Journal of Symbolic Computation 75 (2016) 56–73

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Dual bases for noncommutative symmetric and quasi-symmetric functions via monoidal factorization

V.C. Bui [a,b], G.H.E. Duchamp [b], V. Hoang Ngoc Minh [c,b], L. Kane [b], C. Tollu [b]

[a] College of Sciences, Hue University, 77, Nguyen Hue street, Huê city, Viet Nam
[b] LIPN, UMR 7030, CNRS, Université Paris 13, F-93430 Villetaneuse, France
[c] Université Lille II, 1, Place Déliot, 59024 Lille, France

A R T I C L E I N F O

Article history:
Received 8 December 2014
Accepted 10 July 2015
Available online 4 November 2015

Keywords:
Noncommutative symmetric and quasi-symmetric functions
Lie elements
Monoidal factorization
Transcendence bases

A B S T R A C T

We propose effective constructions of dual bases for the noncommutative symmetric and quasi-symmetric functions. To this end, we use an effective variation of Schützenberger's factorization adapted to the diagonal pairing between a graded space and its dual.

1. Introduction

Originally, "symmetric functions" are thought of as "functions of the roots of some polynomial" (Gelfand et al., 1995). The factorization formula

$$P(X) = \prod_{\alpha\in\mathcal{O}(P)} (X-\alpha) = \sum_{j=0}^{n} X^{n-j}(-1)^j \Lambda_j(\mathcal{O}(P)), \tag{1}$$

where $\mathcal{O}(P)$ is the (multi-)set of roots of P (a polynomial), invites one to consider $\Lambda_j(.)$ as a "multiset (endo)functor"[1] rather than a function $K^n \to K$ (K is a field where P splits). But, here,

E-mail address: hoang@lille2.fr (V. Hoang Ngoc Minh).

[1] We will not touch here on this categorical aspect.

http://dx.doi.org/10.1016/j.jsc.2015.11.007
0747-7171/

$\Lambda_k(X) = 0$ whenever $k > |X|$ and one would like to get the universal formulas *i.e.* which hold true whatever the cardinality of $|X|$. This set of formulas is obtained as soon as the alphabet is infinite and, there, this calculus appears as an art of computing symmetric functions without using any variable. With this point of view, one sees that the algebra of symmetric functions (Macdonald, 1979) comes equipped with many additional structures (comultiplications, λ-ring, transformations of alphabets, internal product, …). As far as we are concerned, the most important of these features is the fact that the (commutative) Hopf algebra of symmetric functions is self-dual. With the exception of self-duality, many features of the (Hopf) algebra of symmetric functions carry over to the noncommutative level (Gelfand et al., 1995). This loss of self-duality has however a benefit: allowing to separate the two sides in the factorization of the diagonal series,[2] thus giving a meaning to what could be considered a complete system of local coordinates for the Hausdorff group of the quasi-shuffle Hopf algebra. Indeed, the elements of the Hausdorff group of the (shuffle or quasi-shuffle) algebras are exactly, through the isomorphism $\mathbf{k}\langle\langle Y\rangle\rangle \simeq (\mathbf{k}\langle Y\rangle)^*$, the characters of the algebra (Bui et al., 2013a; Hoang Ngoc Minh, 2013a, 2013b). Then, letting S be a character and applying $S \otimes \mathrm{Id}$ (necessarily continuous[3]) to the factorization

$$\sum_{w\in Y^*} w \otimes w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(s_l \otimes p_l), \tag{2}$$

S can be decomposed through this complete system of local coordinates:

$$S = \sum_{w\in Y^*} \langle S|w\rangle w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\langle S|s_l\rangle p_l). \tag{3}$$

This fact is better understood when one considers Sweedler's dual of the (shuffle or quasi-shuffle) Hopf algebra $\mathcal{H}$, which contains also, here, the group of characters[4] and its Lie algebra, the space of infinitesimal characters. Such a character is here a series T such that

$$\Delta_*(T) = T \otimes \epsilon + \epsilon \otimes T \tag{4}$$

and one sees from this definition that such a series, as well as the characters, satisfies an identity of the type

$$\Delta_*(S) = \sum_{i=1}^{N} S_i^{(1)} \otimes S_i^{(2)} \tag{5}$$

for some double family $(S_i^{(1)}, S_i^{(2)})_{1\le i\le N}$. Then in (3), the character S is factorized as a product of *elementary* exponentials. This shows firstly, that one can reconstruct a character from its projections onto the free Lie algebra[5] and secondly, that we get a resolution of unity from the process

$$\text{character} \to \text{projection} \to \text{coordinate splitting},$$
$$\text{coordinate splitting} \to \text{exponentials} \to \text{infinite product}.$$

---

2 *I.e.* the left hand side of (2) which is an expression of the identity. Note that, in a "diagonal" tensor $w \otimes w$, the left factor serves as a linear form whereas the right factor is a vector. This is why the left hand side will be endowed with the convolution of linear forms and the right hand side with the concatenation.

3 The series of the form $\sum_{u,v\in Y^*,\mathrm{gr}(u)=\mathrm{gr}(v)} \alpha(u,v) u \otimes v$ where "gr" is a suitable grading (multihomogeneous degree for the shuffle, weight for the quasi-shuffle), form a closed subalgebra of $\mathbf{k}\langle\langle Y^* \otimes Y^*\rangle\rangle$ filtered by the "diagonal" valuation $\mathrm{gr}^{\mathrm{iso}}(u \otimes v) = \mathrm{gr}(u) = \mathrm{gr}(v)$. Then $S \otimes I$, isometric on the monomials, is necessarily continuous.

4 *I.e.* group-like elements for the dual structure.

5 For each $l \in \mathcal{L}ynY$ the map $S \mapsto \langle S \mid s_l\rangle\, p_l$ is a projection into the free Lie algebra and these projectors are orthogonal between themselves.

The key point of this resolution is precisely the coordinate system provided by the dual family of Poincaré–Birkhoff–Witt (PBW) homogeneous bases.

