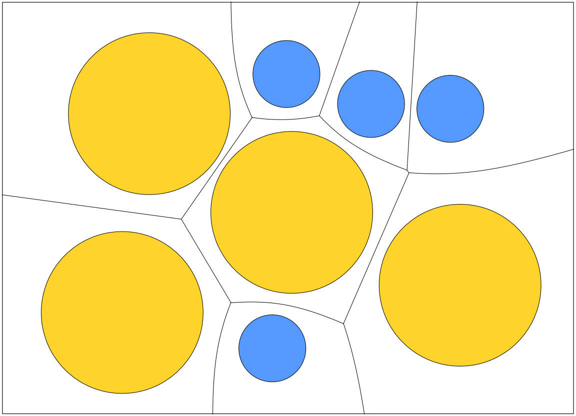
Upper bound on the density:

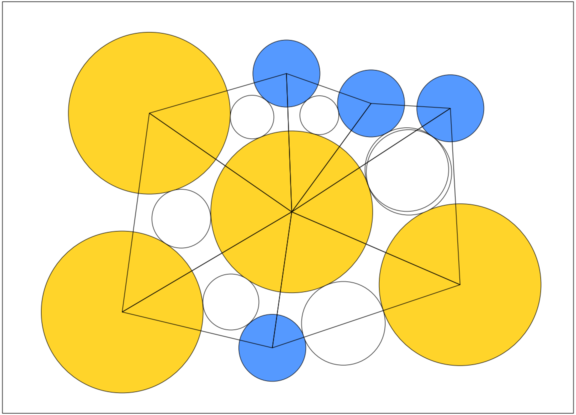
$$\delta \leq \delta^*.$$

FM-triangulation



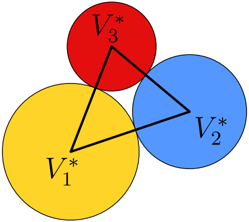
Cell of a circle: the points closer to it than to any other circle.

FM-triangulation



Dual: Fejes-Mólnar (or weighted Delaunay) triangulation.

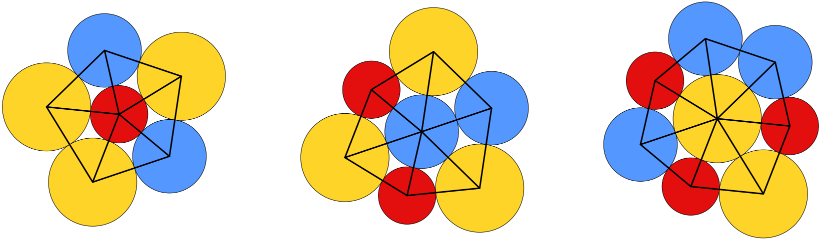
Weight distribution



The circles of a tight triangle R^* receive weights V_i^*'s which depend only on the 3 circle sizes and satisfy the linear equation:

$$V_1^* + V_2^* + V_3^* = \delta^* \mathrm{vol}(R^*) - \mathrm{cov}(R^*).$$

Weight distribution



The total weight received by any circle in the (candidate) densest triangulated packing must be nonnegative in order for the local density inequality to be satisfied for this packing.

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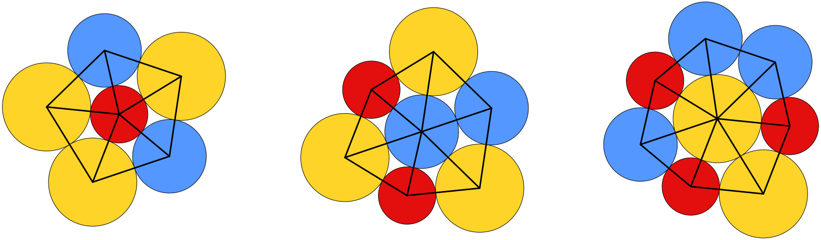
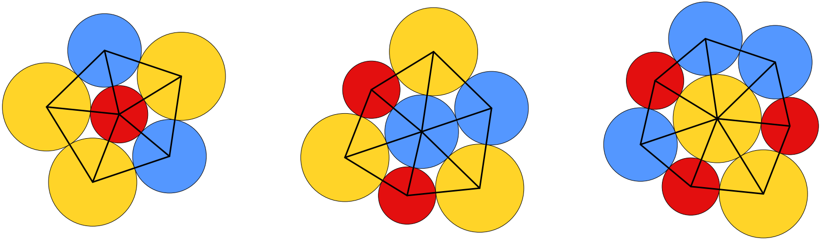


The total weight received by any circle in the (candidate) densest triangulated packing must be equal to 0 in order for the local density inequality to be optimal for this packing.

Weight distribution



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This yields further linear equations on the V_i^*'s.

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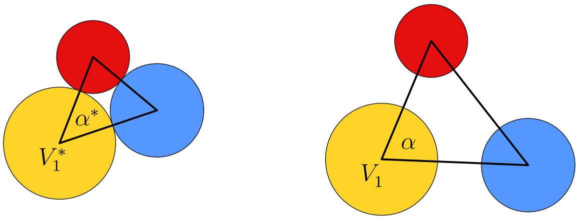
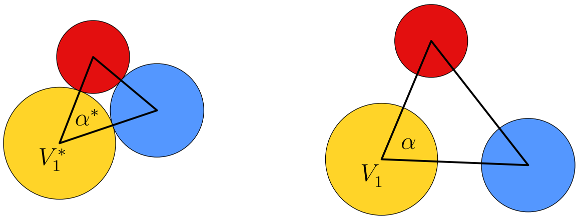


Triangle deformation $\rightsquigarrow$ weight deviation: $V_i := V_i^* + m|\alpha - \alpha^*|$.
m big enough to ensure the local density inequality in any packing.
m small enough to have R of weight at most $\delta^*\mathrm{vol}(R) - \mathrm{cov}(R)$.

