Maximally Dense Sphere Packings

Thomas Fernique

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Outline

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Thue's theorem

A packing of spheres in $\mathbb{R}^n$ is a union of interior disjoint balls.
Its density is defined by

$$\delta := \limsup_{k\to\infty} \frac{\text{area of } B_n(0,k) \text{ covered by the spheres}}{\text{volume of } B_n(0,k)}.$$

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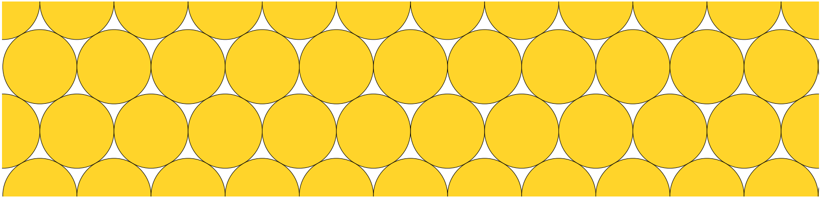

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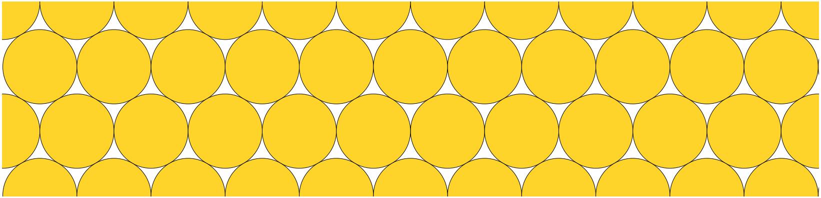

Theorem (Thue 1910, Tóth 1943)

Any packing of equal circles in $\mathbb{R}^2$ has density at most $\frac{\pi}{2\sqrt{3}}$.

Proof (following H.-C. Chang & L.-C. Wang, 2010)

Delaunay triangulation: circumscribed circles have empty interior.

Saturated packing: no further circle can be added.

Lemma

In a Delaunay triangulation of the centers of a saturated packing of unit circles, the density within each triangle is at most $\frac{\pi}{2\sqrt{3}}$.

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- If $\widehat{A} \geq \frac{2\pi}{3}$, then its smallest angle, say $\widehat{B}$, satisfies $\widehat{B} \leq \frac{\pi}{6}$. This yields a circumradius $R = \frac{\overline{AC}}{2\sin\widehat{B}} \geq 2$: a new circle can thus be added at the circumcenter. Contradiction $\rightsquigarrow \widehat{A} < \frac{2\pi}{3}$.

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- The area is $\frac{1}{2}\sin\widehat{A}\cdot\overline{AB}\cdot\overline{AC} \geq \sqrt{3}$, covered by half a unit disc.

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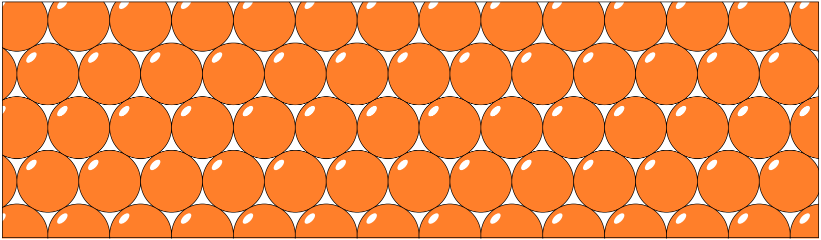
The Kepler conjecture

In $\mathbb{R}^3$, close-packings of equal spheres achieve the density $\frac{\pi}{3\sqrt{2}}$:



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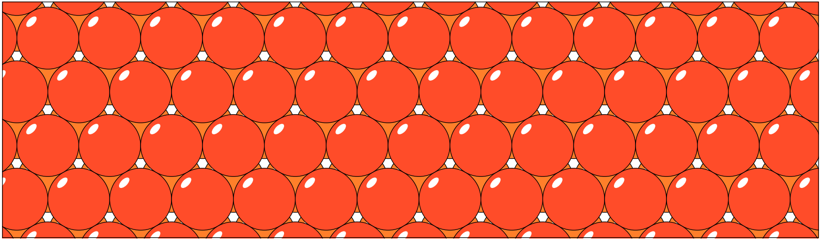

