Effective S-adic symbolic dynamical systems

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Outline

- Symbolic dynamics
- Substitutions and S-adic systems
- Effectiveness for S-adic systems
- Sturmian and planar tilings

Symbolic dynamics

1	2	1	2	1	2	3	1	2	1	2	3	1	3
3	1	3	1	2	1	2	3	1	2	1	2	1	2
2	1	2	3	1	2	1	2	3	1	3	1	2	1
1	2	1	2	3	1	3	1	2	1	2	3	1	2
3	1	2	1	2	1	2	3	1	2	1	2	3	1
2	3	1	3	1	2	1	2	3	1	2	1	2	1
1	2	1	2	3	1	2	1	2	3	1	3	1	2
3	1	2	1	2	3	1	3	1	2	1	2	3	1

Patterns and configurations

We fix a dimension d

- Let $\mathcal{A}$ be finite alphabet.
- A configuration u is an element of $\mathcal{A}^{\mathbb{Z}^d}$ (an $\mathcal{A}$-coloring of $\mathbb{Z}^d$)
- A pattern p is an element of $\mathcal{A}^D$, where $D \subset \mathbb{Z}^d$ is a finite set, called its support.
- A translate of the pattern p by $\mathbf{m} \in \mathbb{Z}^d$ is denoted $p + \mathbf{m}$ and has $D + \mathbf{m}$ for support.
- The language of a configuration u is the set of finite patterns that occur in u (up to translation).

2	1	2	3	1	3	1	2	1	2	3	1	2	1	[illegible]
1	2	1	2	1	2	3	1	2	1	2	3	1	3	[illegible]
3	1	3	1	2	1	2	3	1	2	1	2	1	2	[illegible]
2	1	2	3	1	2	1	2	3	1	3	1	2	1	[illegible]
1	2	1	2	3	1	3	1	2	1	2	3	1	2	[illegible]
3	1	2	1	2	1	2	3	1	2	1	2	3	1	[illegible]
2	3	1	3	1	2	1	2	3	1	2	1	2	1	[illegible]
1	2	1	2	3	1	2	1	2	3	1	3	1	2	[illegible]
3	1	2	1	2	3	1	3	1	2	1	2	3	1	[illegible]

Subshifts

- The set $\mathcal{A}^{\mathbb{Z}^d}$ endowed with the product topology is a compact metric space.

$$d(u,v) \leq 2^{-n} \text{ if } \ u_{|[-n,n]^d} = v_{|[-n,n]^d}.$$

- The shifts $\sigma_{\mathbf{m}}$, $\mathbf{m} \in \mathbb{Z}^d$, act on configurations

$$\sigma_{\mathbf{m}} \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}, \ (u_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d} \mapsto (u_{\mathbf{n}+\mathbf{m}})_{n \in \mathbb{Z}^d}.$$

- A d-dimensional subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a closed and shift-invariant set of configurations in $\mathcal{A}^{\mathbb{Z}^d}$.
- A subshift X can be defined by providing its language, that is, the set of patterns that occur (up to translation) in configurations in X.
- It can be defined equivalently by providing the set of forbidden patterns.
- Example: $\{0,1\}^{\mathbb{Z}}$ with 11 not allowed.

SFT and sofic subshifts

- Subshifts of finite type (SFT) are the subshifts defined by a finite set of forbidden patterns.
- Sofic subshifts are images of SFT under a factor map.
- A factor map $\pi : X \to Y$ between two subshifts X and Y is a continuous, surjective map such that

$$\pi \circ \sigma_{\mathbf{m}} = \sigma_{\mathbf{m}} \circ \pi,$$

  for all $\mathbf{m} \in \mathbb{Z}^d$.
- A factor map is a sliding block code (defined by a local rule/CA) [Curtis-Hedlund-Lyndon].
- Example: add colorations. Take a larger alphabet for X: X is the SFT, Y is the sofic shift.
- Wang tiles

Computable subshifts

A subshift is said to be

- Π_1-computable or effective if its language is co-recursively enumerable;
- Σ_1-computable if its language is recursively enumerable;
- Δ_1-computable or decidable if its language is recursive.

cf. E. Jeandel's lecture.

