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Kleene stars of the plane, polylogarithms and symmetries

G.H.E. Duchamp [a], V. Hoang Ngoc Minh [b], Ngo Quoc Hoan [c]

[a] LIPN, University Paris Nord, 93800, Villetaneuse, France
[b] Lille II University, 59024 Lille, France
[c] University of Hai Phong, 171, Phan Dang Luu, Kien An, Hai Phong, Viet Nam

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ABSTRACT

We extend the definition and construct several bases for polylogarithms Li_T, where T are some series, recognizable by a finite state (multiplicity) automaton of alphabet[1] $X = \{x_0, x_1\}$. The kernel of this new "polylogarithmic map" $\mathrm{Li}_\bullet$ is also characterized and provides a rewriting process which terminates to a normal form. We concentrate on algebraic and analytic aspects of this extension allowing index polylogarithms at non-positive multi-indices, by rational series and regularize polyzetas at non-positive multi-indices[2].

1. Introduction

As a matter of fact the interest of rational series, over the alphabets $Y_0 = \{y_n\}_{n\in\mathbb{N}}$, $Y = Y_0 \setminus \{y_0\}$ and $X = \{x_0, x_1\}$, is twofold: algebraic and analytic.

Firstly, (from the algebraic point of view) these series are closed under shuffle products and the shuffle exponential of letters (and their linear combinations) is precisely their Kleene star.[3] Secondly, the growth of their coefficients is tame[4] [11,25,26]. [11,25,26] and as such their associated polylogarithms can be rightfully computed [20,23,24].

Doing this, we recover many functions (as simple polynomials, for instance) forgotten in the straight algebra of polylogarithms at positive indices, which can be viewed as the image of the following isomorphism of algebras [21]

$$\begin{aligned} \mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle, ⧢, 1_{X^*}) &\longrightarrow (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1),\\ x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 &\longmapsto \mathrm{Li}_{s_1,\ldots,s_r},\\ \forall n\geq 0, x_0^n &\longmapsto \frac{\log^n(z)}{n!}. \end{aligned}$$

E-mail addresses: gheduchamp@gmail.com (G.H.E. Duchamp), hoang@univ-lille2.fr (V. Hoang Ngoc Minh), quochoan_ngo@yahoo.com.vn (N.Q. Hoan).

[1] The space of rational series considered here is $(\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle, ⧢, 1_{X^*})$.

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[3] i.e. for any $S \in \mathbb{C}\langle\langle X\rangle\rangle$ such that $\langle S \mid 1_{X^*}\rangle = 0$, S^* denotes the sum $1_{X^*} + S + S^2 + S^3 + \ldots$ and is called its Kleene star (see [1]).

[4] i.e. for such a rational series S over X, there exists a real number $K > 0$ such that, for any $w \in X^*$, the coefficient $|\langle S \mid w\rangle|$ is bounded from above by $K|w|$.

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To study multi-indexed polylogarithms, one relies on the one-to-one correspondence between the multi-indices $(-s_1,\dots,-s_r)$, in $\mathbb{Z}^r_{\le 0}$ (or $(s_1,\dots,s_r)\in\mathbb{N}^r_+$), and the words $y_{s_1}\dots y_{s_r}$, in the monoid Y_0^*, indexing polylogarithms by $y_{s_1}\dots y_{s_r}$ as follows [10,11,21,22]:

$$\mathrm{Li}_{y_{s_1}\dots y_{s_r}}=\mathrm{Li}_{s_1,\dots,s_r} \text{ and } \mathrm{Li}^-_{y_{s_1}\dots y_{s_r}}=\mathrm{Li}_{-s_1,\dots,-s_r}.$$

We will explain the whole project to extend $\mathrm{Li}_\bullet$ over a sub algebra of rational power series. In particular, we study here various aspects of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$, where $\mathcal{C}$ denotes the ring of polynomials in z,z^{-1} and $(1-z)^{-1}$, with coefficients in $\mathbb{C}$, and we will express polylogarithms (resp. harmonic sums) at negative multi-indices as polynomials in $(1-z)^{-1}$ (resp. $N\in\mathbb{N}$), with coefficients in $\mathbb{Z}$ (resp. $\mathbb{Q}$).

We will concentrate, in particular, on algebraic and analytic aspects of this extension allowing index polylogarithms at non-positive multi-indices by rational series and regularize (divergent) polyzetas at non-positive multi-indices.

The paper is structured as follows:

(i) In Section 2, we will provide some background consisting in the algebraic combinatorial framework on which the first structures of polylogarithms and harmonic sums rely, namely their indexation by words and then by (non-commutative) polynomials.
(ii) In Section 3, we observe that, as such, the derivations, $z\frac{d}{dz}$ and $(1-z)\frac{d}{dz}$ are continuous (for the standard topology) and so are their sections $\int_{z_0}^{z}\frac{ds}{s}\bullet$ and $\int_{z_0}^{z}\frac{ds}{1-s}\bullet$ (for z_0 in a suitable non-void open domain) but, in order to satisfy the standard asymptotic condition (see eq. 14), we need other sections which we develop in this paragraph. We will then study the bi-integro-differential algebra and some functional analysis aspects of polylogarithms, here defined on the cleft complex plane $\mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$. We will give also a linear basis of the algebra of polylogarithms with respect to coefficients being $\mathbb{C}$-polynomials of $\{z,1/z,1/(1-z)\}$.
(iii) In Section 4, in order to extend, to classes of rational series, the indexation of $\mathrm{Li}_\bullet$, we review properties of rational series and the notation of rational expressions.
(iv) In Section 5, in order to study Kummer functional equations on polylogarithms, via their non-commutative generating series satisfying noncommutative differential equation, we define polylogarithms on the universal covering of $\mathbb{C}\setminus\{0,1\}$.
(v) Finally, in Section 6, the extended double regularization of divergent polyzetas, on $\mathbb{C}\langle X\rangle\sqcup\!\sqcup\,\mathbb{C}[x_0^*]\sqcup\!\sqcup\,\mathbb{C}[(-x_0^*)]\sqcup\!\sqcup\,\mathbb{C}[x_1^*]$ and $\mathbb{C}\langle Y\rangle \uplus \mathbb{C}[y_1^*]$, is obtained.

These studies will be applied to obtain solutions of KZ_3 and examples of associators with rational coefficients [9,27].

2. Polylogarithms and algebraic combinatorial framework

Let us, now, go into details, using the notations of [1,33],

(i) We construct the bialgebras.

$$(\mathbb{C}\langle X\rangle, conc, \Delta_{\sqcup\!\sqcup}, 1_{X^*},\varepsilon) \text{ and } (\mathbb{C}\langle Y_0\rangle, conc, \Delta_{\uplus}, 1_{Y_0^*},\varepsilon)$$

in which, for any $i=0,1$ and $j\ge 0$, one has

$$\Delta_{\uplus}(y_j)=y_j\otimes 1_{Y^*}+1_{Y^*}\otimes y_j+\sum_{k+l=j} y_k\otimes y_l,$$
$$\Delta_{\sqcup\!\sqcup}(x_i)=x_i\otimes 1_{X^*}+1_{X^*}\otimes x_i.$$

and *conc* is the usual *concatenation product* between noncommutative polynomials. Out of these two, only the first one is Hopf, the last one contains $1+y_0$ which is group-like and has no inverse (see infiltration product phenomenon in [2]) and therefore has no antipode.
(ii) Let $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ denote the closure of $\mathbb{C}\langle X\rangle$ by rational operations $\{+,.,^*\}$ [1,8] (it is closed by shuffle products). By Kleene-Schützenberger's theorem (see below paragraph 4.1), any power series S belongs to $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ if and only if it is *recognizable* by an automaton admitting a *linear representation* (β,μ,η) of dimension n, with

$$\beta\in\mathcal{M}_{1,n}(\mathbb{C}),\ \mu: X^*\longrightarrow\mathcal{M}_{n,n}(\mathbb{C}),\ \eta\in\mathcal{M}_{n,1}(\mathbb{C})$$

such that, for any $w\in X^*$, one has (see [1,8] and paragraph 4.1)

$$\langle S\mid w\rangle=\beta\mu(w)\eta.$$

(iii) Let us consider the following morphism of algebras, defined by

$$\pi_X:(\mathbb{C}\langle Y\rangle, conc, 1_{Y^*})\longrightarrow(\mathbb{C}\langle X\rangle, conc, 1_{X^*}),$$
$$y_{s_1}\dots y_{s_r}\longmapsto x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1.$$

It admits an adjoint π_Y for the two standard scalar products, i.e.

$$\forall p \in \mathbb{C}\langle X\rangle, \forall q \in \mathbb{C}\langle Y\rangle, \langle \pi_Y(p) \mid q\rangle_Y = \langle p \mid \pi_X(q)\rangle_X.$$

One checks immediately that $\pi_Y(x_0^{s-1}x_1) = y_s$, $\ker(\pi_Y) = \mathbb{C}\langle X\rangle x_0$ and π_Y restricted to the subalgebra $(\mathbb{C}\,1_{X^*} \oplus \mathbb{C}\langle X\rangle x_1, .)$ is an isomorphism, inverse of π_X.

In this work, unless symmetries are involved (i.e. until section 5) Ω denotes the cleft plane $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and $\mathcal{H}(\Omega)$, the set of holomorphic functions over the simply connected domain Ω.

The principal object of the present paper, as in [10,11], is the *polylogarithm* well defined, for any $(s_1, \ldots, s_r) \in \mathbb{C}^r, r \in \mathbb{N}_+$ and for any $z \in \mathbb{C}$ such that $|z| < 1$, by

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}.$$

So is the following Taylor expansion

$$\frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{s_1,\ldots,s_r}(N)\, z^N,$$

where the arithmetic function

$$\mathrm{H}_{s_1,\ldots,s_r} : \mathbb{N} \longrightarrow \mathbb{Q}$$

is expressed by

$$\mathrm{H}_{s_1,\ldots,s_r}(N) := \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}.$$

Here, $\mathrm{Li}_{s_1,\ldots,s_r}$ can also be obtained by iterated integrals (which provide for free their analytic continuations), along paths in Ω and with respect to the differential forms

$$\omega_0(z) = \frac{dz}{z} \text{ and } \omega_1(z) = \frac{dz}{1-z}.$$

Let $\mathcal{H}_r$ [17,36] denote the following domain

$$\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r; \Re(s_1) + \ldots + \Re(s_m) > m\}.$$

After a theorem by Abel, for any $r \geq 1$, if $(s_1, \ldots, s_r) \in \mathcal{H}_r$, we have

$$\zeta(s_1, \ldots, s_r) := \lim_{z \to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \lim_{N \to \infty} \mathrm{H}_{s_1,\ldots,s_r}(N).$$

These limits are no longer valid in the divergent cases and requires the renormalization of the corresponding divergent polyzetas. This has been done for the case of polyzetas at positive multi-indices [3,5,25] and in [15,18,30] and completed in [10,11] for the case of negative multi-indices.