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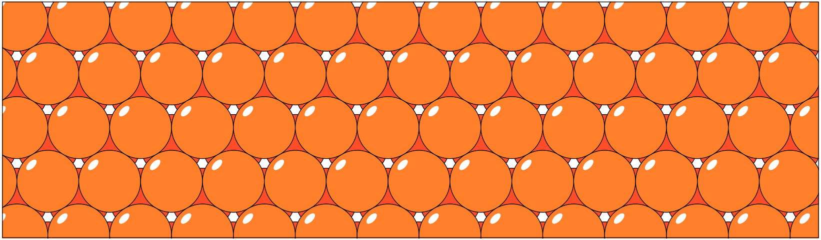

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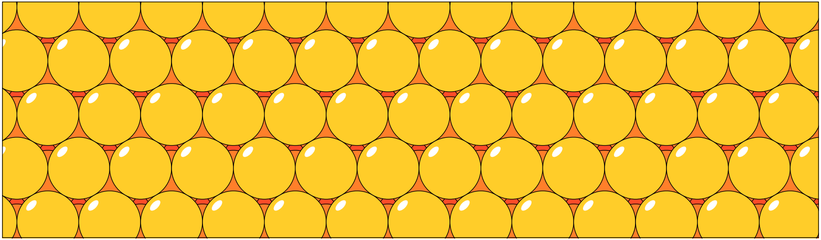

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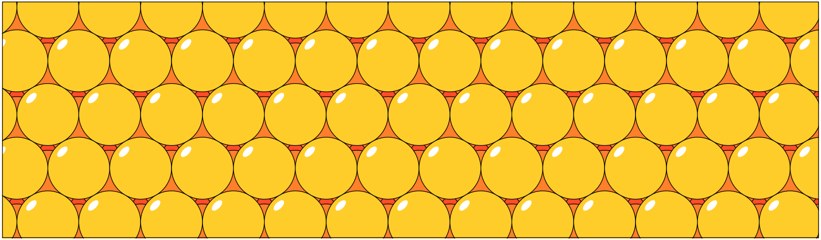

Theorem (Hales-Ferguson, 1992–2014)

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One could try to mimic the proof of Thue's theorem:

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The global optimum (densest packing) is not locally optimal.

This lemma only yields an upper bound on the optimal density:

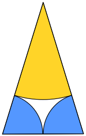
$$\sqrt{2}\arccos\tfrac{23}{27} \approx 0.7796 > \tfrac{\pi}{3\sqrt{2}} \approx 0.7405.$$

More frustration

Already in $\mathbb{R}^2$, this situation is the norm for unequal circles:

Theorem (Florian, 1960)

In a Delaunay triangulation of the centers of a saturated packing of unequal circles, the densest possible triangle connects the centers of two smallest circles and a largest one, all mutually tangent.



There is no decomposition made of such triangles only (exercise). Again, this only yields an upper bound on the optimal density.

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Even better:
Can we express it in the elementary language of the real numbers (quantifiers, logical connectives, $+$, $-$, $\times$, variables, constants)?

$\rightsquigarrow$ This would, at least, be decidable (Tarski, 1931)

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Each local density inequality yields an upper bound on the density.

Localization in practice

We have to find a local density inequality which

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For unequal circles (see suite), I used an enhanced mix of the localizations proposed by Heppes in 2003 and Kennedy in 2004.

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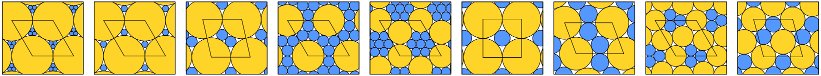
There are exactly 9 sizes allowing a triangulated binary packing.



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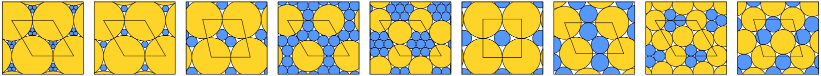

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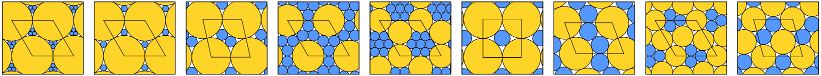

Theorem (Heppes 2000-03, Kennedy 2004, Bédaride-F. 2020)