This paper is devoted to a detailed exposition of the machinery and morphisms surrounding this resolution (Equation (2)) and it is organized as follows: in Section 2 we give a reminder on noncommutative symmetric and quasi-symmetric functions (Gelfand et al., 1995, Duchamp et al., 1997; Krob and Thibon, 1997, 1999; Krob et al., 1997); in Section 3 we focus on the combinatorial aspects of the quasi-shuffle Hopf algebra which will be introduced to obtain – via Schützenberger's monoidal factorization – a pair of bases in duality for the noncommutative symmetric and quasi-symmetric functions, encoded by words.[6]

2. Background

2.1. Notations, statistics about compositions

For any *composition* $I=(i_1,\ldots,i_k)$ of strictly positive integers,[7] called the *parts* of I, the mirror image of I, denoted by $\tilde{I}$, is the composition $(i_k,\ldots,i_1)$. Let $I=(i_1,\ldots,i_k)\in(\mathbb{N}_+)^*$, the *length* and the *weight* of the composition I are defined respectively as the numbers $l(I)=k$ and $w(I)=i_1+\ldots+i_k$. The last part and the product of the partial sum of the entries of I are defined respectively as the numbers $lp(I)=i_k$ and $\pi_u(I)=i_1(i_1+i_2)\ldots(i_1+\ldots+i_k)$. One defines also

$$\pi(I)=\prod_{p=1}^{k} i_p \quad \text{and} \quad sp(I)=\pi(I)l(I)!. \tag{6}$$

Let J be a composition which is finer than I, *i.e.* J can be decomposed as $J=(J_1,\ldots,J_k)$, where each composition J_p, for $p=1,\ldots,k$, satisfies $w(J_p)=i_p$.[8] Let I and J be as above. One defines

$$l(J,I)=\prod_{i=1}^{k} l(J_i), \quad lp(J,I)=\prod_{i=1}^{k} lp(J_i), \tag{7}$$

$$\pi_u(J,I)=\prod_{i=1}^{k} \pi_u(J_i), \quad sp(J,I)=\prod_{i=1}^{k} sp(J_i). \tag{8}$$

2.2. Noncommutative symmetric functions

Let $\mathbf{k}$ be a commutative associative and unital $\mathbb{Q}$-algebra. The algebra of noncommutative symmetric functions, denoted by $\mathbf{Sym}_{\mathbf{k}}=(\mathbf{k}\langle S_1,S_2,\ldots\rangle,\bullet,1)$, is the free associative algebra generated by an infinite sequence $\{S_n\}_{n\geq1}$ of noncommuting indeterminates also called *complete* homogeneous symmetric functions (Gelfand et al., 1995). Let t be another variable commuting with all the $\{S_n\}_{n\geq1}$. Introducing the ordinary generating series one sets (Gelfand et al., 1995)

$$\sigma(t)=\sum_{n\geq0} S_n t^n, \text{ with } S_0=1, \tag{9}$$

other noncommutative symmetric functions can be derived by the following relations

$$\lambda(t)=\sigma(-t)^{-1}, \quad \sigma(t)=\exp(\Phi(t)), \quad \frac{d}{dt}\sigma(t)=\sigma(t)\psi(t)=\psi^*(t)\sigma(t), \tag{10}$$

[6] We will show how to derive the generating series of an analog of Hall–Littlewood functions (Gelfand et al., 1995), recalled in Examples 5 and 6, as a direct application (see (66)) of the factorization of the diagonal series given in (2).

[7] *I.e.* I is an element of the free monoid $(\mathbb{N}_+)^*$ and the empty composition will be denoted here by $\emptyset$.

[8] For example, $J=(3,1,2,2)=((3,1),(2,2))$ is finer than $I=(4,4)$. In the sequel, we will use the standard notation $J\succeq I$.

where Φ, λ, ψ are respectively the following ordinary generating series

$$\Phi(t)=\sum_{n\geq 1}\Phi_n\frac{t^n}{n},\quad \lambda(t)=\sum_{n\geq 0}\Lambda_n t^n,\quad \psi(t)=\sum_{n\geq 1}\Psi_n t^{n-1}. \tag{11}$$

The noncommutative symmetric functions $\{\Lambda_n\}_{n\geq 0}$ (with $\Lambda_0=1$) are called *elementary* functions. The elements $\{\Psi_n\}_{n\geq 1}$ and $\{\Phi_n\}_{n\geq 1}$ are called *power sums of first* and *second kind* respectively.

Let $I=(i_1,\ldots,i_k)\in(\mathbb{N}_+)^*$, one defines the products of complete and elementary symmetric functions, and the products of power sums as follows (Gelfand et al., 1995)

$$S^I=S_{i_1}\ldots S_{i_k},\quad \Lambda^I=\Lambda_{i_1}\ldots\Lambda_{i_k},\quad \Psi^I=\Psi_{i_1}\ldots\Psi_{i_k},\quad \Phi^I=\Phi_{i_1}\ldots\Phi_{i_k}. \tag{12}$$

It is established (Gelfand et al., 1995) that

$$S^I=\sum_{J\succeq I}(-1)^{l(J)-w(I)}\Lambda^J,\qquad \Lambda^I=\sum_{J\succeq I}(-1)^{l(J)-w(I)}S^J; \tag{13}$$

$$S^I=\sum_{J\succeq I}\frac{\Psi^J}{\pi_u(J,I)},\qquad \Psi^I=\sum_{J\succeq I}(-1)^{l(J)-l(I)}lp(J,I)S^J; \tag{14}$$

$$S^I=\sum_{J\succeq I}\frac{\Phi^J}{sp(J,I)},\qquad \Phi^I=\sum_{J\succeq I}(-1)^{l(J)-l(I)}\frac{\pi(I)}{l(J,I)}S^J; \tag{15}$$

$$\Lambda^I=\sum_{J\succeq I}(-1)^{w(J)-l(I)}\frac{\Psi^J}{\pi_u(\tilde{J},\tilde{I})},\qquad \Psi^I=\sum_{J\succeq I}(-1)^{w(I)+l(J)}lp(\tilde{J},\tilde{I})\Lambda^J; \tag{16}$$

$$\Lambda^I=\sum_{J\succeq I}(-1)^{w(J)-l(I)}\frac{\Phi^J}{sp(J,I)},\qquad \Phi^I=\sum_{J\succeq I}(-1)^{w(J)-l(I)}\frac{\pi(I)}{l(J,I)}\Lambda^J. \tag{17}$$