The technique used in [10,11] (based on encoding polylogarithms, harmonic sums at positive multi-indices by words in Y^*) allows renormalize globally polyzetas at non-positive multi-indices (via their noncommutative generating series) but the regularization is not achieved yet. To do that, in the present work, we introduce the rational series as a new and extended encoding suitable to regularize these functions (see [27] for the analytical justification of such algebraic process). This technique is already presented in [12], as a preprint, but never was published before.

3. Bi-integro-differential algebra of polylogarithms

3.1. Differential rings

Let us consider the following group of transformations of $B = \mathbb{C} \setminus \{0, 1\}$ which permutes the singularities in $\{0, 1, +\infty\}$

$$\mathcal{G} := \{z \mapsto z, z \mapsto 1-z, z \mapsto z^{-1}, z \mapsto (1-z)^{-1}, z \mapsto 1-z^{-1}, z \mapsto z(z-1)^{-1}\}$$

and let us also consider the following rings :

$$\mathcal{C}_0' := \mathbb{C}[z^{-1}], \mathcal{C}_1' := \mathbb{C}[(1-z)^{-1}], \mathcal{C}_0 := \mathbb{C}[z, z^{-1}], \mathcal{C}_1 := \mathbb{C}[z, (1-z)^{-1}],$$
$$\mathcal{C}' := \mathbb{C}[z^{-1}, (1-z)^{-1}], \mathcal{C} := \mathbb{C}[z, z^{-1}, (1-z)^{-1}], \tag{1}$$

which are differential rings, endowed with the differential operator $\partial_z := d/dz$ and with the neutral element $1_\Omega : \Omega \to \mathbb{C}$, mapping z to $1_\Omega(z) = 1$. It follows that

Lemma 3.1. *One has the following properties*

(i) *For* $i = 0$ *or* 1,

$$\mathcal{C}'_i \subsetneq \mathcal{C}_i \subsetneq \mathcal{C} \text{ and } \mathcal{C}'_i \subsetneq \mathcal{C}' \subsetneq \mathcal{C}.$$

(ii) *The differential ring* $\mathcal{C}$ *is closed under action of* $\mathcal{G}$*:*

$$\forall G(z) \in \mathcal{C}, \forall g \in \mathcal{G}, G(g(z)) \in \mathcal{C}.$$

(iii) *The subrings* $\mathcal{C}_0, \mathcal{C}_1$ *are closed by the involutions* $\{z \mapsto z^{-1}, z \mapsto 1 - z\}$ *and are exchanged by* $\{z \mapsto 1 - z^{-1}, z \mapsto z(z-1)^{-1}\}$*, respectively.*

Proof.

(i) It is immediate from the definitions of $\mathcal{C}, \mathcal{C}_i, \mathcal{C}'_i, i = \{0, 1\}$ in (1).
(ii) This is an easy consequence of the fact that the element $G \in \mathcal{C}$ can be represented in the form:

$$G(z) = \sum_{n=1}^{N_1} \frac{1}{z^n} + \sum_{m=-N_2}^{N_3} (1-z)^m, N_1, N_2, N_3 \in \mathbb{N}.$$

(iii) It is immediate from the definitions. □

3.2. Differential and integration operators

Now, let us consider also the differential operators, acting on $\mathcal{H}(\Omega)$ [26]:

$$\theta_0 = z\frac{d}{dz} \text{ and } \theta_1 = (1-z)\frac{d}{dz}$$

and integration operators

$$\iota_0^{z_0}(f) = \int_{z_0}^{z} f(s)\omega_0(s) \text{ and } \iota_1^{z_0}(f) = \int_{z_0}^{z} f(s)\omega_1(s).$$

One has $\theta_i \iota_i^{z_0} = \mathrm{Id}_{\mathcal{H}(\Omega)}$ (sections of the θ_i).
One has other sections of θ_i, defined on $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ named ι_i (without superscripts).

They are in fact, much more interesting (and adapted to the explicit computation of associators), these operators (ι_i without superscripts), mentioned in the introduction are (more rigorously) defined by means of a $\mathbb{C}$-basis of

$$\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*} = \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

Now, we recall that a word is *Lyndon* if it is always less (for the lexicographic ordering defined by $x_0 < x_1$) than its proper right factors. Their set, noted $\mathcal{L}yn(X)$, is a transcendance basis of the shuffle algebra $(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})$ (Radford's theorem [31]). Then

$$\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \cong \mathbb{C}[\mathcal{L}yn(X)],$$

one can partition the alphabet of this polynomial algebra in

$$(\mathcal{L}yn(X) \cap X^*x_1) \sqcup \{x_0\}$$

and then get the decomposition

$$\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*} \simeq \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*x_1} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in x_0^*}.$$

Using the following identity [20],

$$ux_1x_0^n = ux_1 ⧢ x_0^n - \sum_{k=1}^{n} (u ⧢ x_0^k)x_1x_0^{n-k},$$

we get

$$ux_1x_0^n = \sum_{m=0}^{n} P_m x_1 \sqcup\!\sqcup\, x_0^m,$$

where $P_m \in \mathbb{C}\langle X\rangle$ is uniquely defined by the above. Thus

$$\mathrm{Li}_{ux_1x_0^n}(z) = \sum_{m\le n} \mathrm{Li}_{P_m x_1}(z)\frac{\log^m(z)}{m!}\ .$$

This means that

$$\begin{aligned}\mathcal{B} := &(z^k \,\mathrm{Li}_{ux_1}(z)\,\mathrm{Li}_{x_0^n}(z))_{(k,n,u)\in\mathbb{Z}\times\mathbb{N}\times X^*}\\ &\sqcup\ ((1-z)^{-l}\,\mathrm{Li}_{ux_1}(z)\,\mathrm{Li}_{x_0^n}(z))_{(l,n,u)\in\mathbb{N}^+\times\mathbb{N}\times X^*}\\ &\sqcup\ (z^k\,\mathrm{Li}_{x_0^n}(z))_{(k,n)\in\mathbb{Z}\times\mathbb{N}}\\ &\sqcup\ ((1-z)^{-l}\,\mathrm{Li}_{x_0^n}(z))_{(l,n)\in\mathbb{N}^+\times\mathbb{N}},\end{aligned}$$

is a $\mathbb{C}$-basis of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$. With this basis, we can define the operator ι_0 as follows

Definition 3.2. Define the index map $\mathrm{ind} : \mathcal{B} \to \mathbb{Z}$ by

$$\mathrm{ind}(z^k(1-z)^{-l}\,\mathrm{Li}_{x_0^n}(z)) = k \text{ and } \mathrm{ind}(z^k(1-z)^{-l}\,\mathrm{Li}_{ux_1}(z)\log^n(z)) = k + |ux_1|.$$

Then ι_0 is computed as follows

$$\iota_0(b) = \begin{cases} \displaystyle\int_0^z b(s)\omega_0(s), & \text{if } \mathrm{ind}(b) \ge 1,\\ \displaystyle\int_1^z b(s)\omega_0(s), & \text{if } \mathrm{ind}(b) \le 0.\end{cases}$$

and, as $z \ne 1$, ι_1 is defined by

$$\iota_1(f) = \int_0^z f(s)\omega_1(s)$$

We will see in section 3.3 that ι_0 is discontinuous. Nevertheless the pair $\{\iota_0, \iota_1\}$ is adapted to computation of the special solution $\mathrm{Li}_\bullet$. One can check easily the following properties.

Proposition 3.3 *([10,11,23,24]). One has the following properties*

(i) *The operators $\{\theta_0, \theta_1, \iota_0, \iota_1\}$ satisfy in particular,*

$$\theta_1 + \theta_0 = [\theta_1, \theta_0] = \partial_z \ \textit{and}\ \forall k = 0, 1, \theta_k\iota_k = \mathrm{Id},$$
$$[\theta_0\iota_1, \theta_1\iota_0] = 0 \ \textit{and}\ (\theta_0\iota_1)(\theta_1\iota_0) = (\theta_1\iota_0)(\theta_0\iota_1) = \mathrm{Id}.$$

(ii) *The subspace $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is closed under the action of $\{\theta_0, \theta_1\}$ and $\{\iota_0, \iota_1\}$. This means that, for any $w = y_{s_1}\dots y_{s_r} \in Y^*$ (then $\pi_X(w) = x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1$) and $u = y_{t_1}\dots y_{t_r} \in Y_0^*$, the functions Li_w and Li_u^- satisfy*

$$\mathrm{Li}_w = (\iota_0^{s_1-1}\iota_1\dots\iota_0^{s_r-1}\iota_1)1_\Omega \ \textit{and}\ \mathrm{Li}_u^- = (\theta_0^{t_1+1}\iota_1\dots\theta_0^{t_r+1}\iota_1)1_\Omega,$$
$$\iota_0\,\mathrm{Li}_{\pi_X(w)} = \mathrm{Li}_{x_0\pi_X(w)} \ \textit{and}\ \iota_1\,\mathrm{Li}_w = \mathrm{Li}_{x_1\pi_X(w)},$$
$$\theta_0\,\mathrm{Li}_{x_0\pi_X(w)} = \mathrm{Li}_{\pi_X(w)} \ \textit{and}\ \theta_1\,\mathrm{Li}_{x_1\pi_X(w)} = \mathrm{Li}_{\pi_X(w)}\,.$$

(iii) *The bi-integro differential ring $(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \theta_0, \iota_0, \theta_1, \iota_1)$ is stable under the action of $\mathcal{G}$*[5]

$$\forall h \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \forall g \in \mathcal{G}, h(g(z)) \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

[5] When the functions Li_w and $\mathcal{C}$ are extended to $\widetilde{B}$.