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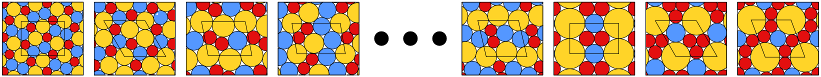
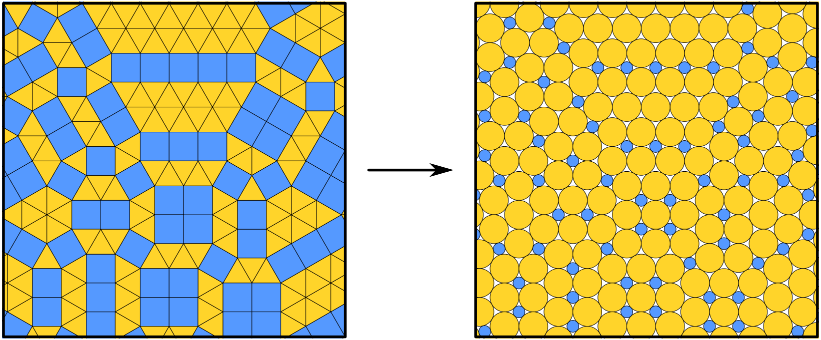
Theorem (Fernique-Hashemi-Sizova, 2019)

There are exactly 164 *sizes allowing a triangulated ternary packing.*



In some cases (not all), the density is maximized by such a packing.

Playing with stoichiometry



Theorem (F. 2020)

The densest packings with circles of size 1 *and* $\sqrt{2}-1$ *are*

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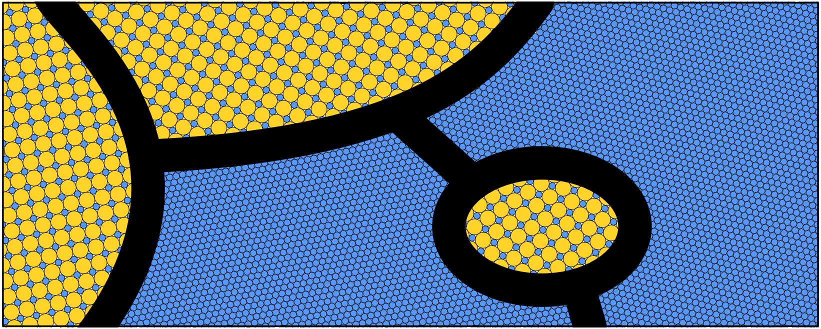

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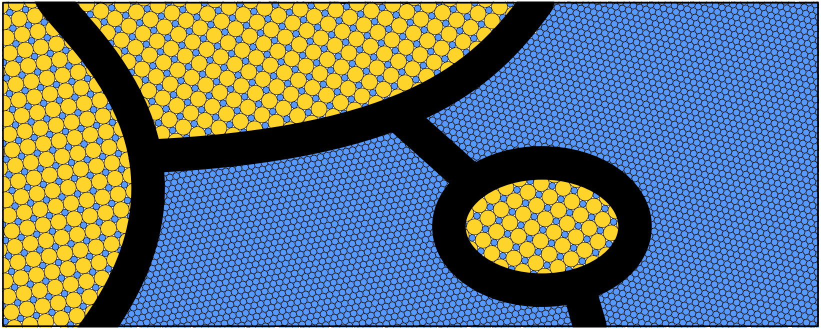

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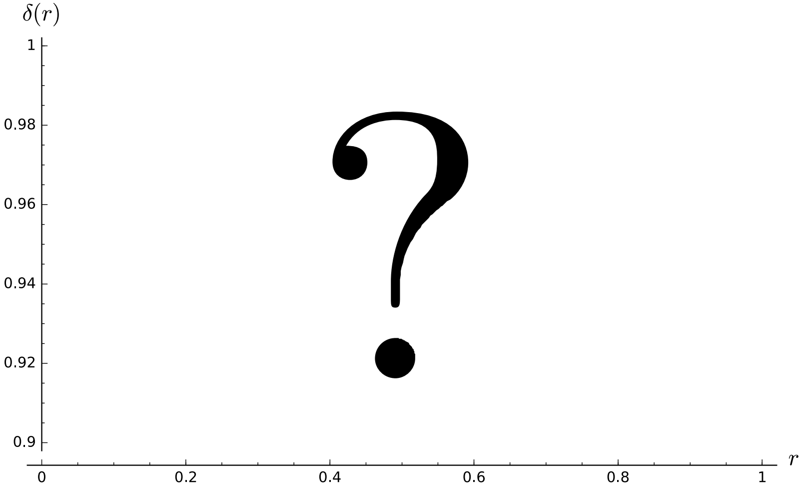
The maximum density as a function of the stoichiometry follows.