Frequencies and measures

- The frequency $f(p)$ of a pattern p in a d-dimensional configuration u is defined as the limit (if it exists) of

$$\lim_n \frac{|x_{|[-n,n]^d}|_p}{(2n+1)^d}$$

  where $|x_{|[-n,n]^d}|_p$ stands for the number of occurrences of p in x_n.
- A subshift is said to be uniquely ergodic if it admits a unique shift-invariant measure; in this case, pattern frequencies do exist (and the convergence is uniform).
- A subshift is said to be minimal if every non-empty closed shift-invariant subset is equal to the whole set. A minimal and uniquely ergodic subshift is said strictly ergodic.
- Any pattern which appears in a strictly ergodic subshift has a positive frequency.

About the computability of frequencies

Computable frequencies: there exists an algorithm that takes as input a pattern and a precision, and that outputs an approximation of this frequency with respect to this precision

$\rightsquigarrow$ Computable pattern frequencies/shift-invariant measure

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Computability of letter frequencies does not say much on the algorithmic complexity of a subshift.

- Take a subshift $X \subset \{0,1\}^{\mathbb{Z}}$ and consider the subshift Y obtained by applying to each configuration of X the substitution

$$0 \mapsto 01, 1 \mapsto 10.$$

  The subshift Y admits letter frequencies (they are both equal to $1/2$), and it has the same algorithmic complexity as X.

Effectiveness for shifts

Theorem Let X be a subshift.

- If X is effective and uniquely ergodic, then its invariant measure is computable and X is decidable.
- If X is minimal and its frequencies are computable, then its language is recursively enumerable.
- If X is minimal and effective, then X is decidable.

Effectiveness for shifts

We assume X effective and uniquely ergodic. Let us prove that the frequency of any pattern is computable.

- Consider the following algorithm that takes as an argument the parameter e for the precision. We consider a finite pattern p.
- At step n, one produces all ‘square’ patterns of size n that do not contain the n first forbidden patterns.
- For each of these square patterns, one computes the number of occurrences of p in it, divided by $(2n+1)^d$.
- We continue until these quantities belong to an interval of length e.
- This algorithm then stops (compactness=subshift + unique ergodicity=uniform frequencies), and taking an element of the interval provides an approximation of the frequency of p up to precision e.

We assume X minimal with computable pattern frequencies. We prove that the language is recursively enumerable.

- Frequencies are positive by minimality.
- Even if the frequencies are computable, one cannot decide whether the frequency of a given pattern is equal to zero or not, hence we cannot decide whether this pattern belongs to the language or not.
- However, one can decide whether the frequency of a pattern is larger than a given value. This thus implies that the language is recursively enumerable.

Substitutions and S-adic systems

Substitutions

- Substitutions on words : symbolic dynamical systems
- Substitutions on tiles : inflation/subdivision rules, tilings and point sets

Substitutions

- Substitutions on words : symbolic dynamical systems

  Morphism of the free monoid (no cancellations)

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 1$$

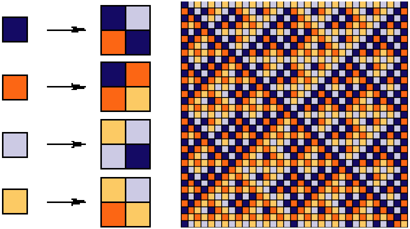
1
12
121
12112
12112121

Fibonacci word $\sigma^{\infty}(1) = 121121211211212\cdots$

- Substitutions on tiles : inflation/subdivision rules, tilings and point sets

Substitutions

- Substitutions on words : symbolic dynamical systems
- Substitutions on tiles : inflation/subdivision rules, tilings and point sets



Tilings Encyclopedia http://tilings.math.uni-bielefeld.de/
[E. Harriss, D. Frettlöh]

S-adic expansions

- Let $\mathcal{S}$ be a set S of substitutions on the alphabet $\mathcal{A}$
- Let $s = (\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ a sequence of substitutions (directive sequence)
- Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of letters in $\mathcal{A}$

We say that the infinite word $u \in \mathcal{A}^{\mathbb{N}}$ admits $(\sigma_n, a_n)_n$ as an S-adic representation if

$$u = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n)$$

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The terminology comes from Vershik adic transformations
Bratteli diagrams

S stands for substitution, adic for the inverse limit
powers of the same substitution= partial quotients

Geometrical substitutions and tilings

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear map

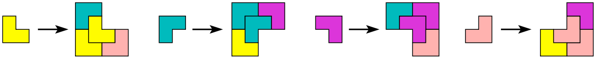
Principle One takes

- a finite number of prototiles $\{T_1, T_2, \ldots, T_m\}$
- an expansive transformation ϕ (the inflation factor)
- a rule that allows one to divide each ϕT_i into copies of the $T_1, T_2, \ldots, T_m$