The $\mathbf{k}$-algebra $\mathbf{Sym_k}$ possesses a finite-dimensional grading by the weight function defined, for any composition $I=(i_1,\ldots,i_k)$, by the number $w(S_I)=w(I)$. Its homogeneous component of weight n (free and finite-dimensional) will be denoted by $\mathbf{Sym}_{\mathbf{k}n}$ and one has

$$\mathbf{Sym_k}=\mathbf{k}1_{\mathbf{Sym_k}}\oplus\bigoplus_{n\geq 1}\mathbf{Sym}_{\mathbf{k}n}. \tag{18}$$

The families $\{S^I\}_{I\in(\mathbb{N}_+)^*}$, $\{\Lambda^I\}_{I\in(\mathbb{N}_+)^*}$, $\{\Psi^I\}_{I\in(\mathbb{N}_+)^*}$ and $\{\Phi^I\}_{I\in(\mathbb{N}_+)^*}$ are then homogeneous bases of $\mathbf{Sym_k}$. Recall that $S^\emptyset=\Lambda^\emptyset=\Psi^\emptyset=\Phi^\emptyset=1$.

One can also endow $\mathbf{Sym_k}$ with a structure of Hopf algebra, the coproduct $\Delta_\star$ being defined by one of the following equivalent formulas, with the convention that $S_0=\Lambda_0=1$ (Gelfand et al., 1995)

$$\Delta_\star S_n=\sum_{i=0}^{n}S_i\otimes S_{n-i},\qquad \Delta_\star\Lambda_n=\sum_{i=0}^{n}\Lambda_i\otimes\Lambda_{n-i}; \tag{19}$$

$$\Delta_\star\Psi_n=1\otimes\Psi_n+\Psi_n\otimes 1,\qquad \Delta_\star\Phi_n=1\otimes\Phi_n+\Phi_n\otimes 1. \tag{20}$$

In other words, for the coproduct $\Delta_\star$, the power sums of the first kind $\{\Psi_n\}_{n\geq 1}$ and of the second kind $\{\Phi_n\}_{n\geq 1}$ are primitive. The noncommutative symmetric function $S_1=\Lambda_1$ is primitive but $\{S_n\}_{n\geq 2}$ and $\{\Lambda_n\}_{n\geq 2}$ are neither primitive nor group-like. Moreover, by (13), (14) and (15), one has

$$S_1=\Lambda_1=\Phi_1=\Psi_1. \tag{21}$$

With $\Delta_\star$, the concatenation and the counit ϵ defined by

$$\forall I\in(\mathbb{N}_+)^*,\qquad \epsilon(S^I)=\langle S^I\mid 1\rangle, \tag{22}$$

one gets the bialgebra, $(\mathbf{k}\langle S_1,S_2,\ldots\rangle,\bullet,1,\Delta_\star,\epsilon)$, over the $\mathbf{k}$-algebra $\mathbf{Sym_k}$. This algebra, $\mathbb{N}$-graded by the weight is, as we will see in Theorem 3.2, the *concatenation* Hopf algebra.

2.3. Quasi-symmetric functions

Let us consider also an infinite sequence $\{M_n\}_{n\geq 1}$ of noncommuting indeterminates generating the free associative algebra[9]

$$\mathbf{QSym_k} \equiv (\mathbf{k}\langle M_1, M_2, \ldots\rangle, \bullet, 1) \tag{23}$$

and define the elements $\{M_I\}_{I\in(\mathbb{N}_+)^*}$ as follows[10]

$$M_\emptyset = 1, \forall I = (i_1, \ldots, i_k) \in (\mathbb{N}_+)^*, M_I = M_{i_1} \ldots M_{i_k}. \tag{24}$$

The elements $\{M_I\}_{I\in(\mathbb{N}_+)^*}$ of $\mathbf{QSym_k}$ are also called the *monomial quasi-symmetric* functions. They are homogeneous polynomials of degree $w(I)$. This family is then a homogeneous basis of $\mathbf{QSym_k}$.

With the pairing

$$\forall I, J \in (\mathbb{N}_+)^*, \qquad \langle S^I \mid M_J\rangle_{\text{ext}} = \delta_{I,J}, \tag{25}$$

one endows the $\mathbf{k}$-algebra $\mathbf{QSym_k}$ with the structure of a bialgebra dual to $\mathbf{Sym_k}$, $(\mathbf{k}\langle M_1, M_2, \ldots\rangle, \star, 1, \Delta_\bullet, \varepsilon)$. Here,

1. the coproduct $\Delta_\bullet$ is defined by

$$\forall I \in (\mathbb{N}_+)^*, \qquad \Delta_\bullet(M_I) = \sum_{\substack{I_1,I_2\in(\mathbb{N}_+)^* \\ I_1\bullet I_2=I}} M_{I_1} \otimes M_{I_2}, \tag{26}$$

2. the counit ε is defined by

$$\forall I \in (\mathbb{N}_+)^*, \qquad \varepsilon(M_I) = \langle M_I \mid 1\rangle, \tag{27}$$

3. the product $\star$ is commutative and is associated, by (external) duality, to the coproduct $\Delta_\star$. It can also be defined, for any $I \in (\mathbb{N}_+)^*$, by

$$M_I \star M_\emptyset = M_\emptyset \star M_I = M_I \tag{28}$$

and for any $I = (i, I')$ and $J = (j, J') \in (\mathbb{N}_+)^*$

$$M_I \star M_J = M_i(M_{I'} \star M_J) + M_j(M_I \star M_{J'}) + M_{i+j}(M_{I'} \star M_{J'}). \tag{29}$$

Since the bialgebra $\mathbf{QSym_k}$ is $\mathbb{N}$-graded by the weight (as the dual of the $\mathbb{N}$-graded bialgebra $\mathbf{Sym_k}$):

$$\mathbf{QSym_k} = \mathbf{k}.1_{\mathbf{QSym_k}} \oplus \bigoplus_{n\geq 1} \mathbf{QSym}_{\mathbf{k}n} \tag{30}$$

then it is, in fact, the *convolution* Hopf algebra. From the definitions, one has, for any $K, I, J \in (\mathbb{N}_+)^*$,

$$\langle \Delta_\star S^K \mid M_I \otimes M_J\rangle_{\text{ext}} = \langle S^K \mid M_I \star M_J\rangle_{\text{ext}}; \tag{31}$$

$$\langle \Delta_\bullet M_K \mid S^I \otimes S^J\rangle_{\text{ext}} = \langle M_K \mid S^I S^J\rangle_{\text{ext}}. \tag{32}$$

[9] We here use the symbol $\equiv$ to warn the reader that the structure of free algebra is used to construct the basis of $\mathbf{QSym_k}$ which will be later free *as a commutative algebra* (with the quasi-shuffle product) and by no means as a noncommutative algebra (with the concatenation product).