Global picture for two sizes of circle



Maximal density $\delta(r)$ of packings of circles of size 1 and r?

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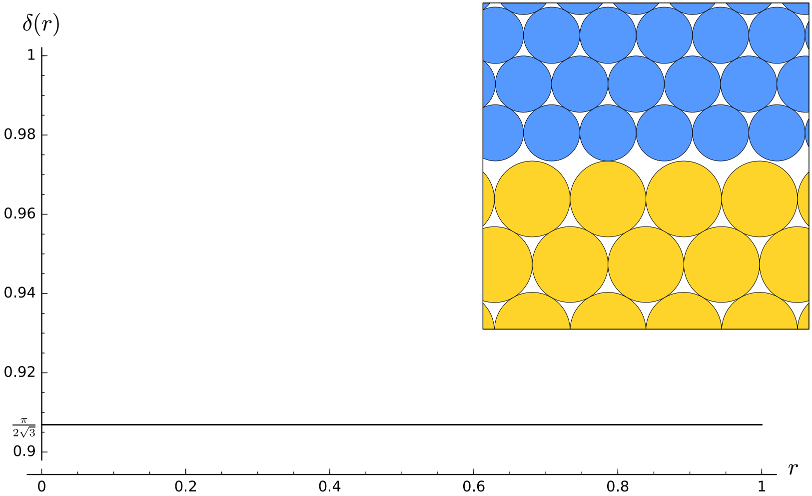

Trivial lower bound: the density of the hexagonal compact packing.

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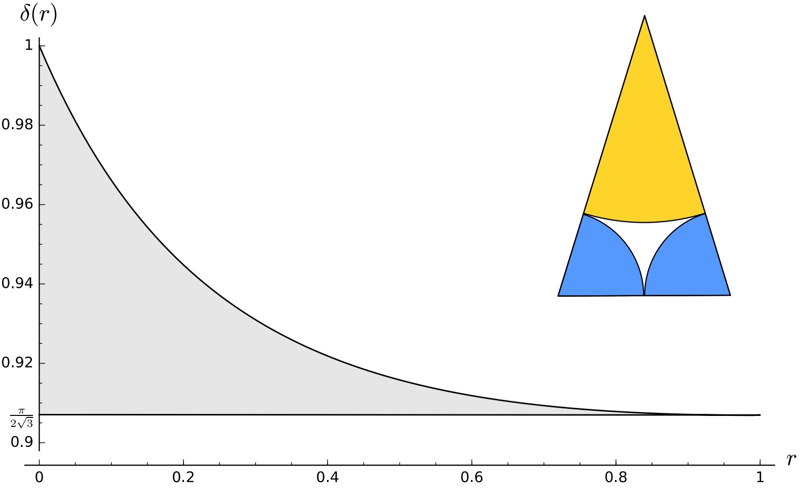

The already mentionned Florian's upper bound.

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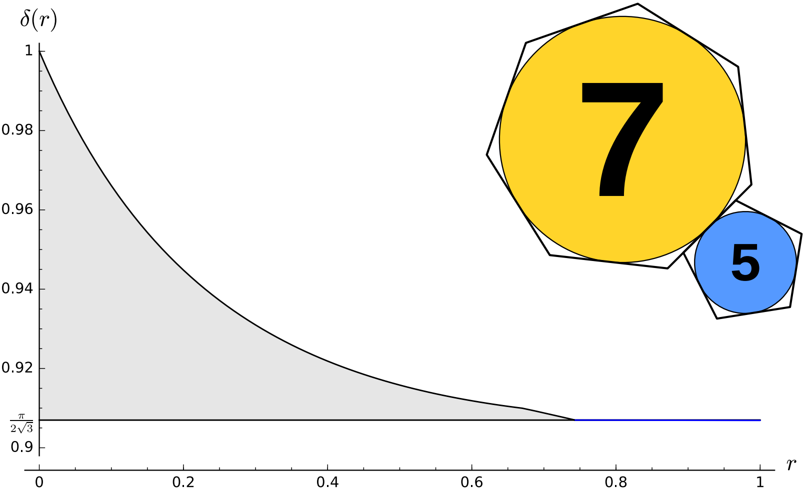

Blind's upper bound (1969) yields a sharp bound over $[0.74\ldots, 1]$.