(iv) $\theta_0\iota_1$ *and* $\theta_1\iota_0$ *are scalar operators within* $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$*, respectively with eigenvalues* $\lambda := z \to z(1-z)$ *and* $1/\lambda$*, i.e.*

$$\forall f \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \quad (\theta_0\iota_1)f = \lambda f \text{ and } (\theta_1\iota_0)f = (1/\lambda)f.$$

Proof. The three first points can be checked by (more or less) straightforward computations. The last point needs on the one hand identification of each Li_w with its unique lifting to $\widetilde{\mathbb{C}\setminus\{0,1\}}$ which coincides with Li_w on Ω and on the other hand to lift the elements of $\mathcal{G}$ so that this group acts on $\widetilde{\mathbb{C}\setminus\{0,1\}}$.[6] □

3.3. Topology on $\mathcal{H}(\Omega)$ and continuity or discontinuity of the sections ι_i

The algebra $\mathcal{H}(\Omega)$ is that of analytic functions defined over Ω. We quickly describe the standard topology on it (see also [32]), namely that of *compact convergence* whose seminorms are indexed by compact subsets of Ω, and defined by

$$p_K(f) := ||f||_K = \sup_{s\in K} |f(s)|.$$

Of course,

$$p_{K_1\cup K_2} = \sup(p_{K_1}, p_{K_2}),$$

and therefore the same topology is defined by extracting a *fundamental subset of seminorms*, which can be choosen denumerable. As $\mathcal{H}(\Omega)$ is complete with this topology it is a Fréchet space (see [34]).[7]

With the standard topology above, an operator $\phi \in \mathrm{End}(\mathcal{H}(\Omega))$ is continuous iff, with K_i compacts of Ω,

$$(\forall K_2)(\exists K_1)(\exists M_{21} > 0)(\forall f \in \mathcal{H}(\Omega))(||\phi(f)||_{K_2} \le M_{21}||f||_{K_1}),$$

the algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (and $\mathcal{H}(\Omega)$) is closed under the operators $\theta_i, i = 0, 1$. We have build sections of them $\iota_i^{z_0}, \iota_1$, which are continuous and, ι_0 which is discontinuous and adapted to renormalization and the computation of associators.

For $z_0 \in \Omega$, let us define $\iota_i^{z_0} \in \mathrm{End}(\mathcal{H}(\Omega))$ by

$$\iota_0^{z_0}(f) = \int_{z_0}^{z} f(s)\omega_0(s) \text{ and } \iota_1^{z_0}(f) = \int_{z_0}^{z} f(s)\omega_1(s).$$

It is easy to check that $\theta_i\iota_i^{z_0} = \mathrm{Id}_{\mathcal{H}(\Omega)}$ and that they are continuous on $\mathcal{H}(\Omega)$ (for the topology of compact convergence) because for all $K \subset_{\text{compact}} \Omega$, we have

$$|p_K(\iota_i^{z_0}(f)| \le p_K(f)[\sup_{z\in K}|\int_{z_0}^{z}\omega_i(s)|],$$

and this is sufficient to prove continuity. The operators $\iota_i^{z_0}$ are also well defined on $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ and it is easy to check that

$$\iota_i^{z_0}(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}) \subset \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

Due to the decomposition of $\mathcal{H}(\Omega)$ into a direct sum of closed subspaces

$$\mathcal{H}(\Omega) = \mathcal{H}_{z_0\mapsto 0}(\Omega) \oplus \mathbb{C}1_\Omega,$$

it is not hard to see that the graphs of θ_i are closed, thus, the θ_i are also continuous.

To show discontinuity of ι_0, one of the possibilities consists in exhibiting two sequences $f_n, g_n \in \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$ converging to the same limit but such that

$$\lim \iota_0(f_n) \ne \lim \iota_0(g_n).$$

Here, we choose the function z for being approached in a twofold way and if ι_0 were continuous, we would have equality of the limits of the image-sequences (and this is not the case). We first remark that

$$z = \sum_{n\ge 0}\frac{\log^n(z)}{n!} = \sum_{n\ge 1}(-1)^{n+1}\frac{\log^n((1-z)^{-1}}{n!}$$

[6] For details, see section 5.1.
[7] It is even a Fréchet algebra with unit, but we will not use the multiplicative structure here.

Set

$$f_n=\sum_{0\le m\le n}\frac{\log^m(z)}{m!}\text{ and }g_n=\sum_{1\le m\le n}(-1)^{m+1}\frac{\log^m((1-z)^{-1}))}{m!}$$

(these two sequences are in $\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$). It is easily seen that

$$\iota_0(f_n)=f_{n+1}-1$$

and then

$$\lim_{n\to+\infty}\iota_0(f_n)(z)=z-1.$$

Now, for any $s\in[0,z]$ with $z\in]0,1[$, one has

$$\begin{aligned}|g(s)|&=|\sum_{m=1}^{n}(-1)^{m+1}\frac{\log^n(1-s)}{m!}|\\&\le\sum_{m=1}^{n}\frac{|\log^n(1-s)|}{m!}\\&\le\exp(-\log(1-s))-1\\&=\frac{s}{1-s}.\end{aligned}$$

In order to exchange limits, we apply *Lebesgue's dominated convergence theorem* to the measure space $(]0,z],\mathcal{B},dz/z)$ ($\mathcal{B}$ is the usual Borel σ-algebra) and the function $p(x)=s(1-s)^{-1}$ which is - as are the functions g_n - integrable on $]0,z]$ (for every $z\in]0,1[$). Then

$$\lim(\iota_0(g_n))=\lim_{n\to+\infty}\int_0^z g_n(s)\frac{ds}{s}=\int_0^z\lim_{n\to+\infty}g_n(s)\frac{ds}{s}=\int_0^z s\frac{ds}{s}=z.$$

Hence, for $z\in]0,1[$, we obtain,

$$\lim(\iota_0(f_n))=z-1\neq z=\lim(\iota_0(g_n))\ .$$

This completes the proof.

4. Extension of $\mathrm{Li}_\bullet$ to its rational domain

4.1. Rational series

Rational series arise from an extension of finite state (boolean) automata to graphs with costs or weights [14,35]. They have many connections with computer science [1,35] but also with operator and Hopf algebras [8,13]. In short a *weighted graph* is a finite directed graph with edges marked by weights (taken in a semiring, ring or field) and letters (taken in an alphabet X) as follows

$$<tail>\xrightarrow{x|\alpha}<head>$$

this amounts to giving a map $\mu:X\longrightarrow\mathcal{M}_{n,n}(\mathbb{C})$ which is extended to words as

$$\mu:X^*\longrightarrow\mathcal{M}_{n,n}(\mathbb{C})$$

by morphism. Along a graph path, weights multiply and letters concatenate, this gives the *behavior* of the automaton, which has an initial vector as input $\beta\in\mathcal{M}_{1,n}(\mathbb{C})$ and a final vector $\eta\in\mathcal{M}_{n,1}(\mathbb{C})$ as output ; this *behavior* is a series $S\in\mathbb{C}\langle\langle X\rangle\rangle$. It can be proved the following theorem

Theorem 4.1 *([1,8,35]). Let X be a finite alphabet and $S\in\mathbb{C}\langle\langle X\rangle\rangle$. The following are equivalent*

i) *S admits a linear representation (β,μ,η) of dimension n i.e. it exists*

$$\beta\in\mathcal{M}_{1,n}(\mathbb{C}),\ \mu:X^*\longrightarrow\mathcal{M}_{n,n}(\mathbb{C}),\ \eta\in\mathcal{M}_{n,1}(\mathbb{C})$$

such that, for all $w\in X^*$

$$\langle S\mid w\rangle=\beta\mu(w)\eta\qquad(2)$$

ii) *S belongs to the smallest (concatenation) subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$, containing $\mathbb{C}\langle X\rangle$ and closed by $S \to S^{-1}$ (rational closure[8] of $\mathbb{C}\langle X\rangle$).*

4.2. *Domain of* Li$_\bullet$

Under suitable conditions of convergence (see below), the extension of $\mathrm{Li}_\bullet$ in general and to some subdomain of $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ can be done as follows: call $\mathrm{Dom}(\mathrm{Li}_\bullet)$ the set of series

$$S = \sum_{n\geq 0} S_n \text{ with } S_n := \sum_{|w|=n} \langle S \mid w\rangle w$$

such that $\sum_{n\geq 0} \mathrm{Li}_{S_n}$ converges uniformly any compact of Ω. Then

Proposition 4.2. *One has*

(i) *The set* $\mathrm{Dom}(\mathrm{Li}_\bullet)$ *is closed by shuffle products.*
(ii) *For any* $S, T \in \mathrm{Dom}(\mathrm{Li}_\bullet)$, *one has* $\mathrm{Li}_{S ш T} = \mathrm{Li}_S \mathrm{Li}_T$.
(iii) *One has* $\mathbb{C}\langle X\rangle ш \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ш \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \subset \mathrm{Dom}(\mathrm{Li}_\bullet)$.