A tile-substitution s with expansion ϕ is a map $T_i \mapsto s(T_i)$, where $s(T_i)$ is a patch made of translates of the prototiles and

$$\phi(T_i) = \bigcup_{T_j \in s(Ti)} T_j$$

Example



Combinatorial tiling substitutions

- The substitution rule replaces a tile by some configuration of tiles that may not bear any geometric resemblance to the original.
- The difficulty with such a rule comes when one wishes to iterate it: we need to be sure that the substitution can be applied repeatedly so that all the tiles fit together without gaps or overlaps.
- Combinatorial substitutions map a tiling by tiles onto a tiling by super-tiles so that the super-tiles of the latter are arranged as the tiles of the former.
- We can introduce concatenation rules which specify how the respective images of two adjacent tiles must be glued.

[Priebe-Frank, Fernique-Ollinger]

Some decision problems for substitutions

Decision problems for word substitutions

Some classical decision problems for primitive substitutions can be solved using return words and derived sequences [Durand]

Let $\mathcal{A}$, $\mathcal{B}$, be finite alphabets. We consider two morphisms $\sigma\colon \mathcal{A}^* \to \mathcal{A}^*$, $\phi\colon \mathcal{A}^* \to \mathcal{B}^*$; an infinite word of the form

$$\lim_n \sigma^n(u)$$

is is a D0L word and

$$\phi(\lim_n \sigma^n(u))$$

an HD0L or morphic word, for u finite word.

Decision problems for word substitutions

Some classical decision problems for primitive substitutions can be solved using return words and derived sequences [Durand]

Let σ be a primitive substitution. It generates a minimal subshift X_σ. A return word to a word u of its language is a word w of the language such that

uw admits exactly two occurrences of u, with the second occurrence of u being a suffix of uw.

One can recode sequences of the subshift via return words, obtaining derived sequences.

[cf D. Perrin's lecture]

Decision problems for word substitutions

Some classical decision problems for primitive substitutions can be solved using return words and derived sequences [Durand]

- The HD0L ω-equivalence problem for primitive morphisms: it is decidable to know whether two HD0L words are equal.
- The decidability of the ultimate periodicity of HD0L infinite sequences: it is decidable to know whether an HD0L word is ultimately periodic.
- The uniform recurrence of morphic sequences is decidable.

Decision problems for word substitutions

- Constant-length substitutions (automatic sequences): decision procedures are produced based on the connections between first-order logic and automata [Shallit-Walnut]
- It is decidable if the fixed-points of a morphism avoid (long) abelian powers (no eigenvalue equal to 1) [Rao-Rosenfeld]
- Consistency of multidimensional combinatorial substitutions [Jolivet-Kari]
- Decidability of topological properties for two-dimensional self-affine tiles [Jolivet-Kari]

Effectiveness for S-adic systems

Effectiveness for S-adic subshifts

- Directive sequences
- Pattern frequencies/invariant measure
- Language
- Existence of (decorated) local rules (being sofic or an SFT)

There are mainly two difficulties which come from

- the notion of substitution in dimension d
- the S-adic framework

A natural viewpoint since the characterization of entropy as right-recursively enumerable numbers [Hochman-Meyerovitch] [cf. R. Pavlov's lecture]

Effectiveness for S-adic shifts

Theorem [B.-Fernique-Sablik] Let X_S be a strictly ergodic S-adic subshift defined with respect to a directive sequence $S \in \mathfrak{S}^{\mathbb{N}}$ such that $\mathfrak{S}$ satisfies the good growing property. The following conditions are equivalent:

- there exists a computable sequence S' such that $X_S = X_{S'}$;
- the unique invariant measure of X_S is computable;
- the subshift X_S is decidable.

Good growing substitution

- A finite set of substitutions $\mathfrak{S}$ has a good growing property if
  - there are finitely many ways of gluing super-tiles: there exists a finite set of patterns $\mathcal{P} \subset \mathcal{A}^*$ such that if a pattern formed by a super-tile of order n surrounded by super-tiles of order n is in the language of $X_{\mathfrak{S}^{\mathbb{N}}}$, then it appears as the n-iteration of a pattern of $\mathcal{P}$
  - the size of the super-tiles of order n grows with n: for every ball of radius R, there exist $n \in \mathbb{N}$ such a translate of this ball is contained in all the supports of super-tiles of order n.
- Non-trivial rectangular substitutions or geometrical tiling substitutions verify this property.