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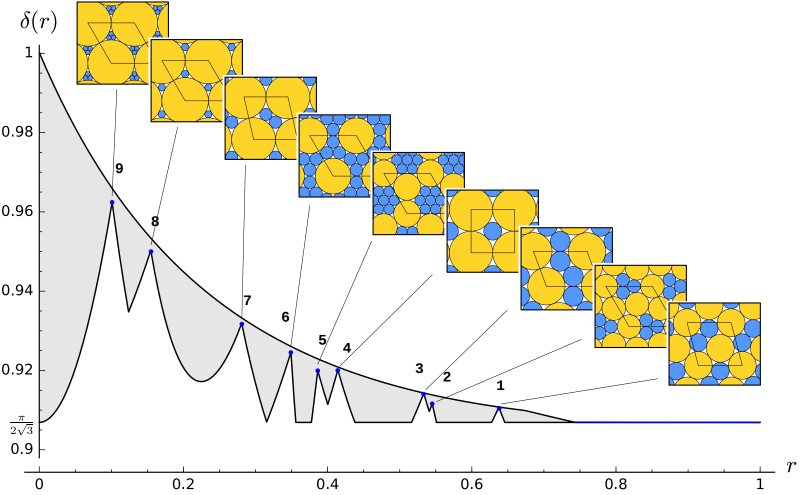

Flip & flow on the 9 triangulated binary packing $\rightsquigarrow$ lower bound.

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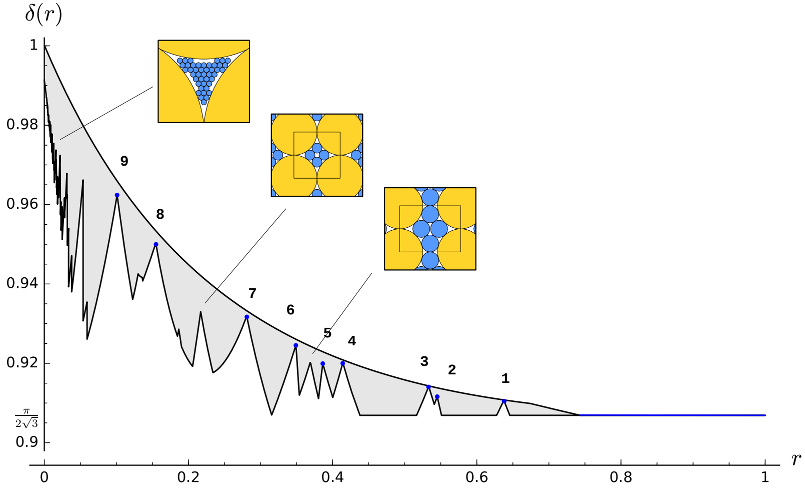

Flip & flow on any suitable packing improves the lower bound.

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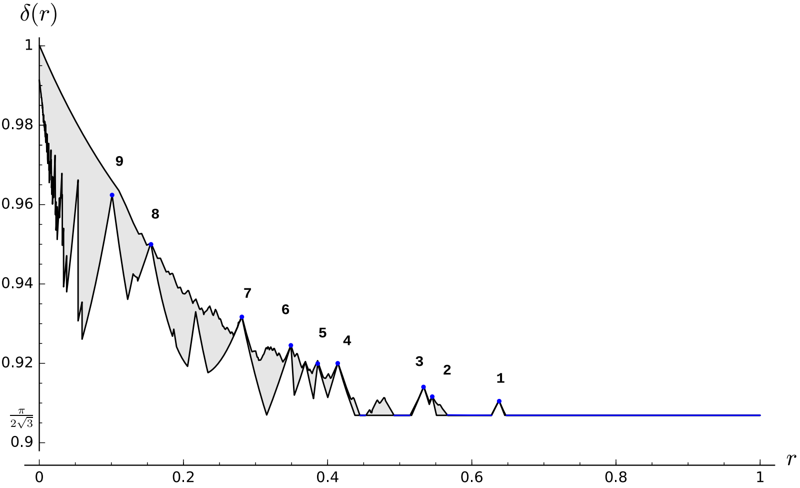
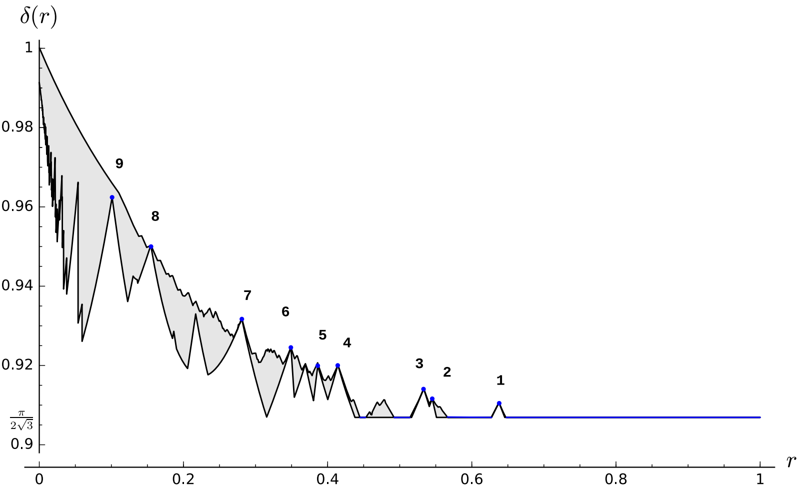