Proof. (i) and (ii): Suppose $S, T \in \mathrm{Dom}(\mathrm{Li}_\bullet)$ and $S = \sum_{p\geq 0} S_p$ (resp. $T = \sum_{q\geq 0} T_q$) their decomposition in homogeneous components, then the family $(\mathrm{Li}_{S_p})_{p\geq 0}$ (resp. $(\mathrm{Li}_{T_q})_{q\geq 0}$) is summable in $\mathcal{H}(\Omega)$. This implies ([4] Ch III §6) that the families $(\mathrm{Li}_{S_p} \mathrm{Li}_{T_q})_{p,q\geq 0}$ and then $(\sum_{p+q=n} \mathrm{Li}_{S_p} \mathrm{Li}_{T_q})_{n\geq 0}$ are summable in $\mathcal{H}(\Omega)$. As

$$\sum_{p+q=n} \mathrm{Li}_{S_p} \mathrm{Li}_{T_q} = \sum_{p+q=n} \mathrm{Li}_{S_p ш T_q} = \mathrm{Li}_{(S ш T)_n}$$

one gets that $S ш T \in \mathrm{Dom}(\mathrm{Li}_\bullet)$ and $\mathrm{Li}_{S ш T} = \mathrm{Li}_S \mathrm{Li}_T$.
(iii): In view of (i,ii) it suffices to check that each of $\mathbb{C}\langle X\rangle, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ is in $\mathrm{Dom}(\mathrm{Li}_\bullet)$. The first being given, the property for the last two is a consequence of Kronecker's theorem [37] i.e. the fact that

$$\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P/Q\}_{\substack{P,Q\in\mathbb{C}[x]\\ Q(0)\neq 0}}$$

and the partial fraction decomposition. □

This extension is compatible with identities between rational series as *Lazard's elimination*,[9] for instance, for all $S \in \mathbb{C}^{\mathrm{rat}}\langle\langle S\rangle\rangle \cap \mathrm{Dom}(\mathrm{Li}_\bullet)$:

$$\mathrm{Li}_S(z) = \sum_{n\geq 0} \langle S \mid x_0^n\rangle \frac{\log^n(z)}{n!} + \sum_{k\geq 1} \sum_{w\in(x_0^* x_1)^k x_0^*} \langle S \mid w\rangle \mathrm{Li}_w(z),$$

Remark 4.3. Here we will be mostly interested by rational series within $\mathrm{Dom}(\mathrm{Li}_\bullet)$. But there are other series as the following (infinite sum of rational series).

$$T = \sum_{n\geq 0} \frac{(nx_0)^*}{n!} = \sum_{n\geq 0} \frac{1}{n!(1-nx_0)} \overset{Treves}{=} e \sum_{k\geq 0} B_k x_0^k$$

where the Treves topology is just the product topology and therefore limits, for it, are computed term by term.[10]

Now, it is easy to see that we have compact convergence because on Ω (or, below, $\tilde{B}$) as for all $\phi \in \mathcal{H}(\Omega)$, and $K \subset_{compact} \Omega$ (or $\tilde{B}$) one gets

$$||e \sum_{k=0}^{N} B_k \frac{\phi^k}{k!}||_K \leq e \sum_{k=0}^{N} B_k \frac{||\phi||_K^k}{k!} \leq e \sum_{k=0}^{\infty} B_k \frac{||\phi||_K^k}{k!} = e e^{e^{||\phi||_K} - 1} = e^{(e^{||\phi||_K})}.$$

---

8 Of course $S \to S^{-1}$ is only partially defined, its domain is the set of series such that $\langle S \mid 1_{X^*}\rangle \neq 0$.
9 i.e. $X^* = (x_0^* x_1)^* x_0^*$. In other words, in $\mathbb{C}\langle\langle X\rangle\rangle$, $(1-(x_0+x_1))^{-1} = (1-(1-x_0)^{-1}x_1))^{-1}(1-x_0)^{-1}$.
10 i.e. $S_n \to S$ iff

$$(\forall w \in X^*)(\lim_n \langle S_n \mid w\rangle = \langle S \mid w\rangle).$$

Now, remarking that $\mathrm{Li}_{x_0^k}(z) = \log^k(z)/k!$, this proves that

$$T \in \mathrm{Dom}(\mathrm{Li}_\bullet) \text{ and } \mathrm{Li}_T(z) = e^{(e^z)}.$$

The morphism $\mathrm{Li}_\bullet$ is no longer injective on its domain but the family $(\mathrm{Li}_w)_{w\in X^*}$ is still $\mathcal{C}$-linearly independent [23]. We will use several times the following lemma which is characteristic-free.

4.3. Stars of the plane

Lemma 4.4. *Let $(\mathcal{A}, d)$ be a commutative differential ring without zero divisors, and $R = \ker(d)$ be its subring of constants. Let $z \in \mathcal{A}$ such that $d(z) = 1$ and $S = \{e_\alpha\}_{\alpha\in I}$ be a set of eigenfunctions of d, with all different eigenvalues (for example, take $I \subset R$) i.e., $e_\alpha \neq 0$ and $d(e_\alpha) = \alpha e_\alpha$; $\forall\alpha \in I$. Then the family $(e_\alpha)_{\alpha\in I}$ is $R[z]$-linearly free.*[11]

Proof. If there is no non-trivial $R[z]$-linear relation, we are done. Otherwise let us consider relations

$$\sum_{j=1}^{N} P_j(z) e_{\alpha_j} = 0, \tag{3}$$

with[12] $P_j \in R[t]_{pol}$, $P_j(z) \neq 0$ for all j (we will, in the sequel refer to these expressions as *packed linear relations*). We choose one of them minimal with respect to the triplet $[N, \deg(P_N), \sum_{j<N} \deg(P_j)]$, lexicographically ordered from left to right.[13] Remarking that $d(P(z)) = P'(z)$ (because $d(z) = 1$), we apply the operator $d - \alpha_N Id$ to both sides of (3) and get

$$\sum_{j=1}^{N} (P_j'(z) + (\alpha_j - \alpha_N) P_j(z)) e_{\alpha_j} = 0. \tag{4}$$

Minimality of (3) implies that (4) is trivial i.e.

$$P_N'(z) = 0 \text{ and } (\forall j = 1..N-1)\ (P_j'(z) + (\alpha_j - \alpha_N) P_j(z) = 0). \tag{5}$$

Now (3) implies

$$\prod_{j=1}^{n-1} (\alpha_N - \alpha_j) \sum_{j=1}^{N} P_j(z) e_{\alpha_j} = 0,$$

which, because $\mathcal{A}$ has no zero divisors, is a packed linear relation and has the same associated triplet as (3). From (5), we see that for any $k < N$, one has

$$\prod_{j=1}^{n-1} (\alpha_N - \alpha_j) P_k(z) = \prod_{j=1, j\neq k}^{n-1} (\alpha_N - \alpha_j) P_k'(z),$$

so, if $N \geq 2$, we would get a relation of lower triplet. This being impossible, we get $N = 1$ and (3) boils down to $P_N(z) e_N = 0$ which, as $\mathcal{A}$ has no zero divisors, implies $P_N \equiv 0$, a contradiction. Then the $(e_\alpha)_{\alpha\in I}$ are $R[z]$-linearly independent. □

Remark 4.5. If $\mathcal{A}$ is of characteristic zero, $d(z) = 1$ implies that z is transcendent over R and the two notations $R[z]$ and $R[z]_{pol}$ coincide.

First of all, let us prove

Lemma 4.6. *Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra (associative, unital, commutative) and z an indeterminate, then $e^z \in \mathcal{A}[[z]]$ is transcendent over $\mathcal{A}[z]$.*

Proof. It is a straightforward consequence of Remark 4.5. Note that this can be rephrased as $[z, e^z]$ are algebraically independent over $\mathcal{A}$. □

---

11 Here $R[z]$ is understood as ring adjunction i.e. the smallest subring generated by $R \cup \{z\}$.

12 Here, $R[t]_{pol}$ means the formal univariate polynomial ring (the subscript is here to avoid confusion).

13 i.e. consider the ones with N minimal and among these, we choose one with $\deg(P_N)$ minimal and among these we choose one with $\sum_{j<N} \deg(P_j)$ minimal.

Proposition 4.7. *Let $Z=\{z_n\}_{n\in\mathbb{N}}$ be an alphabet, then $[z_0^*, z_1^*]$ is algebraically independent on $\mathbb{C}[Z]$ within $(\mathbb{C}[[Z]], ш, 1_{Z^*})$.*

Proof. Recall that, for x a letter, one has

$$x^* := \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} \frac{x^{ш\, n}}{n!} = e_ш^x. \tag{6}$$

By using Lemma 4.6, one can prove by induction that $[e_ш^{z_0}, e_ш^{z_1}, \cdots, e_ш^{z_k}, z_0, z_1, \cdots, z_k]$, are algebraically independent. This implies that $Z \sqcup \{e_ш^z\}_{z\in Z}$ is an algebraically independent set and, by restriction $Z \sqcup \{e_ш^{z_0}, e_ш^{z_1}\}$ whence the proposition. □

Using Lemma 4.7, we obtain the following proposition.

Proposition 4.8. *One has*

(i) *The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, ш, 1_{X^*})$.*
(ii) *The module $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is free over $\mathbb{C}\langle X\rangle$ and the family $\{(x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.*
(iii) *As a consequence, $\{w ш (x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{\substack{w\in X^* \\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.*

Now, we can construct the following morphism

Definition 4.9. The following morphism

$$\mathrm{Li}_\bullet^{(1)} : (\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] \longrightarrow \mathcal{H}(\Omega)$$

can be defined, for any $w \in X^*$ and $\mathrm{Li}_w^{(1)} = \mathrm{Li}_w$, by

$$\mathrm{Li}_{x_0^*}^{(1)} = z,\ \mathrm{Li}_{(-x_0)^*}^{(1)} = z^{-1},\ \mathrm{Li}_{x_1^*}^{(1)} = (1-z)^{-1}.$$

In fact existence and uniqueness of this morphism obtained as a consequence of Proposition 4.8. Moreover, its kernel and image are given by the following result[14]:

Theorem 4.10. *We have $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ and $\ker(\mathrm{Li}_\bullet^{(1)})$ is the ideal generated by $x_0^* ш x_1^* - x_1^* + 1_{X^*}$.*

Proof. As $\mathbb{C}\langle X\rangle[x_0^*, x_1^*, (-x_0)^*]$ admits $\{(x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{k\in\mathbb{Z},\in\mathbb{N}}$ as a basis for its structure of $\mathbb{C}\langle X\rangle$-module, it suffices to remark that

$$\mathrm{Li}^{(1)}_{(x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}}(z) = \frac{z^k}{(1-z)^l},$$

is a generating system of $\mathcal{C}$. As regards the second assertion, let us prove the following lemma (in this lemma and its proof, all sums are supposed finitely supported)

Lemma 4.11. *Let M_1 and M_2 be K-modules (K is a unitary commutative ring). Suppose the following map is linear*

$$\phi : M_1 \longrightarrow M_2$$

Let $N \subset \ker(\phi)$ be a submodule. If there is a system of generators in M_1, namely $\{g_i\}_{i\in I}$ and $J \subset I$, such that

(i) *For any $i \in I \setminus J$, $g_i \equiv \sum_{j\in J\subset I} c_i^j g_j\ [\mathrm{mod}\ N]$, ($c_i^j \in K; \forall j \in J$);*
(ii) *$\{\phi(g_j)\}_{j\in J}$ is K-free in M_2;*

then $N = \ker(\phi)$.