Weight distribution

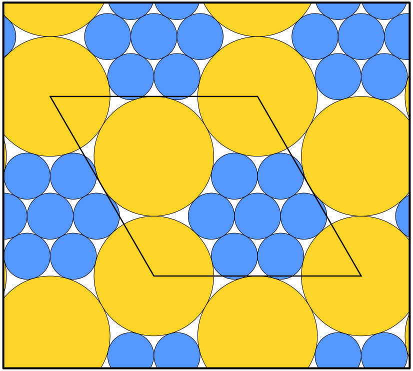


Triangle deformation $\rightsquigarrow$ weight deviation: $V_i := V_i^* + m|\alpha - \alpha^*|$.
m big enough to ensure the local density inequality in any packing.
m small enough to have R of weight at most $\delta^* \mathrm{vol}(R) - \mathrm{cov}(R)$.

$$\left.\begin{array}{r}\text{Finitely many configurations}\\ \text{Compact set of FM-triangles}\end{array}\right\} \rightsquigarrow \text{computer check!}$$

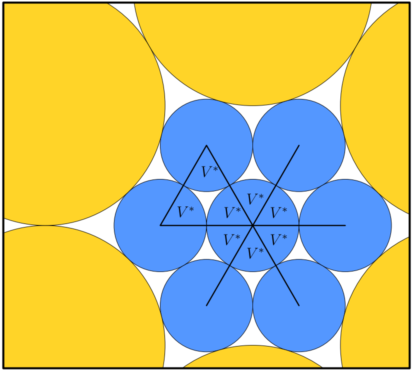
Outline

The case c_5



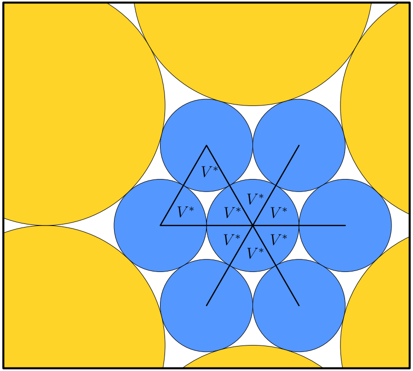
Consider the m-localization for this triangulated binary packing.

The case c_5



Each blue triangle gives weight V^* to each of its 3 circles.
The total weight received by the central blue circle is $6V^*$.
Hence $V^* = 0$ and the total weight of a blue triangle R is 0.

The case c_5



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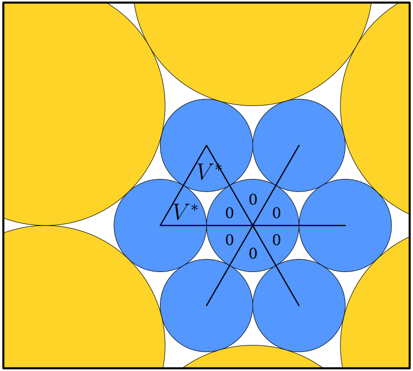
The total weight received by the central blue circle is $6V^*$.
Hence $V^* = 0$ and the total weight of a blue triangle R is 0.
But it should be $\delta^* \mathrm{vol}(R) - \mathrm{cov}(R) > \frac{\pi}{2\sqrt{3}} \mathrm{vol}(R) - \mathrm{cov}(R) = 0$.

The case c_5



Solution: the weight received by a circle depends on its neighbors.

Stoichiometry

What about the packings with frequency f_r of circles of size r?

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Example (Fernique, 2020)

Characterization of the densest packings by circles of size 1 and $\sqrt{2}-1$ for each possible stoichiometry.

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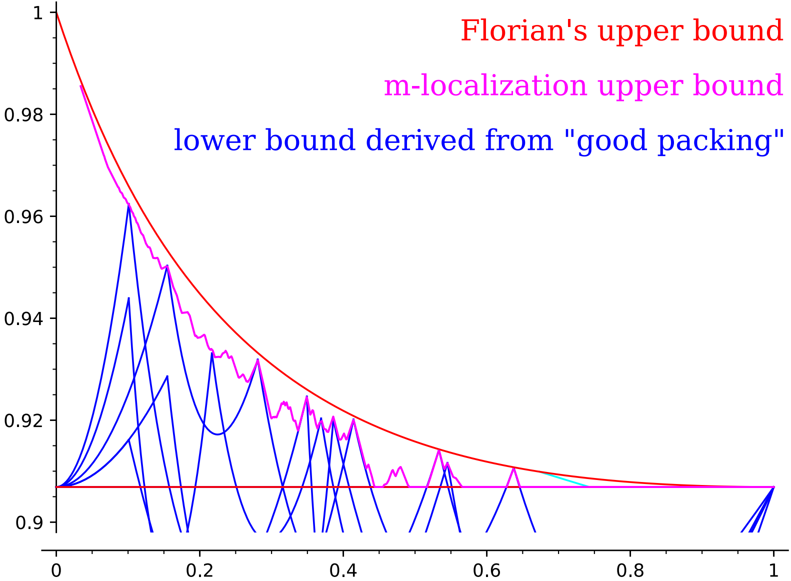
At least we can hope to get an upper bound on the density...

Upper bounds for binary packings



Example: two circles within ratio 0.48