Compact packings with three discs

Thomas Fernique (Paris)
Amir Hashemi (Isfahan)
Olga Sizova (Moscow)
Sphere packings

Sphere packing: interior disjoint unit spheres.
Density: limsup of the proportion of $B(0, r)$ covered.
Question: densest packings?
Sphere packings

**Sphere packing:** interior disjoint unit spheres.

**Density:** limsup of the proportion of \( B(0, r) \) covered.

**Question:** densest packings?

**Theorem (Thue, 1910)**

*The densest packing in \( \mathbb{R}^2 \) is the hexagonal compact packing.*
Sphere packings

Sphere packing: interior disjoint unit spheres.  
Density: limsup of the proportion of $B(0, r)$ covered.  
Question: densest packings?

Theorem (Hales, 1998) 

*The densest packings in $\mathbb{R}^3$ are the close-packings.*
Sphere packings

**Sphere packing**: interior disjoint unit spheres.
**Density**: limsup of the proportion of $B(0, r)$ covered.
**Question**: densest packings?

**Theorem (Hales, 1998)**

*The densest packings in $\mathbb{R}^3$ are the close-packings.*
Sphere packings

Sphere packing: interior disjoint unit spheres.
Density: limsup of the proportion of $B(0, r)$ covered.
Question: densest packings?

Theorem (Hales, 1998)

*The densest packings in $\mathbb{R}^3$ are the close-packings.*
Sphere packings

Sphere packing: interior disjoint unit spheres.
Density: limsup of the proportion of $B(0, r)$ covered.
Question: densest packings?

Theorem (Hales, 1998)

*The densest packings in $\mathbb{R}^3$ are the close-packings.*
Sphere packings

**Sphere packing**: interior disjoint unit spheres.
**Density**: limsup of the proportion of $B(0, r)$ covered.
**Question**: densest packings?

**Theorem (Vyazovska et al., 2017)**

*The densest packings are known in $\mathbb{R}^8$ and $\mathbb{R}^{24}$.***
Unequal sphere packings

The density becomes parametrized by the ratios of sphere sizes. Natural problem in materials science!
Unequal sphere packings

The density becomes parametrized by the ratios of sphere sizes. Natural problem in materials science!

Theorem (Heppes-Kennedy, 2004–2006)

The densest packings with two discs are known for seven ratios.
Two discs

The maximal density is a function $\delta(r)$ of the ratio $r \in [0, 1]$. 
The hexagonal compact packing yields a uniform lower bound.
Any given packing yields a lower bound for a specific $r$. 

Two discs
It actually yields a lower bound in a neighborhood of $r$. 
\[
\lim_{r \to 0} \delta(r) = \delta(1) + (1 - \delta(1))\delta(1) \simeq 0.99133.
\]
Two discs

The density in the r1r triangle is an upper bound (Florian, 1960).
Two discs

For $r \geq 0.74$, two discs do not pack better than one (Blind, 1969).
The seven "magic" ratios

0.41, root of $X^2 - 2$. 

$\delta(r)$
The seven ”magic” ratios

\[ \delta(r) \]

0.15, root of \( 3X^2 + 6X - 1 \).
The seven ”magic” ratios

0.28, root of $2X^2 + 3X - 1$. 

$\delta(r)$
The seven "magic" ratios

$0.53$, root of $8X^3 + 3X^2 - 2X - 1$. 

The seven "magic" ratios

0.64, root of $X^4 - 10X^2 - 8X + 9$. 
The seven "magic" ratios

0.35, root of $X^4 - 28X^3 - 10X^2 + 4X + 1$. 
The seven "magic" ratios

\[ 0.55, \text{root of } X^8 - 8X^7 - 24X^6 - 232X^5 - 482X^4 - 120X^3 + 388X^2 - 44X + 9. \]
Compact packings

The contact graphs of the 7 previous packings are triangulated.
Compact packings

The *contact graphs* of the 7 previous packings are triangulated. The one of the hexagonal compact packing of unit spheres also.
Compact packings

The *contact graphs* of the 7 previous packings are triangulated. The one of the hexagonal compact packing of unit spheres also.

**Definition**
A packing of discs is *compact* if its contact graph is triangulated.
Compact packings

The *contact graphs* of the 7 previous packings are triangulated. The one of the hexagonal compact packing of unit spheres also.

**Definition**
A packing of spheres is *compact* if its contact graph is simplicial.
Compact packings

The *contact graphs* of the 7 previous packings are triangulated. The one of the hexagonal compact packing of unit spheres also.