Proof. Suppose $P \in \ker(\phi)$. Then

$$P \equiv \sum_{j\in J} p_j g_j\ [\mathrm{mod}\ N]$$

[14] This result is already presented in [12], as a preprint, but never was published before.

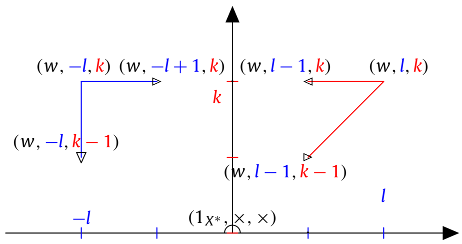

Fig. 1. Rewriting mod$\mathcal{J}$ of $\{w \sqcup\!\sqcup (x_0^*)^{\sqcup\!\sqcup\, l} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, k}\}_{k\in\mathbb{N},l\in\mathbb{Z},w\in X^*}$.

with $\{p_j\}_{j\in J} \subset K$. Thus

$$0 = \phi(P) = \sum_{j\in J} p_j \phi(g_j).$$

From the fact that $\{\phi(g_j)\}_{j\in J}$ is $K-$ free in M_2, we obtain $p_j = 0$ for any $j \in J$. This means that $P \in N$. Thus $\ker(\phi) \subset N$ and, finally, $N = \ker(\phi)$. □

Let now $\mathcal{J}$ be the ideal generated by $x_0^* \sqcup\!\sqcup x_1^* - x_1^* + 1_{X^*}$. It is easily checked, from the following formulas,[15] for $l \geq 1$,

$$w \sqcup\!\sqcup x_0^* \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l} \equiv w \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l} - w \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l-1}\ [\mathcal{J}],$$
$$w \sqcup\!\sqcup (-x_0)^* \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l} \equiv w \sqcup\!\sqcup (-x_0)^* \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l-1} + w \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l} [\mathcal{J}],$$

that one can rewrite [mod$\mathcal{J}$] any monomial $w \sqcup\!\sqcup (x_0^*)^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l}$ as linear combination of such monomials with $kl = 0$. Applying Lemma 4.11 with the following data:

- the morphism $\phi = \mathrm{Li}_\bullet^{(1)}$,
- the modules $M_1 = \mathbb{C}\langle X\rangle[x_0^*, x_1^*, (-x_0)^*]$, $M_2 = \mathcal{H}(\Omega)$, $N = \mathcal{J}$,
- the families $\{g_i\} = \{w \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, n} \sqcup\!\sqcup (x_0^*)^{\sqcup\!\sqcup\, m}\}_{(w,n,m)\in I}$,
- and the indices $I = X^* \times \mathbb{N} \times \mathbb{Z}$, $J = (X^* \times \mathbb{N} \times \{0\}) \sqcup (X^* \times \{0\} \times \mathbb{Z})$,

we have the second point of Theorem 4.10. □

4.4. Examples of polylogarithms indexed by rational series

Proposition 4.12 *([19,20]). One has*

(i) *For $x \in X, i \in \mathbb{N}_+, a \in \mathbb{C}, |a| < 1,$*

$$\mathrm{Li}^{(1)}_{(ax_0)^{*i}}(z) = z^a \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(a\log(z))^k}{k!},$$

$$\mathrm{Li}^{(1)}_{(ax_1)^{*i}}(z) = \frac{1}{(1-z)^a} \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(a\log((1-z)^{-1})^k}{k!}.$$

(ii) *For any $(s_1,\ldots,s_r) \in \mathbb{N}_+^r$ and $|t_1| < 1, \ldots, |t_r| < 1$,*

$$\mathrm{Li}^{(1)}_{(t_1x_0)^{*s_1}x_0^{s_1-1}x_1\ldots(t_rx_0)^{*s_r}x_0^{s_r-1}x_1}(z) = \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}.$$

In particular,

$$\mathrm{Li}^{(1)}_{(t_1x_0)^*x_1\ldots(t_rx_0)^*x_1}(z) = \sum_{n_1,\ldots,n_r>0} \mathrm{Li}_{x_0^{n_1-1}x_1\ldots x_0^{n_r-1}x_1}(z)\, t_0^{n_1-1}\ldots t_r^{n_r-1}.$$

To prove this proposition, we use the following easy lemma:

[15] In Fig. 1, (w,l,k) codes the element $w \sqcup\!\sqcup (x_0^*)^{\sqcup\!\sqcup\, l} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, k}$.

Lemma 4.13. *For any $i, n \in \mathbb{N}^*$, we have*

$$\binom{n+i-1}{n} = \sum_{k=0}^{n} \binom{i-1}{k}\binom{n}{n-k}.$$

Now, we give the proof of Proposition 4.12.

Proof.

(i) Let's choose $i \in \mathbb{N}^*, a \in \mathbb{C}$ and $x \in X$. Note that

$$(ax)^* = \sum_{n=0}^{\infty} (ax)^n = \frac{1}{1-ax},$$
$$((ax)^*)^i = \sum_{n=0}^{\infty} \binom{n+i-1}{n} (ax)^n ,$$
$$(ax)^* ш (1+ax)^{i-1} = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} \binom{i-1}{k}\binom{n}{k})(ax)^n,$$

and use Lemma 4.13, we obtain, for $i \in \mathbb{N}^*, a \in \mathbb{C}, x \in X$,

$$((ax)^*)^i = (ax)^* ш (1+ax)^{i-1}.$$

Thus, for $i \in \mathbb{N}^*$, $|a| < 1, x \in X$,

$$\mathrm{Li}^{(1)}_{((ax)^*)^i} = \mathrm{Li}^{(1)}_{(ax)^*} \mathrm{Li}^{(1)}_{(1+ax)^{i-1}} = \mathrm{Li}^{(1)}_{(ax)^*} \sum_{k=0}^{i-1} \binom{i-1}{k} a^k \mathrm{Li}^{(1)}_{x^k}.$$

It follows then the expected results.

(ii) Using Lemma 4.13, we obtain this statement by direct calculation. □

Corollary 4.14. *One has*

$$\{\mathrm{Li}_S\}_{S \in \mathbb{C}\langle X\rangle ш \mathbb{C}[x_0^*] ш \mathbb{C}[(-x_0^*)] ш \mathbb{C}[x_1^*]} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{z^a}{(1-z)^b} \mathrm{Li}_w(z) \right\}_{w\in X^*}^{a\in\mathbb{Z}, b\in\mathbb{N}}$$
$$\subset \mathrm{span}_{\mathbb{C}} \{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r \in \mathbb{Z}^r}$$
$$\oplus \mathrm{span}_{\mathbb{C}} \{z^a | a \in \mathbb{Z}\},$$
$$\{\mathrm{Li}_S\}_{S \in \mathbb{C}\langle X\rangle ш \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ш \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{z^a}{(1-z)^b} \mathrm{Li}_w(z) \right\}_{w\in X^*}^{a,b\in\mathbb{C}}$$
$$\subset \mathrm{span}_{\mathbb{C}} \{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r \in \mathbb{C}^r}$$
$$\oplus \mathrm{span}_{\mathbb{C}} \{z^a | a \in \mathbb{C}\}.$$

5. Symmetries and transition elements

5.1. Framework

Up to now (and historically [29]), the polylogarithms are computed in $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$, cleft in order to cope with the two singularities $\{0, 1\}$. But $B = \mathbb{C} \setminus \{0, 1\}$ is acted on by the following group of symmetries (which permutes, in fact, $\{0, 1, +\infty\}$).

$$\mathcal{G} := \{z \mapsto z, z \mapsto 1-z, z \mapsto z^{-1}, z \mapsto (1-z)^{-1}, z \mapsto 1-z^{-1}, z \mapsto z(z-1)^{-1}\}$$

To this end and because $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ is not stable by this group we have now to work on $\Omega = \tilde{B}$.

5.2. Monodromy Principle

For convenience, we consider the following situation

$$\begin{array}{ccc} & & Y \\ & \overset{g}{\nearrow} & \big\downarrow p \\ X' & \xrightarrow{f} & X \end{array}$$

and recall the *monodromy principle* (see [7] 16.28.8)

Theorem 5.1 *(Monodromy Principle). Let (Y, X, p) be a covering of a differential manifold X and let $f : X' \to X$ be a C^∞-mapping of a simply-connected,*[16] *differential manifold. Let $a' \in X'$ and $b \in p^{-1}(f(a'))$. Then there exists a unique C^∞-mapping $g : X' \to Y$ such that $g(a') = b$ and $p \circ g = f$ (the mapping g is said to be a lifting of f).*

Here we specialize this to $B = \mathbb{C} \setminus \{0, 1\}$ choose a universal covering $(B, \widetilde{B}, p)$ and a section $s : \Omega \to \widetilde{B}$ of p, lifted from the canonical embedding $j : \Omega \hookrightarrow B$

$$\begin{array}{ccc} & & \widetilde{B} \\ & \overset{s}{\nearrow} & \big\downarrow p \\ \Omega & \overset{j}{\hookrightarrow} & B \end{array}$$

We first remark that any $g \in \mathcal{G}$ maps in fact $B = \mathbb{C} \setminus \{0, 1\}$ to itself and apply the Monodromy Principle to the following situation

$$\begin{array}{ccc} \widetilde{B} & \overset{\tilde{g}}{\dashrightarrow} & \widetilde{B} \\ \big\downarrow p & & \big\downarrow p \\ B & \xrightarrow{g} & B \end{array}$$

where $(\widetilde{B}, B, p)$ is any universal covering of B.

First remark that $\mathcal{G}$ is a copy of $\mathfrak{S}_3$ as it permutes the three singularities and choose an orbit in Ω (as the orbit of i), for instance now, we can pinpoint the lifting (i.e. find points like a', b in Theorem 5.1). See Figs. 2 and 3.