[10] For the readers who are familiar with the standard representation, this (noncommutative) product corresponds to the *ordered* product of series.

3. Monoidal factorization for $\mathrm{Sym}_{\mathbf{k}}$ and $\mathrm{QSym}_{\mathbf{k}}$

3.1. Combinatorics in shuffle Hopf algebras

Let $Y = \{y_i\}_{i\geq 1}$ be a totally ordered alphabet.[11] The free monoid and the set of Lyndon words, over Y, are denoted respectively by Y^* and $\mathcal{L}ynY$. The neutral element of Y^* is denoted by 1_{Y^*}. Let $u = y_{i_1} \ldots y_{i_k} \in Y^*$, the *length* and the *weight* of the word u are defined respectively as the numbers $l(u) = k$ and $w(u) = i_1 + \ldots + i_k$. Let us define the commutative product over $\mathbf{k}Y$, denoted by μ, as follows (Bui et al., 2013b; Enjalbert and Hoang Ngoc Minh, 2012)

$$\forall y_n, y_m \in Y, \qquad \mu(y_n, y_m) = y_{n+m}, \tag{33}$$

or by its associated coproduct, Δ_μ, defined by

$$\forall y_n \in Y, \qquad \Delta_\mu y_n = \sum_{i=1}^{n-1} y_i \otimes y_{n-i} \tag{34}$$

satisfying,

$$\forall x, y, z \in Y, \qquad \langle \Delta_\mu x \mid y \otimes z\rangle = \langle x \mid \mu(y, z)\rangle. \tag{35}$$

Let $\mathbf{k}\langle Y\rangle$ be equipped with

1. The concatenation $\bullet$ (or with its associated coproduct, $\Delta_\bullet$).
2. The *shuffle* product, *i.e.* the commutative product defined by Reutenauer (1993), for any $w \in Y^*$,

$$w \sqcup\!\sqcup 1_{Y^*} = 1_{Y^*} \sqcup\!\sqcup w = w \tag{36}$$

and, for any $x, y \in Y$ and $u, v \in Y^*$,

$$xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v) \tag{37}$$

or with its associated coproduct, $\Delta_{\sqcup\!\sqcup}$, defined, on the letters, by

$$\forall y_k \in Y, \qquad \Delta_{\sqcup\!\sqcup} y_k = y_k \otimes 1 + 1 \otimes y_k \tag{38}$$

and extended by morphism. It satisfies

$$\forall u, v, w \in Y^*, \qquad \langle \Delta_{\sqcup\!\sqcup} w \mid u \otimes v\rangle = \langle w \mid u \sqcup\!\sqcup v\rangle. \tag{39}$$

3. The *quasi-shuffle* product, *i.e.* the commutative product defined by Hoffman (2000), for any $w \in Y^*$,

$$w \uplus 1_{Y^*} = 1_{Y^*} \uplus w = w, \tag{40}$$

and, for any $y_i, y_j \in Y$ and $u, v \in Y^*$,

$$y_i u \uplus y_j v = y_j(y_i u \uplus v) + y_i(u \uplus y_j v) + \mu(y_i, y_j)(u \uplus v) \tag{41}$$

or with its associated coproduct, $\Delta_{\uplus}$, defined, on the letters, by

$$\forall y_k \in Y, \qquad \Delta_{\uplus} y_k = \Delta_{\sqcup\!\sqcup} y_k + \Delta_\mu y_k \tag{42}$$

and extended by morphism. It satisfies

$$\forall u, v, w \in Y^*, \qquad \langle \Delta_{\uplus} w \mid u \otimes v\rangle = \langle w \mid u \uplus v\rangle. \tag{43}$$

[11] By $y_1 > y_2 > y_3 > \ldots$.

Note that $\Delta_{⧢}$ and $\Delta_{\sqcup\!\sqcup}$ are morphisms for the concatenation (by definition) whereas their corresponding Δ_+ is not a morphism[12] for the concatenation (for example, for the two delta(s), one has

$$\Delta_+(y_1^2)=\Delta(y_1^2)-y_1^2\otimes 1-1\otimes y_1^2=2y_1\otimes y_1\,, \tag{44}$$

whereas $\Delta_+(y_1)^2=0$).

Hence, with the counit e defined by

$$\forall P\in\mathbf{k}\langle Y\rangle,\qquad \mathrm{e}(P)=\langle P\mid 1_{Y^*}\rangle, \tag{45}$$

one gets two pairs of mutually dual bialgebras

$$\mathcal{H}_{\sqcup\!\sqcup}=(\mathbf{k}\langle Y\rangle,\bullet,1,\Delta_{\sqcup\!\sqcup},\mathrm{e}),\qquad \mathcal{H}_{\sqcup\!\sqcup}^{\vee}=(\mathbf{k}\langle Y\rangle,\sqcup\!\sqcup,1,\Delta_{\bullet},\mathrm{e}) \tag{46}$$

$$\mathcal{H}_{⧢}=(\mathbf{k}\langle Y\rangle,\bullet,1,\Delta_{⧢},\mathrm{e}),\qquad \mathcal{H}_{⧢}^{\vee}=(\mathbf{k}\langle Y\rangle,⧢,1,\Delta_{\bullet},\mathrm{e}) \tag{47}$$