**Definition**

A packing of spheres is *compact* if its contact graph is simplicial.

The densest packings of unit spheres in $\mathbb{R}^8$ and $\mathbb{R}^{24}$ are compact!
Compact packings

The *contact graphs* of the 7 previous packings are triangulated. The one of the hexagonal compact packing of unit spheres also.

**Definition**

A packing of spheres is *compact* if its contact graph is simplicial.

The densest packings of unit spheres in $\mathbb{R}^8$ and $\mathbb{R}^{24}$ are compact! Not in $\mathbb{R}^3$... where no compact packing of unit spheres does exist!
Compact packings

The *contact graphs* of the 7 previous packings are triangulated. The one of the hexagonal compact packing of unit spheres also.

**Definition**
A packing of spheres is *compact* if its contact graph is simplicial.

The densest packings of unit spheres in $\mathbb{R}^8$ and $\mathbb{R}^{24}$ are compact! Not in $\mathbb{R}^3$... where no compact packing of unit spheres does exist!

Compact packings are candidates to provably maximize the density.
Compact packings with two discs

Theorem (Kennedy, 2006)

There are nine ratios allowing a compact packing with two discs.
Compact packings with two discs

Theorem (Kennedy, 2006)

*There are nine ratios allowing a compact packing with two discs.*

Remark: two have (still?) not been proven to maximize the density.
Compact packings with three discs

Theorem (F.-Hashemi-Sizova)

There are 164 ratios allowing a compact packing with three discs.
Theorem (F.-Hashemi-Sizova)
There are 164 ratios allowing a compact packing with three discs.
Compact packings with three discs

Theorem (F.-Hashemi-Sizova)

There are 164 ratios allowing a compact packing with three discs.
Compact packings with three discs

Theorem (F.-Hashemi-Sizova)

There are 164 ratios allowing a compact packing with three discs.
Compact packings with three discs

Theorem (F.-Hashemi-Sizova)

There are 164 ratios allowing a compact packing with three discs.
Compact packings with three discs

Theorem (F.-Hashemi-Sizova)

There are 164 ratios allowing a compact packing with three discs.
Proof sketch

Let $s < r < 1$ be the three sizes of discs.

Definition

An $x$-corona is a sequence of sizes of discs around a disc of size $x$.
Proof sketch

Let $s < r < 1$ be the three sizes of discs.

**Definition**

An \textit{x-corona} is a sequence of sizes of discs around a disc of size $x$.

**Claim**

There are finitely many different $s$-coronas.
Proof sketch

Let $s < r < 1$ be the three sizes of discs.

Definition
An $x$-corona is a sequence of sizes of discs around a disc of size $x$.

Claim
There are finitely many different $s$-coronas.

Lemma
There are finitely many different $r$-coronas in a compact packing.
Proof sketch

Let $s < r < 1$ be the three sizes of discs.

**Definition**
An $x$-corona is a sequence of sizes of discs around a disc of size $x$.

**Claim**
There are finitely many different $s$-coronas.

**Lemma**
*There are finitely many different $r$-coronas in a compact packing.*

**Proposition**
*Each pair of $s$- and $r$-coronas yields a polynomial system in $r$ and $s$.*
Proof sketch

Let $s < r < 1$ be the three sizes of discs.

**Definition**
An $x$-corona is a sequence of sizes of discs around a disc of size $x$.

**Claim**
There are finitely many different $s$-coronas.

**Lemma**
*There are finitely many different $r$-coronas in a compact packing.*

**Proposition**
*Each pair of $s$- and $r$-coronas yields a polynomial system in $r$ and $s$.*

**Strategy**
For each pair of $s$- and $r$-coronas, solve the polynomial system, then find all the possible coronas and finally find the packings.
During the poster sessions, you can get more details about...

the proof:

- how to associate a polynomial system with s- and r-coronas?
- why (and how) we used resultants and interval arithmetic?
- which alternative strategies do exist?
- is it that easy, given the coronas, to find a packing?
During the poster sessions, you can get more details about... 

the proof:
- how to associate a polynomial system with $s$- and $r$-coronas?
- why (and how) we used resultants and interval arithmetic?
- which alternative strategies do exist?
- is it that easy, given the coronas, to find a packing?

the packings:
- admire a compact packing for each pair $(r, s)$.
- what diversity of packings allows each pair $(r, s)$?
- try yourself to pack it compact (javascript)!