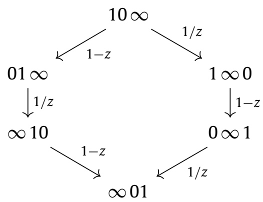

Fig. 2. Orbit of the singularities.

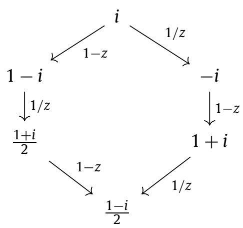

Fig. 3. Orbit of i.

Let us note $g_{1-z}(z) = 1 - z$, $g_{1/z}(z) = z^{-1} \in \mathfrak{S}_B$ and set $\widetilde{g}_\phi \in \mathfrak{S}_{\widetilde{B}}$ such that

$$p \circ \widetilde{g}_\phi = g_\phi \circ p \text{ with } p \circ \widetilde{g}_\phi(s(z_0)) = g_\phi(z_0).$$

[16] Nowadays simply-connected implies path-connected.

where $z_0 = i$.

Remark 5.2. An involutive bi-homomorphism cannot always be lifted as a permutation of finite order as shows the example of $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by $a(z) = -z$. It can be shown (exercise left to the reader) that every lifting $\tilde{a}$ of a i.e.

$$\begin{array}{ccc} \widetilde{\mathbb{C}^*} & \overset{\tilde{a}}{\dashrightarrow} & \widetilde{\mathbb{C}^*} \\ \downarrow p & & \downarrow p \\ \mathbb{C}^* & \xrightarrow{a} & \mathbb{C}^* \end{array}$$

is of infinite order.

Before computing the transition maps for $\widetilde{\mathrm{L}}$ (extension of L to $\widetilde{B}$) under $\widetilde{g}_{1-z}$ and $\widetilde{g}_{1/z}$, we must take an excursion to noncommutative differential equations.

5.3. Noncommutative differential equations

Let (V, d) be a one-dimensional C^{ω} connected complex manifold[17] with its derivation $d = d/dz$. We endow $\mathcal{H}(V)\langle\langle X\rangle\rangle$ with $\mathbf{d}$

$$\mathbf{d}(S) = \sum_{w \in X^*} d(\langle S \mid w\rangle) w \ . \tag{7}$$

It can be easily checked that $\mathbf{d}$ is a derivation of the $\mathbb{C}$-algebra $\mathcal{H}(V)\langle\langle X\rangle\rangle$.

We are now able to define noncommutative differential equations (left multiplier case).

Definition 5.3. A noncommutative differential equation, on V, with left multiplier $M \in \mathcal{H}(V)_+\langle\langle X\rangle\rangle$ is an equality

$$\mathbf{d}(S) = MS. \tag{8}$$

An initial condition can be pinpointed (with $z_0 \in V$ and $S_0 \in \mathbb{C}\langle\langle X\rangle\rangle$)

$$\begin{cases} \mathbf{d}(S) = MS, \\ S(z_0) = S_0. \end{cases} \tag{9}$$

It can be asymptotic[18]

$$\begin{cases} \mathbf{d}(S) = MS, \\ \lim_{\mathfrak{F}} S(z) = S_0. \end{cases} \tag{10}$$

We gather here the needed results

Proposition 5.4 *([6,9]). We have the following properties:*

(i) *If V is simply connected, equation (9) has a unique solution:*
- *with $S_0 = 1_{X^*}$, it can be computed, through Picard's process, by iterated integrals with lower bond z_0, this solution will be noted $S_{Pic}^{z_0}$.*
- *in the general case (initial condition S_0) the solution is $S_{Pic}^{z_0} S_0$.*

(ii) *If V is connected, solutions to equation (10) may not exist, but if it does, the solution is unique.*

(iii) *(V is connected) The set of solutions of (8) is a vector space. Two solutions which coincide at a point - actual (9) or asymptotic (10) - coincide everywhere.*

(iv) *(V is simply connected) The set of invertible solutions of (9) is the following orbit on the right*

$$\mathcal{S} = S_{Pic}^{z_0} (\mathbb{C}\langle\langle X\rangle\rangle)^{\times}.$$

---

[17] A Riemann surface in short.

[18] Below and in general $\mathfrak{F}$ is a filter, the reader who is not familiar with these objects can replace $\mathfrak{F}$ by any *mode of convergence with respect to a subset (e.g. a cone, cluster point, to infinity - full or with restrictions - etc.).*

5.4. Equivariance of polylogs on $\widetilde{B}$

Now, we will explain the property of polylogs expressed by formulas of the type

$$\widetilde{\mathrm{L}}(g.z)=\mu_g(\mathrm{L}(z))Z(g),$$

where μ_g is a morphism of alphabets of the type

$$\begin{pmatrix}\mu_g(x_0)\\ \mu_g(x_1)\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}x_0\\ x_1\end{pmatrix} \tag{11}$$

and $Z(g)\in\mathbb{C}\langle\langle X\rangle\rangle$.

We detail here the computation for $\widetilde{g}_{1-z}$.

(i) $\widetilde{g}_{1-z}$ has been choosen such that

$$p\circ\widetilde{g}_{1-z}=g_{1-z}\circ p \text{ and } p\circ\widetilde{g}_{1-z}(s(i))=1-i. \tag{12}$$

(ii) From $p:\widetilde{B}\to B$, one has $p^*:\mathcal{H}(B)\to\mathcal{H}(\widetilde{B})$ and define the functions $\widetilde{\phi}_z$ (resp. $\widetilde{\phi}_{1/z}$) in $\mathcal{H}(\widetilde{B})$ by

$$\widetilde{\phi}_z(u)=p(u)\in\mathbb{C} \text{ resp. } \widetilde{\phi}_{1/z}(u)=\frac{1}{p(u)}\in\mathbb{C}. \tag{13}$$

(iii) Solve, in $\mathcal{H}(\widetilde{B})\langle\langle X\rangle\rangle$,

$$\begin{cases}\mathbf{d}(\widetilde{S})(u)=\Big(\dfrac{x_0}{\widetilde{\phi}_z(u)}+\dfrac{x_1}{\widetilde{\phi}_{1-z}(u)}\Big)\widetilde{S}(u),\\ \lim_{\substack{z\to 0\\ z\in\Omega}}\widetilde{S}(s(z))e^{-x_0\log(z)}=1_{X^*}.\end{cases} \tag{14}$$

The uniqueness of such a solution is a result of Proposition 5.4 (iii). The existence can be obtained by lifting or remarking that the analog of (14) (first row) can be solved at the level of Ω and, for any choice of $z_0\in\Omega$, one has

$$\widetilde{S}_{Pic}^{s(z_0)}\circ s=S_{Pic}^{z_0} \tag{15}$$

then the solution of (14) is given by

$$\widetilde{\mathrm{L}}=\widetilde{S}_{Pic}^{s(z_0)}\mathrm{L}(z_0)$$

and satisfies

$$\widetilde{\mathrm{L}}\circ s=\mathrm{L}.$$

(iv) Now

$$\mathbf{d}(\widetilde{\mathrm{L}}\circ\widetilde{g}_{1-z})\circ s=\frac{d}{dz}(\widetilde{\mathrm{L}}\circ\widetilde{g}_{1-z})\circ s$$

and, in a neighborhood of i, one has

$$\widetilde{\mathrm{L}}\circ\widetilde{g}_{1-z}\equiv\mathrm{L}(1-z).$$

(v) Setting $\mathrm{L}_1(z)=\mathrm{L}(1-z)$, one gets

$$\frac{d}{dz}\mathrm{L}_1=\Big(-\frac{x_0}{1-z}-\frac{x_1}{z}\Big)\mathrm{L}_1=\mu\Big(\frac{x_0}{z}+\frac{x_1}{1-z}\Big)\mathrm{L}_1,$$

where μ is the morphism of alphabets

$$\mu\begin{pmatrix}x_0\\ x_1\end{pmatrix}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x_0\\ x_1\end{pmatrix}$$

(vi) as μ permutes with the derivation and is invertible, one has

$$\frac{d}{dz}\mu^{-1}(\mathrm{L}_1)=\Big(\frac{x_0}{z}+\frac{x_1}{1-z}\Big)\mu^{-1}(\mathrm{L}_1).$$

Hence,

$$\mu^{-1}(\mathrm{L}_1)=\mathrm{L}Z_1 \text{ and } \mathrm{L}_1=\mu(\mathrm{L})\mu(Z_1).$$

(vii) Finally, as they coincide on an open subset and are analytic, one gets

$$\widetilde{\mathrm{L}}_1=\mu(\widetilde{\mathrm{L}})\mu(Z_1).$$

6. Applications

In this section, we will give some applications of this new presentation of polylogarithms at negative integer multi-indices.

6.1. A new presentation of harmonic sums with non-positive multi-indices

From Corollary 4.14, for any $w \in Y_0^*$, the corresponding polylogarithm Li_w^- is an element of the algebra $\mathbb{Q}[(1-z)^{-1}]$. Thus, for $w \in Y_0^*$, we suppose that Li_w^- can be expanded as follows

$$\mathrm{Li}_w^-(z) = \sum_{k=0}^{(w)+|w|} \frac{a_k^w}{(1-z)^k} \text{ with } a_i^w \in \mathbb{Q}. \tag{16}$$

We note that $a_0^{1_{Y_0^*}} = 1$ and, for any $n > 0$, $a_n^{1_{Y_0^*}} = 0$. Then, using Proposition 3.3, the sequences $\{a_i^w\}_{w \in Y_0^+, n \in \mathbb{N}}$ are computed as follows:

Lemma 6.1 *(Algorithm to compute* $\mathrm{Li}_\bullet^-$*).*
Let $w = y_k u \in Y_0^+ = Y_0^* \setminus \{1_{Y_0^*}\}$.[19] *We have*

(i) *If* $k = 0$ *then*

$$a_i^{y_0 u} := \begin{cases} a_{i-1}^u & \text{for } i = (u) + |u| + 1, \\ a_{i-1}^u - a_i^u & \text{for } 1 \le i \le (u) + |u|, \\ -a_i^u & \text{for } i = 0, \\ 0 & \text{otherwise}. \end{cases}$$

(ii) *If* $k > 0$ *then*

$$a_i^{y_k u} := \begin{cases} (i-1) a_{i-1}^{y_{k-1} u} & \text{for } i = (u) + |u| + k + 1, \\ (i-1) a_{i-1}^{y_{k-1} u} - i a_i^{y_{k-1} u} & \text{for } 2 \le i \le (u) + |u| + k, \\ -a_i^{y_{k-1} u} & \text{for } i = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof.