Massive computations for localization improve the upper bound.

Outline

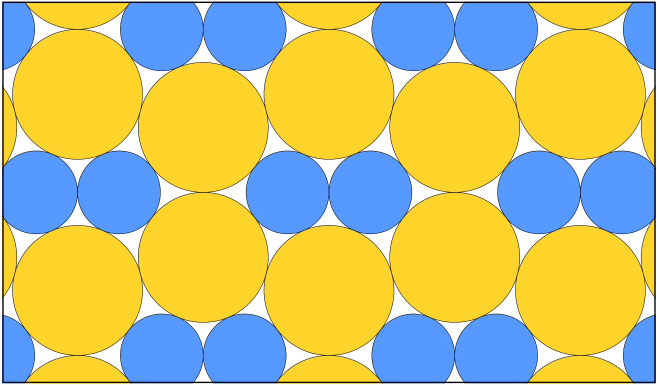
The Complete picture for two sizes of circle



The localization is challenged by nontriangulated optimal packings and circles that fit well locally (around a circle), but not globally.

The most uniform packing

What is the largest r such that circles with sizes in $[r, 1]$ can be packed more densely than all equal circles?



$r \geq 0.6375\ldots$ (Tóth, 1964) $r \leq 0.7430\ldots$ (Blind, 1969).

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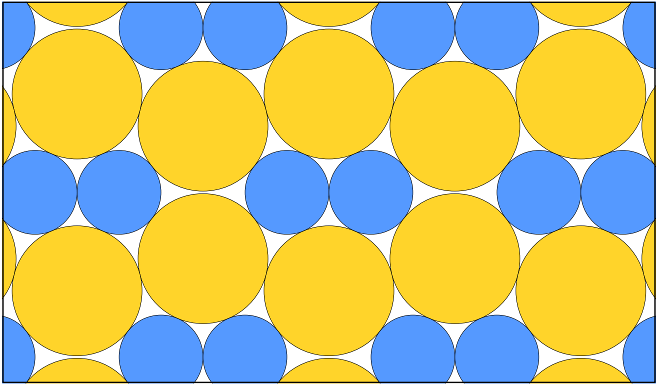

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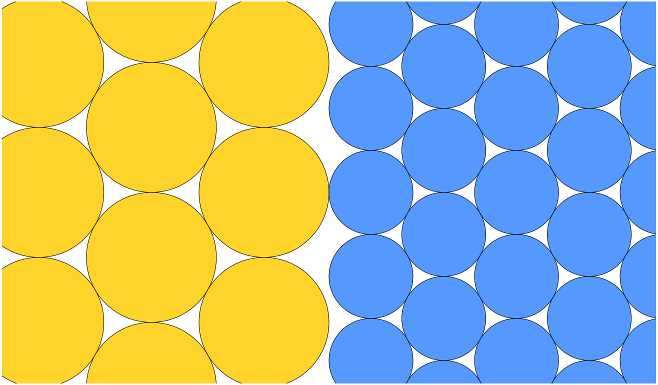

$r < 0.6464$ if there are only two sizes of circle (Fernique, 2020).