Local rules and substitutions

- In dimension $d \geq 2$, under natural assumptions, it is known for different types of substitutions that substitutive tilings can be enforced with (colored) local rules.
- The ideas is always to force a hierarchical structure, as in Robison's tiling, where each change of level is marked by the type of the super-tile of this level, and the rule used is transmitted for super-tiles of lower order.
  - Rectangular substitutions: [Mozes'89]
  - Geometrical substitutions: [Goodman-Strauss'98]
  - Combinatorial substitutions: [Fernique-Ollinger'2010]

Existence of local rules

A closed subset $\mathbf{S} \subset \mathfrak{S}^{\mathbb{N}}$ is effectively closed if the set of (finite) words which do not appear as prefixes of elements of $\mathbf{S}$ is recursively enumerable.

One enumerates forbidden prefixes.

- Theorem [Aubrun-Sablik] We consider rectangular substitutions. The $\mathbf{S}$-adic subshift $X_{\mathbf{S}}$ is sofic if and only if it can be defined by a set of directive sequences $\mathbf{S}$ which is effectively closed.

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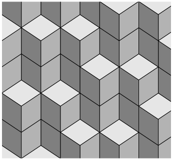
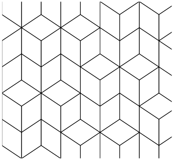
- A similar result for more general substitutions is expected
- The difficulty relies in the ability to exhibit a rectangular grid to use the simulation of a one-dimensional effective subshift by a two-dimensional sofic subshift [Aubrun-Sablik, Durand-Romanschenko-Shen]

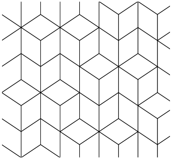
Sturmian and planar tilings

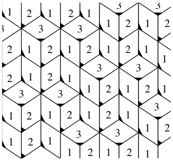
Tilings and symbolic dynamics





From a discrete plane to a tiling by projection....





....and from a tiling by lozenges to a ternary coding

Two-dimensional Sturmian words

Theorem [B.-Vuillon]

Let $(u_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^2} \in \{1,2,3\}^{\mathbb{Z}^2}$ be a 2d Sturmian word, that is, a coding of a discrete plane. Then there exist $x \in \mathbb{R}$, and α, $\beta \in \mathbb{R}$ such that $1, \alpha, \beta$ are $\mathbb{Q}$-linearly independent and $\alpha + \beta < 1$ such that

$$\forall \mathbf{m} = (m, n) \in \mathbb{Z}^2,\ U_{\mathbf{m}} = i \Longleftrightarrow R_\alpha^m R_\beta^n(x) = x + m\alpha + n\beta \in I_i \pmod 1,$$

with

$$I_1 = [0, \alpha[,\ I_2 = [\alpha, \alpha+\beta[,\ I_3 = [\alpha+\beta, 1[$$

or

$$I_1 =]0, \alpha],\ I_2 =]\alpha, \alpha+\beta],\ I_3 =]\alpha+\beta, 1].$$

Coding of a $\mathbb{Z}^2$-action

Factors

- The block $W = [w_{i,j}]$, defined on $\{1,2,3\}$ and of size (m,n), is a factor of u if and only if

$$I_W := \bigcap_{1\leq i\leq m, 1\leq j\leq n} R_\alpha^{-i+1} R_\beta^{-j+1} I_{w_{i,j}} \neq \emptyset.$$

- The sets I_W are connected.
- The frequency of every factor W of U exists and is equal to the length of $I(W)$.

Effective $2d$ Sturmian shifts

Theorem [B.-Bourdon-Jolivet-Siegel] A $2d$ Sturmian shift is S-adic with an expansion provided by Brun continued fraction algorithm

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Theorem [B.-Fernique-Sablik] The following conditions are equivalent:

- its normal vector is computable;
- its unique invariant measure is computable;
- its language is decidable;
- Its Brun S-adic directive sequence is computable.

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Theorem [Fernique-Sablik] A Euclidean plane E admits colored weak local rules if and only if it is computable: there is a sofic shift that contains planar tilings with a slope parallel to E with a bounded thickness.

Conclusion and perspectives

The following effectiveness notions for S-adic symbolic dynamical systems are intimately related

- Effectiveness of the directive sequences
- Computability of pattern frequencies/invariant measures
- Decidability of the language
- Existence of (colored) local rules (being sofic or an SFT)

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How to extend these results?

- Can one extend Durand's approach in a multidimensional setting?
- Extend Aubrun-Sablik result on the connections between soficity and effectiveness of the directive sequence
- One has to formulate suitable assumptions on substitutions