Let us then consider the following diagonal series[13]

$$\mathcal{D}_{\sqcup\!\sqcup}=\sum_{w\in Y^*}w\otimes w\quad\text{and}\quad \mathcal{D}_{⧢}=\sum_{w\in Y^*}w\otimes w. \tag{48}$$

By the Cartier–Quillen–Milnor and Moore (CQMM in the sequel) theorem (see Bui et al., 2013b), the connected $\mathbb{N}$-graded, co-commutative Hopf algebra $\mathcal{H}_{\sqcup\!\sqcup}$ is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements which is equal to $\mathcal{L}ie_{\mathbf{k}}\langle Y\rangle$:

$$\mathcal{H}_{\sqcup\!\sqcup}\cong\mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle Y\rangle)\quad\text{and}\quad \mathcal{H}_{\sqcup\!\sqcup}^{\vee}\cong\mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle Y\rangle)^{\vee}. \tag{49}$$

Hence, let us consider

1. the PBW–Lyndon basis $\{p_w\}_{w\in Y^*}$ for $\mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle Y\rangle)$ constructed recursively as follows (Chen et al., 1958)

$$\begin{cases} p_y &= y & \text{for } y\in Y,\\ p_l &= [p_s,p_r] & \text{for } l\in\mathcal{L}ynY\setminus Y,\\ p_w &= p_{l_1}^{i_1}\ldots p_{l_k}^{i_k} & \text{for } w=l_1^{i_1}\ldots l_k^{i_k}, \end{cases} \tag{50}$$

   where (s,r) is the standard factorization[14] of l, $l_1\ldots,l_k$ belong to $\mathcal{L}ynY$ such that $l_1>\ldots>l_k$,
2. and, by duality,[15] the basis $\{s_w\}_{w\in Y^*}$ for $(\mathbf{k}\langle Y\rangle,\sqcup\!\sqcup)$, *i.e.*

$$\forall u,v\in Y^*,\ \langle p_u\mid s_v\rangle=\delta_{u,v}. \tag{51}$$

   It can be shown that this linear basis can be computed recursively as follows (Reutenauer, 1993)

$$\begin{cases} s_y &= y, & \text{for } y\in Y,\\ s_l &= ys_u, & \text{for } l=yu\in\mathcal{L}ynY,\\ s_w &= \dfrac{s_{l_1}^{\sqcup\!\sqcup i_1}\sqcup\!\sqcup\ldots\sqcup\!\sqcup s_{l_k}^{\sqcup\!\sqcup i_k}}{i_1!\ldots i_k!} & \text{for } w=l_1^{i_1}\ldots l_k^{i_k} \end{cases} \tag{52}$$

   and $l_1>\ldots>l_k, l_1\ldots,l_k\in\mathcal{L}ynY$.

[12] In a general bialgebra $\Delta_+(g)=\Delta(g)-g\otimes 1-1\otimes g+\epsilon(g)1\otimes 1$.

[13] Here, in $\mathcal{D}_{\sqcup\!\sqcup}$ and $\mathcal{D}_{⧢}$, the operation on the right factor of the tensor product is the concatenation, and on the left is the shuffle and the quasi-shuffle, respectively. One, of course, has $\mathcal{D}_{\sqcup\!\sqcup}=\mathcal{D}_{⧢}$, but we want to stress the fact the treatment will be different.

[14] This is the factorization of $l=sr$ in two Lyndon words such that r is of maximal length.

[15] The dual family, *i.e.* the set of coordinate forms of a basis lies in the algebraic dual which is here the space of noncommutative series, but as the enveloping algebra under consideration is graded in finite dimensions (by the multidegree), these series are in fact multihomogeneous polynomials.

Hence, we get Schützenberger's factorization of $\mathcal{D}_{\sqcup\sqcup}$

$$\mathcal{D}_{\sqcup\sqcup} = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(s_l \otimes p_l) \in \mathcal{H}_{\sqcup\sqcup}^{\vee} \hat{\otimes} \mathcal{H}_{\sqcup\sqcup}. \tag{53}$$

Similarly, by the CQMM theorem, the connected $\mathbb{N}$-graded, co-commutative Hopf algebra $\mathcal{H}_{⧢}$ is isomorphic to the enveloping algebra of its primitive elements:

$$\mathrm{Prim}(\mathcal{H}_{⧢}) = \mathrm{Im}(\pi_1) = \mathrm{span}_{\mathbf{k}}\{\pi_1(w)|w \in Y^*\}, \tag{54}$$

where, for any $w \in Y^*, \pi_1(w)$ is obtained as follows (Hoang Ngoc Minh, 2013a, 2013b)

$$\pi_1(w) = w + \sum_{k\geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1 ⧢ \ldots ⧢ u_k\rangle\, u_1 \ldots u_k\ . \tag{55}$$

Note that (55) is equivalent to the following identity, which will be used later:

$$w = \sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k\in Y^*} \langle w \mid u_1 ⧢ \ldots ⧢ u_k\rangle\, \pi_1(u_1)\ldots\pi_1(u_k). \tag{56}$$

In particular, for any $y_k \in Y$, the primitive polynomial $\pi_1(y_k)$ is given by

$$\pi_1(y_k) = y_k + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{\substack{j_1,\ldots,j_l\geq 1\\ j_1+\ldots+j_l=k}} y_{j_1}\ldots y_{j_l}. \tag{57}$$

As previously, (57) is equivalent to

$$y_n = \sum_{k\geq 1} \frac{1}{k!} \sum_{s_1+\cdots+s_k=n} \pi_1(y_{s_1})\ldots\pi_1(y_{s_k}). \tag{58}$$

By introducing the new alphabet $\bar{Y} = \{\bar{y}\}_{y\in Y} = \{\pi_1(y)\}_{y\in Y}$, one obtains

$$\mathcal{H}_{⧢} \cong \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle\bar{Y}\rangle) \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{⧢})), \tag{59}$$

$$\mathcal{H}_{⧢}^{\vee} \cong \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle\bar{Y}\rangle)^{\vee} \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{⧢}))^{\vee}. \tag{60}$$