(i) If $k = 0$, we have $\mathrm{Li}_{y_0 u}^- = \mathrm{Li}_{y_0}^- \mathrm{Li}_u^-$. Hence, we obtain this condition.
(ii) This condition is a direct corollary of the identity $\mathrm{Li}_{y_k u}^- = \theta_0 \mathrm{Li}_{y_{k-1} u}^-$. □

Proposition 6.2. *For any* $w \in Y_0^*$, *we have* $\mathrm{Li}_w^- = \mathrm{Li}_{P_w}$, *where*

$$P_w := \sum_{i=0}^{(w)+|w|} a_i^w (x_1^*)^{\text{ш} i} \in (\mathbb{C}[x_1^*], \text{ш}, 1_{X^*})$$

and the coefficients $\{a_i^w\}_{i \in \mathbb{N}}$ *are defined as in Lemma 6.1. See Table 1.*

Now, we note that for any $k \in \mathbb{N}$,

$$\frac{1}{(1-z)^{k+1}} = \frac{1}{k!} \sum_{n \ge k} n \ldots (n-k+1) z^{n-k} = \sum_{N \ge 0} \binom{N+k}{k} z^N.$$

On the other hand, for any $w \in Y_0^*$, denoting $p = (w) + |w|$, we have

$$\sum_{N \ge 0} \mathrm{H}_w^-(N) z^N = \frac{\mathrm{Li}_w^-(z)}{1-z} = \frac{a_p^w}{(1-z)^{p+1}} + \ldots + \frac{a_1^w}{(1-z)^2} + \frac{a_0^w}{1-z}.$$

[19] Such a (non-empty) word w can always be written uniquely $w = y_k u$ where $y_k \in Y_0$ and $u \in Y_0^*$.

Table 1
The value of $a_i^{y_k y_0}$.

$k \setminus i$	0	1	2	3	4	5	6	7	...
0	1	−2	1	0	0	0	0	0	...
1	0	2	−4	2	0	0	0	0	...
2	0	−2	10	−14	6	0	0	0	...
3	0	2	−22	62	−66	24	0	0	...
4	0	−2	46	−230	450	−384	120	0	...
...	...	...	...	...	...	...	...	...	...

Hence,

$$\sum_{N\geq 0} \mathrm{H}^-_w(N) z^N = a_p^w \sum_{N\geq 0} \binom{N+p}{p} z^N + \ldots + a_0^w \sum_{N\geq 0} \binom{N}{0} z^N.$$

This means that,

Proposition 6.3. *For any $w \in Y_0^*$,*

$$\mathrm{H}^-_w(N) = \sum_{k=0}^{(w)+|w|} a_k^w \binom{N+k}{k}.$$

In fact, since the definition of the sequence $\{a_p^w\}_{w\in Y_0^*, p\in\mathbb{N}}$, we can use the formula

$$\binom{n+r}{r} - \binom{n+r-1}{r-1} = \binom{n+r-1}{r}$$

and Proposition 6.3 to obtain an extension of Faulhaber's result [28], i.e., the harmonic sum $\mathrm{H}^-_w(N)$, $w \in Y_0^*$ can be written like a finite linear combination of the elements in the family $\{\binom{N+n}{m}\}_{n,m\in\mathbb{N}}$, where the number of terms is at most $\lfloor \frac{(w)+|w|}{2} \rfloor + 1$.

Example 6.4.

$$\begin{aligned}
\mathrm{H}^-_{y_0}(N) &= \binom{N+1}{1} - \binom{N}{0} \\
&= \binom{N}{1}, \\
\mathrm{H}^-_{y_1}(N) &= \binom{N+2}{2} - \binom{N+1}{1} \\
&= \binom{N+1}{2}, \\
\mathrm{H}^-_{y_2}(N) &= 2\binom{N+3}{3} - 3\binom{N+2}{2} + \binom{N+1}{1} \\
&= 2\binom{N+2}{3} - \binom{N+1}{1}, \\
\mathrm{H}^-_{y_3}(N) &= 6\binom{N+4}{4} - 12\binom{N+3}{3} + 7\binom{N+2}{2} - \binom{N+1}{1} \\
&= 6\binom{N+2}{4} + \binom{N+1}{2}, \\
\mathrm{H}^-_{y_0^2}(N) &= \binom{N+2}{2} - 2\binom{N+1}{1} + \binom{N}{0}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{N}{2},\\
\mathrm{H}^-_{y_1y_0}(N) &= 2\binom{N+3}{3} - 4\binom{N+2}{2} + 2\binom{N+1}{1}\\
&= 2\binom{N+1}{3},\\
\mathrm{H}^-_{y_2y_0}(N) &= 6\binom{N+4}{4} - 14\binom{N+3}{3} + 10\binom{N+2}{2} - 2\binom{N+1}{1}\\
&= 6\binom{N+2}{4} + 2\binom{N+1}{2}.
\end{aligned}
$$

6.2. Regularization of polyzetas at negative multi-indices

Let $\{t_i\}_{i\in\mathbb{N}_+}$ be a family of variables. The symmetric functions $\{\eta_k\}_{k\in\mathbb{N}_+}$ and the power sums $\{\psi_k\}_{k\in\mathbb{N}_+}$ are defined [16] respectively by

$$\eta_k(\underline{t}) := \sum_{n_1>\ldots n_k>0} t_{n_1}\ldots t_{n_k} \text{ and } \psi_k(\underline{t}) := \sum_{n>0} t_n^k.$$

Then the generating series of the family[20] $\{\eta_k\}_{k\in\mathbb{N}}$ is defined by

$$1 + \sum_{k\geq 1} \eta_k(\underline{t}) z^k = \prod_{i>0}(1+t_i z) =: \eta(\underline{t}\mid z).$$

In the same way, we also define

$$\sum_{k\geq 1} \psi_k(\underline{t}) z^k = \sum_{i\geq 1} \frac{t_i z}{1-t_i z} =: \psi(\underline{t}\mid z).$$

Note that the functions $\eta(\underline{t}\mid z)$ and $\psi(\underline{t}\mid z)$ satisfy Euler's identity

$$z\frac{d}{dz}\log(\eta(\underline{t}\mid z)) = \psi(\underline{t}\mid -z). \tag{17}$$

For any $w = y_{s_1}\ldots y_{s_r} \in Y^*$, the quasi-symmetric function of depth $|w| = r$ and of degree (or weight) $(w) := s_1 + \ldots + s_r$,namely F_w, is defined by

$$\mathrm{F}_w(\underline{t}) := \sum_{n_1>\ldots>n_r>0} t_{n_1}^{s_1}\ldots t_{n_r}^{s_r}.$$

Note that $\mathrm{F}_{y_1^k} = \eta_k$ and $\mathrm{F}_{y_k} = \psi_k$, and from equation (17), we obtain

$$\sum_{k\geq 0} \mathrm{F}_{y_1^k} z^k = \exp\Big(-\sum_{k\geq 1} \mathrm{F}_{y_k}\frac{(-z)^k}{k}\Big). \tag{18}$$

Remark that the harmonic sum $\mathrm{H}_w(N)$ can be obtained by specializing, in the quasi-symmetric function F_w, the variables $\{t_i\}_{i\geq 1}$ as follows [24]:

$$t_i = \frac{1}{i} \quad \text{if} \quad 0 < i \leq N \quad \text{else} \quad t_i = 0.$$

On the other hand, we recall the morphisms of regularization [5]

$$
\begin{aligned}
&\gamma_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) \longrightarrow (\mathbb{C}, \cdot, 1),\\
&\zeta_⧢ : (\mathbb{C}\langle X\rangle, ⧢, 1_{X^*}) \longrightarrow (\mathbb{C}, \cdot, 1),
\end{aligned}
$$

which are defined by

$$\gamma_{y_1} = \gamma,\ \ \zeta_⧢(x_0) = \zeta_⧢(x_1) = 0$$

[20] We set $\eta_0(\underline{t}) = 1$.

and

$$\forall l \in \mathcal{L}yn(X) - X,\ \gamma_{\pi_Y(l)} = \zeta(l).$$

Then

$$\forall t \in \mathbb{C},\ \zeta_{\sqcup\!\sqcup}((tx_1)^*) = 1.$$

Note that, by a Newton-Girard like identity (see (18)), one has [5]

$$\sum_{k\geq 0} \mathrm{H}_{y_1^k} z^k = \exp\Big(-\sum_{k\geq 1} \mathrm{H}_{y_k} \frac{(-z)^k}{k}\Big),$$

and by the properties of Gamma function, we obtain, for $|t| < 1$,

$$\gamma_{\pi_Y((tx_1)^*)} = \exp\Big(\gamma t - \sum_{n\geq 2} \zeta(n) \frac{(-t)^n}{n}\Big) = \frac{1}{\Gamma(1+t)}. \tag{19}$$

For $t \in \mathbb{N}$, we put

$$\gamma_{\pi_Y((tx_1)^*)} = \frac{1}{t!}.$$

Proposition 6.5 *([27]). Let $w = y_{s_1} \ldots y_{s_r} \in Y_0^*$. Then there exists a* <u>*unique*</u> *polynomial $P_w \in (\mathbb{C}[x_1^*], \sqcup\!\sqcup, 1_{X^*})$ such that*

$$\mathrm{Li}_w^- = \mathrm{Li}_{P_w}.$$

Setting $\gamma_{-s_1,\ldots,-s_r} := \gamma_{\pi_Y(P_w)}$, we get

$$\gamma_{-s_1,\ldots,-s_r} = \sum_{k=0}^{s_1+\ldots s_r+r} \frac{a_k^w}{k!}.$$

Proof. The first part of this proposition is the direct consequence of (16) and Proposition 6.2. See [27] for the analytical justification of such algebraic process and the proof of the uniqueness. □

