Markov Chains on Tilings: From Chaos to Order

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1. Order

2. Chaos

3. From chaos to order
1 Order

2 Chaos

3 From chaos to order
Quasicrystals and tilings

**Definition (IUCr, 1992)**
Crystal $=$ ordered material $=$ essentially discrete diffraction.


The periodic crystals are usually modelled by patterns on lattices.
The non-periodic ones have quickly been modelled by *tilings*.
Quasicrystals and tilings

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The periodic crystals are usually modelled by patterns on lattices. The non-periodic ones have quickly been modelled by tilings.

Definition

Tiling = covering of the plane by non-overlapping compact sets.

Example: digitizations of affine planes in higher dimensional space.

Quasicrystal stability (at low $T$): finite range energetic interaction. Modelled on tilings by constraints on the way things locally fit.
Example 1: Dimer tilings

Rows alternate rhombi (between their orientation):
Example 1: Dimer tilings

Rows alternate rhombi (between their orientation):
Example 2: Beenker tilings

Rows alternate rhombi, squares are free:
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Rows alternate rhombi, squares are free:
Example 3: generalized Penrose tilings

Rows alternate rhombi of a given type, different types freely mix:
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Melt and random tilings

First quasicrystals: rapid cooling of the melt (quenching). At high $T$: stabilization by entropy rather than energy.
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Definition (Configurational entropy of a tiling $T$)

$$S(T) := \log(\text{nb tilings of the same domain as } T)/\text{nb tiles in } T.$$
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**Definition**

Flip on a vertex $x$: half-turn a hexagon of three rhombi sharing $x$.

Diffusion: flips on random vertices with probability $\exp(-\Delta E / T)$.
Cooling and stochastic flips

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This is the Metropolis-Hastings algorithm for the Gibbs distribution

$$P(\text{tiling}) = \frac{1}{Z(T)} \exp \left( - \frac{E(\text{tiling})}{T} \right).$$

Chaos for high $T$, order for low $T$, but what about convergence?
Example 1: Dimer tilings

Ergodic at $T > 0$, $\Theta(n^2 \ln n)$ mixing at $T = \infty$, $O(n^{2.5})$ at $T = 0$. 
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