We consider

1. the PBW–Lyndon basis $\{\Pi_w\}_{w\in Y^*}$ for $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{⧢}))$ constructed recursively as follows (Hoang Ngoc Minh, 2013a, 2013b)

$$\begin{cases} \Pi_y &= \pi_1(y) & \text{for } y \in Y, \\ \Pi_l &= [\Pi_s, \Pi_r] & \text{for } l \in \mathcal{L}ynY \setminus Y, \\ \Pi_w &= \Pi_{l_1}^{i_1}\ldots\Pi_{l_k}^{i_k} & \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, \end{cases} \tag{61}$$

where (s, r) is the standard factorization of l, and $l_1 \ldots, l_k \in \mathcal{L}ynY$ such that $l_1 > \ldots > l_k$,
2. the basis $\{\Sigma_w\}_{w\in Y^*}$ within[16] $\mathbf{k}\langle Y\rangle$, obtained by duality, *i.e.*

$$\forall u, v \in Y^*,\ \langle\Pi_u \mid \Sigma_v\rangle = \delta_{u,v}\ . \tag{62}$$

It can be shown that this linear basis can be computed recursively as follows (Bui et al., 2013a; Hoang Ngoc Minh, 2013a, 2013b)

[16] Same remark as previously, the grading being here provided by the weight.

$$\begin{cases} \Sigma_y = y, & \text{for } y \in Y, \\ \Sigma_l = \sum\limits_{(!)} \dfrac{y_{s_{k_1}+\cdots+s_{k_i}}}{i!} \Sigma_{l_1\cdots l_n}, & \text{for } l = y_{s_1}\ldots y_{s_k} \in \mathcal{L}ynY, \\ \Sigma_w = \dfrac{\Sigma_{l_1}^{⧢ i_1} ⧢ \ldots ⧢ \Sigma_{l_k}^{⧢ i_k}}{i_1!\ldots i_k!}, & \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, \text{ with} \\ & l_1 > \ldots > l_k \in \mathcal{L}ynY. \end{cases} \tag{63}$$

In (!), the sum is taken over all subsequences $\{k_1,\ldots,k_i\} \subset \{1,\ldots,k\}$ and all Lyndon words $l_1 \geq \cdots \geq l_n$ such that $(y_{s_1},\ldots,y_{s_k}) \stackrel{*}{\Leftarrow} (y_{s_{k_1}},\ldots,y_{s_{k_i}},l_1,\ldots,l_n)$, where $\stackrel{*}{\Leftarrow}$ denotes the transitive closure of the relation on standard sequences, denoted by $\Leftarrow$ (see Bui et al., 2013a).

Example 1 *(of $\{\Pi_w\}_{w\in Y^*}$ and $\{\Sigma_w\}_{w\in Y^*}$).*

$$\begin{aligned}
\Pi_{y_1} &= y_1,\\
\Pi_{y_2} &= y_2 - \tfrac{1}{2}y_1^2,\\
\Pi_{y_2y_1} &= y_2y_1 - y_1y_2,\\
\Pi_{y_3y_1y_2} &= y_3y_1y_2 - \tfrac{1}{2}y_3y_1^3 - y_2y_1^2y_2\\
&+ \tfrac{1}{4}y_2y_1^4 - y_1y_3y_2 + \tfrac{1}{2}y_1y_3y_1^2 + \tfrac{1}{2}y_1^2y_2^2 - \tfrac{1}{2}y_1^2y_2y_1^2\\
&- y_2y_3y_1 + \tfrac{1}{2}y_2^2y_1^2 + y_2y_1y_3 + \tfrac{1}{2}y_1^2y_3y_1 - \tfrac{1}{2}y_1^3y_3 + \tfrac{1}{4}y_1^4y_2,\\
\Pi_{y_3y_1y_2y_1} &= y_3y_1y_2y_1 - y_3y_1^2y_2 - \tfrac{1}{2}y_2y_1^2y_2y_1 - y_1y_3y_2y_1 + y_1y_3y_1y_2\\
&+ \tfrac{1}{2}y_1^2y_2^2y_1 - y_2y_1y_3y_1 - \tfrac{1}{2}y_1^2y_2y_1y_2 + \tfrac{1}{2}y_2y_1y_2y_1^2\\
&+ y_2y_1^2y_3 + y_1y_2y_3y_1 - \tfrac{1}{2}y_1y_2^2y_1^2 - y_1y_2y_1y_3 + \tfrac{1}{2}y_1y_2y_1^2y_2,\\
\Sigma_{y_1} &= y_1,\\
\Sigma_{y_2} &= y_2,\\
\Sigma_{y_2y_1} &= y_2y_1 + \tfrac{1}{2}y_3,\\
\Sigma_{y_3y_1y_2} &= y_3y_2y_1 + y_3y_1y_2 + y_3^2 + \tfrac{1}{2}y_4y_2 + \tfrac{1}{3}y_6 + \tfrac{1}{2}y_5y_1,\\
\Sigma_{y_3y_1y_2y_1} &= y_3y_1y_2y_1 + 2y_3y_2y_1^2 + y_3y_2^2 + \tfrac{3}{2}y_3^2y_1 + \tfrac{1}{2}y_3y_1y_3 + \tfrac{1}{2}y_3y_4\\
&+ \tfrac{1}{2}y_4y_2y_1 + \tfrac{1}{4}y_4y_3 + y_5y_1^2 + \tfrac{1}{2}y_5y_2 + \tfrac{1}{2}y_6y_1 + \tfrac{1}{8}y_7.
\end{aligned}$$

We get the following factorization for $\mathcal{D}_⧢$ (Bui et al., 2013a; Hoang Ngoc Minh, 2013a, 2013b)

$$\mathcal{D}_⧢ = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\Sigma_l \otimes \Pi_l) \in \mathcal{H}_⧢^{\vee} \hat{\otimes} \mathcal{H}_⧢. \tag{64}$$

3.2. Encoding noncommutative symmetric and quasi-symmetric functions by words

Proposition 1. *Let $\mathcal{Y}(t)$ be the following ordinary generating series of $\{y_n\}_{n\geq 1}$:*

$$\mathcal{Y}(t) = 1 + \textstyle\sum_{n\geq 1} y_n\, t^n \;\in\; \mathbb{Q}\langle Y\rangle[\![t]\!].$$

Then $\mathcal{Y}(t)$ is group-like, for the coproduct $\Delta_⧢$.

