Tilings of a polycell : Algorithmic and structural aspects

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May 27, 2004

We consider the following definition of tiling (see Figure 1):

> Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a set of circuits (called *cells*) of a directed graph G. Let E be the set of all (directed) edges in $\mathcal{C}$ and I a subset of E. We call *inner edges* the edges of I and *boudary edges* those of $B = E \backslash I$. For $a \in I$, $Cell(a)$ denotes the subset of $\mathcal{C}$ formed by the cells which use the edge a. A *tiling* of $(\mathcal{C}, I)$ is a subset T of I such that $\{Cell(a)\}_{a \in T}$ is a partition of $\mathcal{C}$.

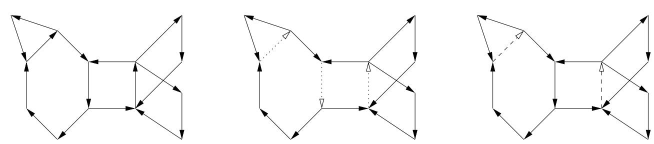

Figure 1: From left to right: a polycell $\mathcal{C}$ of 5 cells (the chordless circuits); 3 inner edges (dashed); a tiling (dashed edges).

This definition, introduced in [BL03], extends the classical planar tilings with dominoes or lozenges ([Thu90], [DMRR03], [Fou03] and [Thi03]), and also the codimension-one tilings ([DMB97]). Figure 2 shows the case with dominoes.

Figure 2: Tiling of a polyomino with dominoes: each square is mapped to a cell of $\mathcal{C}$, whose orientation depends on the color of the square; the inner edge separates two squares, and to each edge of a tiling corresponds the domino formed by the two cells containing this edge.

We are interested in both a **descriptive** approach (conditions for existence of a tiling and structure of the whole set of the tilings for a fixed couple $(\mathcal{C}, I)$)

and a **constructive** approach (algorithms to construct a tiling, to generate the set of the tilings and to perform random sampling upon).

[Pro93] describes the structure of the set of the tilings, within a framework barely less general than ours, but only with non-constructive methods. Recent works ([DMRR03], [BL03], [Thi03] or [Fou03]) provide more algorithmic approaches, but with noticeable loss of generality (planar tilings with dominoes, lozenges or k-regular cells). Our aim is to unify these results within a more general and simplier framework.

In order to do that, we will define some basic notions (see Figures 3 and 4):

- a *counter* is a weight-function on the edges such that the weight over each cell of $\mathcal{C}$ is equal to one. Those with 0-1 values (only 0 for boundary edges) correspond to tilings;
- the *height function* of a counter maps each vertex to the weight (by this counter) of a shortest path from a fixed vertex to this vertex itself;
- a *fence* is a strongly connected component of the graph obtained by removing the edges T of a tiling (one proves that it does not depend on the tiling). In a sense, fences split a polycell into independly-tileable polycells;
- a *flip* (on a fence) is an elementary operation which transforms a tiling into another tiling. There are *local* and *global* flips.

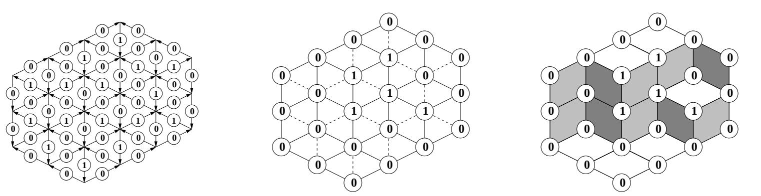

Figure 3: From left to right: a counter with 0-1 values on triangular cells; the associated tiling (dashed edges) and height function; a more visual interpretation of height functions.

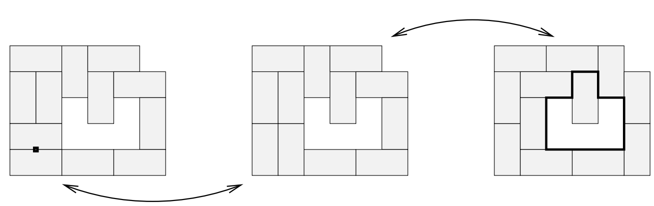

Figure 4: Local flip (left) and global flip (right) on tilings with dominoes of a domain with a hole (used fences are drawn in a thicker way).

We thus obtain the following results:

- the flip operation induces a structure of (finite) distributive lattice over the set of tilings. Consequently, results about this well-studied structure can be applied to our purpose (see [DP90] for general lattice theory);
- an algorithm to construct a tiling (minimal, in a sense to be specified), whose complexity depends on the considered class of graphs. For planar tilings, result in [FR01] leads to a $\mathcal{O}(n \log^3(n))$ algorithm;
- an algorithm to generate the whole set of the tilings and one to perform random sampling on this set ([DR03], [Pro97]).

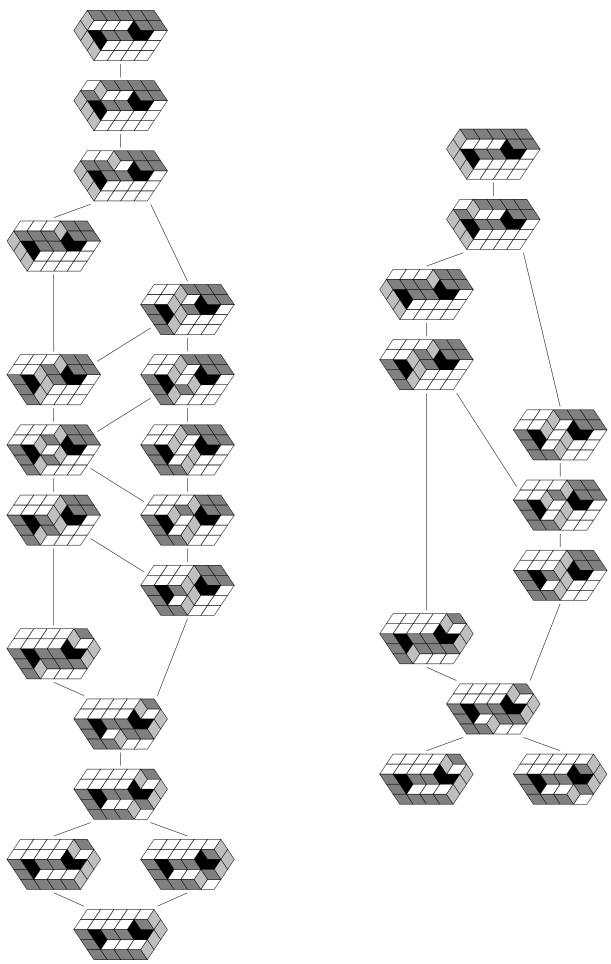

Figure 5: Distributive lattice of the tilings of a polycell with lozenges (left) and a more compact isomorphic representation, namely the Birkhoff's one (right).

References

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