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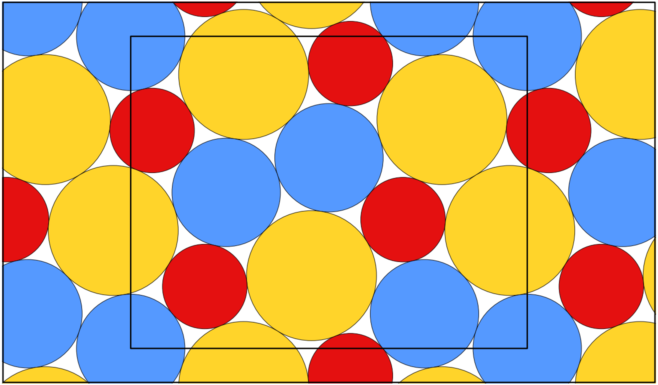

$r \geq 0.6510\ldots$ (Fernique-Hashemi-Sizova, 2018).

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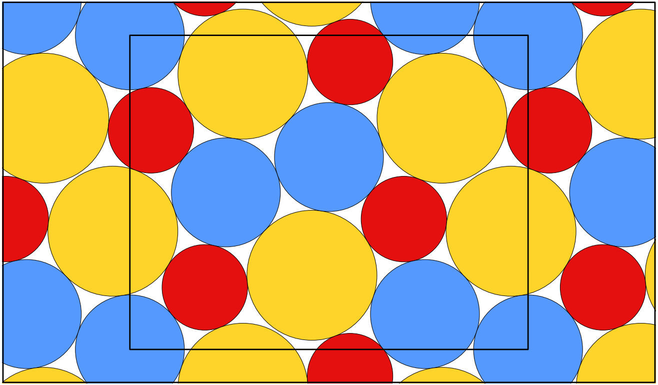

$r \geq 0.6585\ldots$ (Connelly-Pierre, 2019).

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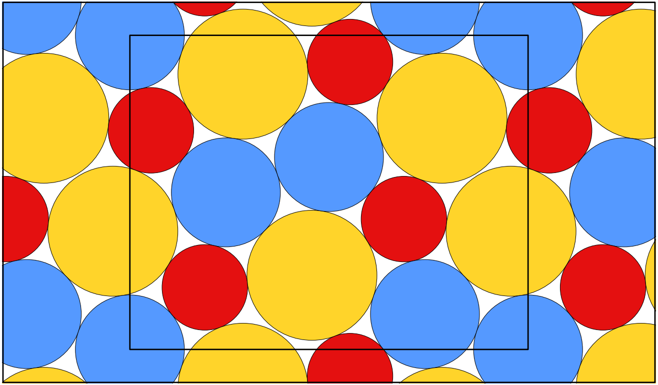

$r \geq 0.6585\ldots$ (Connelly-Pierre, 2019). Optimal? (Connelly-Zhang)

An aperiodic densest packing

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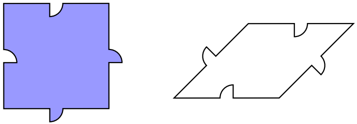
Theorem (Bédaride-Fernique, 2012)

The tilings *with the highest proportion of rhombus are aperiodic:*



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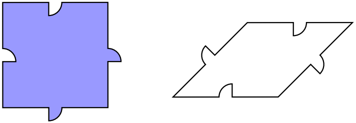
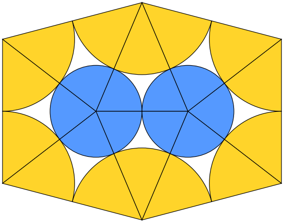


Density for triangulated packings $\sim$ tile proportions for tilings:



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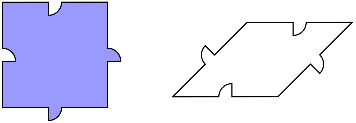

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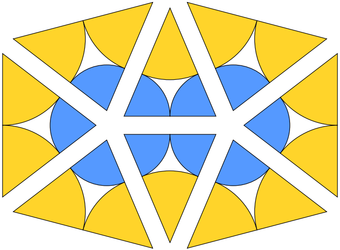

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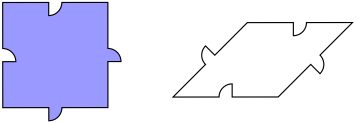

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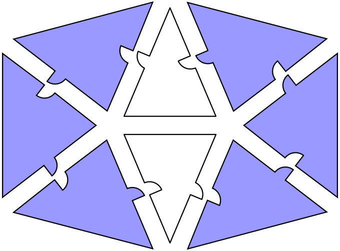

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