Proof. We have successively (here, in order to make complete the correspondence $\mathcal{S}$, we put $y_0 = 1$)

$$\Delta_⧢ \mathcal{Y}(t) = \sum_{n\geq 0}\Big[\sum_{r+s=n} y_s \otimes y_r\Big] t^n = \sum_{n\geq 0}\sum_{r+s=n} (y_s t^s) \otimes (y_r t^r).$$

Thus, $\Delta_⧢ \mathcal{Y} = \mathcal{Y} \hat{\otimes} \mathcal{Y}$ meaning $\mathcal{Y}$ is group-like ($\epsilon(\mathcal{Y}) = 1$). $\square$

Proposition 2. *We have $\mathcal{G} = \mathrm{Prim}(\mathcal{H}_⧢)$, where $\mathcal{G}$ is the Lie algebra generated by $\{\langle \log \mathcal{Y} \mid t^n\rangle\}_{n\geq 1}$.*

Proof. The power series $\log \mathcal{Y} \in \mathbb{Q}\langle Y\rangle[\![t]\!]$ is primitive, so by expanding it, we obtain

$$\log \mathcal{Y}(t) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \left(\sum_{s_1,\ldots,s_k\geq 1} y_{s_1}\ldots y_{s_k} t^{s_1}\ldots t^{s_k} \right).$$

By (57), we get, for any $n \geq 1$, $\langle \log \mathcal{Y} \mid t^n\rangle = \pi_1(y_n)$ and since $\{\pi_1(y_n)\}_{n\geq 1}$ generates freely $\mathrm{Prim}(\mathcal{H}_{⧢})$ (Hoang Ngoc Minh, 2013a, 2013b), the expected results follow. □

Corollary 1.

$$\mathcal{Y}(t) = 1 + \sum_{n\geq 1} \sum_{k\geq 1} \frac{1}{k!} \sum_{s_1+\cdots+s_k=n} \pi_1(y_{s_1})\ldots\pi_1(y_{s_k}) t^n,$$
$$\dot{\mathcal{Y}}^{-1} = -\mathcal{Y}^{-1}\dot{\mathcal{Y}}\mathcal{Y}^{-1}.$$

Proof. By virtue of (57) and (58), we get the first result. By using the identities $\mathcal{Y}\mathcal{Y}^{-1} = \mathcal{Y}^{-1}\mathcal{Y} = 1_{Y^*}$ and then by differentiating, we get the second one. □

Corollary 2. *Let us write the (group-like) power series $\mathcal{Y}^{-1} \in \mathbb{Q}\langle Y\rangle[\![t]\!]$ as follows*

$$\mathcal{Y}(t)^{-1} = 1 + \sum_{n\geq 1} X_n\, t^n.$$

Then, for any $n \geq 1$, one has

$$\sum_{i=0}^{n} y_i X_{n-i} = 0 \quad \textit{and} \quad \sum_{i=0}^{n} X_i y_{n-i} = 0.$$

From this, we obtain an inductive formula of X_n as follows

$$X_0 = 1, \qquad \forall n \geq 1,\, X_n = -\sum_{i=1}^{n} y_i X_{n-i}.$$

Proof. The results follow immediately by identification of the coefficients of t^n in the second identity of Corollary 1. □

Corollary 3. *There exists a unique generating series $L \in \mathbb{Q}\langle Y\rangle[\![t]\!]$ satisfying $\dot{\mathcal{Y}} = L\mathcal{Y}$ and a unique generating series $R \in \mathbb{Q}\langle Y\rangle[\![t]\!]$ satisfying $\dot{\mathcal{Y}} = \mathcal{Y}R$. Moreover, L and R are primitive and, defining*

$$L(t) = \sum_{n\geq 1} L_n\, t^{n-1} \quad \textit{and} \quad R(t) = \sum_{n\geq 1} R_n\, t^{n-1},$$

one has, for any $n \geq 1$,

$$ny_n = \sum_{i=1}^{n} L_i y_{n-i} \quad \textit{and} \quad ny_n = \sum_{i=1}^{n} y_i R_{n-i},$$
$$L_n = \sum_{i=1}^{n} i y_i X_{n-i} \quad \textit{and} \quad R_n = \sum_{i=1}^{n} i X_{n-i} y_i.$$

Proof. On the one hand, by Proposition 1, one has

$$\frac{d}{dt}\mathcal{Y}(t) = \sum_{n\geq 1} ny_n\, t^{n-1}.$$

On the other hand, such generating series exist since

$$\begin{aligned} \dot{\mathcal{Y}} = L\mathcal{Y} &\quad \text{and} \quad \dot{\mathcal{Y}} = \mathcal{Y}R, \\ \iff L = \dot{\mathcal{Y}}\mathcal{Y}^{-1} &\quad \text{and} \quad R = \mathcal{Y}^{-1}\dot{\mathcal{Y}}. \end{aligned}$$

Hence, identifying the coefficients of t^n, the expected results follow. Moreover, since $\Delta_{\sqcup\!\sqcup}$ commutes with d/dt and is a morphism for the concatenation,

$$\begin{aligned} \Delta_{\sqcup\!\sqcup} L &= (\dot{\mathcal{Y}}\hat{\otimes}\mathcal{Y} + \mathcal{Y}\hat{\otimes}\dot{\mathcal{Y}})(\mathcal{Y}^{-1}\hat{\otimes}\mathcal{Y}^{-1}) \\ &= \dot{\mathcal{Y}}\mathcal{Y}^{-1}\hat{\otimes}\mathcal{Y}\mathcal{Y}^{-1} + \mathcal{Y}\mathcal{Y}^{-1}\hat{\otimes}\dot{\mathcal{Y}}\mathcal{Y}^{-1} \\ &= \dot{\mathcal{Y}}\mathcal{Y}^{-1}\hat{\otimes}1_{Y^*} + 1_{Y^*}\hat{\otimes}\dot{\mathcal{Y}}\mathcal{Y}^{-1}, \\ \Delta_{\sqcup\!\sqcup} R &= (\mathcal{Y}^{-1}\hat{\otimes}\mathcal{Y}^{-1})(\dot{\mathcal{Y}}\hat{\otimes}\mathcal{Y} + \mathcal{Y}\hat{\otimes}\dot{\mathcal{Y}}) \\ &= \mathcal{Y}^{-1}\dot{\mathcal{Y}}\hat{\otimes}\mathcal{Y}^{-1}\mathcal{Y} + \mathcal{Y}^{-1}\mathcal{Y}\hat{\otimes}\mathcal{Y}^{-1}\dot{\mathcal{Y}} \\ &= \mathcal{Y}^{-1}\dot{\mathcal{Y}}\hat{\otimes}1_{Y^*} + 1_{Y^*}\hat{\otimes}\mathcal{Y}^{-1}\dot{\mathcal{Y}}. \end{aligned}$$