Example 6.6.

$$\begin{aligned}
\gamma_0 &= -1 + \frac{1}{1!} \\
&= 0, \\
\gamma_{-1} &= -\frac{1}{1!} + \frac{1}{2!} \\
&= -\frac{1}{2}, \\
\gamma_{-2} &= \frac{1}{1!} - \frac{3}{2!} + \frac{2}{3!} \\
&= -\frac{1}{6}, \\
\gamma_{-3} &= -\frac{1}{1!} + \frac{7}{2!} - \frac{12}{3!} + \frac{6}{4!} \\
&= \frac{3}{4}, \\
\gamma_{-4} &= \frac{1}{1!} - \frac{15}{2!} + \frac{50}{3!} - \frac{60}{4!} + \frac{24}{5!} \\
&= -\frac{7}{15}, \\
\gamma_{0,0} &= 1 - \frac{2}{1!} + \frac{1}{2!} \\
&= -\frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
\gamma_{0,-1} &= \frac{1}{1!} - \frac{2}{1!} + \frac{1}{3!} \\
&= -\frac{5}{6}, \\
\gamma_{-2,0} &= -\frac{2}{1!} + \frac{10}{2!} - \frac{14}{3!} + \frac{6}{4!} \\
&= \frac{11}{12}, \\
\gamma_{0,-2} &= -1 + \frac{4}{2!} - \frac{5}{3!} + \frac{2}{4!} \\
&= \frac{1}{4}, \\
\gamma_{-1,-1} &= -1 + \frac{5}{2!} - \frac{7}{3!} + \frac{3}{4!} \\
&= \frac{11}{24}, \\
\gamma_{0,0,0} &= -1 + \frac{3}{1!} - \frac{3}{2!} + \frac{1}{3!} \\
&= \frac{2}{3}, \\
\gamma_{0,-1,0} &= -\frac{2}{1!} + \frac{6}{2!} - \frac{6}{3!} + \frac{2}{4!} \\
&= \frac{1}{12}, \\
\gamma_{0,-2,0} &= \frac{-12}{2!} + \frac{6}{5!} + \frac{2}{1!} - \frac{20}{4!} + \frac{24}{3!} \\
&= -\frac{47}{60}, \\
\gamma_{-4,-4,-6} &= 1 - \frac{16655}{2!} + \frac{5260444}{3!} - \frac{321370622}{4!} + \frac{7519806977}{5!} - \frac{90280292235}{6!} + \frac{647428882810}{7!} \\
&\quad - \frac{3028468246320}{8!} + \frac{9748178974760}{9!} - \frac{22298261594760}{10!} + \frac{36869237126640}{11!} - \frac{44258208343200}{12!} \\
&\quad + \frac{38240776382400}{13!} - \frac{23192869190400}{14!} + \frac{9376213916160}{15!} - \frac{2270032128000}{16!} + \frac{249080832000}{17!} \\
&= -\frac{47315637837661}{137837700}.
\end{aligned}$$

7. Conclusion

In this work, we explained the whole project of extending $\mathrm{Li}_\bullet$ over a shuffle subalgebra of rational power series.

In particular, we have studied different aspects of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$, where $\mathcal{C}$ denotes the ring of polynomials in z, z^{-1} and $(1-z)^{-1}$, with coefficients in $\mathbb{C}$.

On the other hand, we applied this new indexing of $\mathrm{Li}_\bullet$ to express the polylogarithms (resp. harmonic sums) at negative multi-indices as polynomials in $(1-z)^{-1}$ (resp. N), with coefficients in $\mathbb{Z}$ (resp. $\mathbb{Q}$).

We concentrated, particularly, on algebraic and analytic aspects of this extension allowing index polylogarithms, at non-positive multi-indices, by rational series and to regularize divergent polyzetas, at non-positive multi-indices.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] J. Berstel, C. Reutenauer, Rational Series and Their Languages, Springer-Verlag, 1988.
[2] V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, C. Tollu, Q.H. Ngo, (Pure) transcendence bases in ϕ-deformed shuffle bialgebras, Sémin. Lothar. Comb. 74 (2018), 22 pp., Article B74f.

[3] C. Costermans, Hoang Ngoc Minh, Some Results à l'Abel Obtained by Use of Techniques à la Hopf, "Global Integrability of Field Theories and Applications", 2006, Daresbury.
[4] N. Bourbaki, General Topology, Hermann, 1966.
[5] C. Costermans, Hoang Ngoc Minh, Noncommutative algebra, multiple harmonic sums and applications in discrete probability, J. Symb. Comput. 801 (817) (2009).
[6] M. Deneufchâtel, G.H.E. Duchamp, Hoang Ngoc Minh, A.I. Solomon, Independence of hyperlogarithms over function fields via algebraic combinatorics, Lect. Notes Comput. Sci. 6742 (2011) (2011) 127–139.
[7] J. Dieudonné, Treatise on Analysis, vol. 3, Acad. Press, 1972.
[8] G. Duchamp, C. Reutenauer, Un critère de rationalité provenant de la géométrie non-commutative (à la mémoire de Schützenberger), Invent. Math. 128 (1997).
[9] G.H.E. Duchamp, V. Hoang Ngoc Minh, K.A. Penson, About Some Drinfel'd Associators, Lecture Notes in Computer Science, LNCS, vol. 11077, Springer-Verlag, 2018, pp. 146–263.
[10] G.H.E. Duchamp, Hoang Ngoc Minh, Ngo Quoc Hoan, Harmonic sums and polylogarithms at negative multi-indices, J. Symb. Comput. 83 (2017) 166–186.
[11] G.H.E. Duchamp, Hoang Ngoc Minh, Ngo Quoc Hoan, K.A. Penson, P. Simonnet, Mathematical renormalization in QED via noncommutative generating series, in: Kotsireas, Martinez-Moro (Eds.), Applications of Computer Algebra, in: Springer Proceedings in Mathematics & Statistics, vol. 198, 2017, pp. 59–100, Chapter 6.
[12] G.H.E. Duchamp, Hoang Ngoc Minh, Ngo Quoc Hoan, The algebra $\mathbb{C}\langle \mathbf{X}\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle \mathbf{x_0}\rangle\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle \mathbf{x_1}\rangle\rangle$ and polylogarithms, 2015.
[13] G.H.E. Duchamp, C. Tollu, Sweedler's duals and Schützenberger's calculus, in: K. Ebrahimi-Fard, M. Marcolli, W. van Suijlekom (Eds.), Combinatorics and Physics, in: Amer. Math. Soc. (Contemporary Mathematics), vol. 539, 2011, pp. 67–78, arXiv:0712.0125v3.
[14] S. Eilenberg, Automata, Languages, and Machines (vol. A), Acad. Press, New York, 1974.
[15] H. Furusho, Y. Komori, K. Matsumoto, H. Tsumura, Desingularization of Multiple Zeta-Functions of Generalized Hurwitz-Lerch Type, 2014.
[16] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (1995) 218–348.
[17] A.B. Goncharov, Multiple Polylogarithms and Mixed Tate Motives, 2001.
[18] L. Guo, B. Zhang, Renormalization of multiple zeta values, J. Algebra 319 (2008) 3770–3809.
[19] V. Hoang Ngoc Minh, Evaluation transform, Theor. Comput. Sci. 79 (1991) 163–177.
[20] V. Hoang Ngoc Minh, Summations of polylogarithms via evaluation transform, Math. Computers in Simulations 1336 (1996) 707–728.
[21] V. Hoang Ngoc Minh, G. Jacob, N.E. Oussous, M. Petitot, Aspects combinatoires des polylogarithmes et des sommes d'Euler-Zagier, J. SLC B 43e (1998).
[22] V. Hoang Ngoc Minh, M. Petitot, Lyndon words, polylogarithmic functions and the Riemann ζ function, Discrete Math. 217 (2000) 273–292.
[23] V. Hoang Ngoc Minh, Differential Galois groups and noncommutative generating series of polylogarithms, in: "Automata, Combinatorics and Geometry". 7th World Multi-Conference on Systemics, Cybernetics and Informatics, Florida, 2003.
[24] V. Hoang Ngoc Minh, Finite polyzêtas, poly-Bernoulli numbers, identities of polyzêtas and noncommutative rational power series, in: Proceedings of 4th I. C. W, Finland, 2003, pp. 232–250.
[25] V. Hoang Ngoc Minh, Algebraic combinatoric aspects of asymptotic analysis of nonlinear dynamical system with singular inputs, Acta Acad. Abo., Ser. B, Math. Phys. 67 (2) (2007) 117–126.
[26] V. Hoang Ngoc Minh, On a conjecture by Pierre Cartier about a group of associators, Acta Math. Vietnam. 38 (3) (2010) 339–398.
[27] V. Hoang Ngoc Minh, On solutions of KZ_3, (forthcoming).
[28] D.E.V. Knuth, Johann faulhaber and sums of powers, arXiv:math/9207222v1, 1992.
[29] J.A. Lappo-Danilevskij, Mémoires sur la Théorie des Systèmes des Équations Différentielles Linéaires, vol. I, Travaux Inst. Physico-Math. Stekloff, vol. 8, Acad. Sci. USSR, Moscow-Leningrad, 1936, pp. 5–206.
[30] D. Manchon, S. Paycha, Nested sums of symbols and renormalised multiple zeta functions, Int. Math. Res. Not. 24 (2010) 4628–4697.
[31] D.E. Radford, A natural ring basis for the shuffle algebra and an application to group schemes, J. Algebra 58 (1979) 432–454.
[32] R. Remmert, Theory of Complex Functions (translated by Robert B. Burckel) Springer-Verlag New York Inc, 1991.
[33] C. Reutenauer, Free Lie Algebras, London Math. Soc. Monographs, 1993.
[34] W. Rudin, Functional Analysis, second edition, McGraw-Hill Book Co, 1991.
[35] M.P. Schützenberger, On the definition of a family of automata, Inf. Control 4 (1961) 270–275.
[36] J. Zhao, Analytic continuation of multiple zeta functions, Proc. Am. Math. Soc. 128 (5) (1999) 1275–1283.
[37] A. Zygmund, Trigonometric Series, Cambridge University Press, 2002.