Hence, $\Delta_{\sqcup\!\sqcup} L = 1_{Y^*}\hat{\otimes}L + L\hat{\otimes}1_{Y^*}$ and $\Delta_{\sqcup\!\sqcup} R = 1_{Y^*}\hat{\otimes}R + R\hat{\otimes}1_{Y^*}$ meaning that L and R are primitive. □

More generally, with the notations of Corollary 3, one has

Proposition 3. *For any $k \ge 1$, there exists a unique generating series $\mathcal{L}_k \in \mathbb{Q}\langle Y\rangle[\![t]\!]$ satisfying $\mathcal{Y}^{(k)} = \mathcal{L}_k\mathcal{Y}$ and a unique generating series $\mathcal{R}_k \in \mathbb{Q}\langle Y\rangle[\![t]\!]$ satisfying $\mathcal{Y}^{(k)} = \mathcal{Y}R_k$. The families $\{\mathcal{L}_k\}_{k\ge1}$ and $\{\mathcal{R}_k\}_{k\ge1}$ are defined recursively as follows*

$$\begin{aligned} \mathcal{L}_1 = L &\quad \text{and} \quad \mathcal{L}_k = \dot{\mathcal{L}}_{k-1} + \mathcal{L}_{k-1}L, \\ \mathcal{R}_1 = R &\quad \text{and} \quad \mathcal{R}_k = \dot{\mathcal{R}}_{k-1} + R\mathcal{R}_{k-1}. \end{aligned}$$

Proof. (By induction.) For $k = 1$, it is Corollary 3. Then, suppose that the property holds for any $1 \le n \le k-1$, we see that, for $n = k$, such generating series exist since, by the induction hypothesis,

$$\begin{aligned} \mathcal{Y}^{(k)} &= \dot{\mathcal{L}}_{k-1}\mathcal{Y} + \mathcal{L}_{k-1}\dot{\mathcal{Y}} = (\dot{\mathcal{L}}_{k-1} + \mathcal{L}_{k-1}L)\mathcal{Y} \\ \mathcal{Y}^{(k)} &= \dot{\mathcal{Y}}\mathcal{R}_{k-1} + \mathcal{Y}\dot{\mathcal{R}}_{k-1} = \mathcal{Y}(R\mathcal{R}_{k-1} + \dot{\mathcal{R}}_{k-1}). \end{aligned}$$

Hence, $\mathcal{L}_k = \dot{\mathcal{L}}_{k-1} + \mathcal{L}_{k-1}L$ and $\mathcal{R}_k = R\mathcal{R}_{k-1} + \dot{\mathcal{R}}_{k-1}$. □

Corollary 4. *For any proper power series A, B, let $\mathrm{ad}_A^n B$ be the iterated Lie brackets defined by $\mathrm{ad}_A^0 B = B$ and $\mathrm{ad}_A^{n+1} B = [A, \mathrm{ad}_A^n B]$, for $n \ge 0$. Then,*

$$\mathcal{L}_k = \sum_{n\ge0} \frac{\mathrm{ad}^n_{\log\mathcal{Y}}\mathcal{R}_k}{n!} \quad \textit{and} \quad \mathcal{R}_k = \sum_{n\ge0}(-1)^n \frac{\mathrm{ad}^n_{\log\mathcal{Y}}\mathcal{L}_k}{n!}.$$

Proof. Since $\mathcal{L}_k\mathcal{Y} = \mathcal{Y}\mathcal{R}_k$ then

$$\begin{aligned} \mathcal{L}_k &= \mathcal{Y}\mathcal{R}_k\mathcal{Y}^{-1} = \exp(\log\mathcal{Y})\mathcal{R}_k\exp(-\log\mathcal{Y}) = \exp(\mathrm{ad}_{\log\mathcal{Y}})\mathcal{R}_k, \\ \mathcal{R}_k &= \mathcal{Y}^{-1}\mathcal{L}_k\mathcal{Y} = \exp(-\log\mathcal{Y})\mathcal{L}_k\exp(\log\mathcal{Y}) = \exp(\mathrm{ad}_{-\log\mathcal{Y}})\mathcal{L}_k. \end{aligned}$$

Expanding exp, the results follow. □

Proposition 4. *Let $\mathcal{G}$ be the Lie algebra generated by $\{R_n\}_{n\ge1}$ (resp. $\{L_n\}_{n\ge1}$). Then $\mathcal{G} = \mathrm{Prim}(\mathcal{H}_{\sqcup\!\sqcup})$.*

Proof. By Corollary 3, one has on the one hand,

$$\sum_{n\geq 1}\Delta_{\text{⧢}} R_n\, t^{n-1} = 1_{Y^*}\otimes\sum_{n\geq 1}R_n\, t^{n-1}+\sum_{n\geq 1}R_n\, t^{n-1}\otimes 1_{Y^*}$$
$$=\sum_{n\geq 1}(1_{Y^*}\otimes R_n+R_n\otimes 1_{Y^*})t^{n-1}.$$

Thus, by identifying the coefficients of t^{n-1} in the first and last sums, one has $\Delta_{\text{⧢}} R_n = 1_{Y^*}\otimes R_n + R_n\otimes 1_{Y^*}$, meaning that R_n is primitive. On the other hand, according to basic properties of quasi-determinants (Gelfand and Retakh, 1991, 1992, see also Gelfand et al., 1995), one has

$$ny_n=\begin{vmatrix} R_1 & R_2 & \dots & R_{n-1} & \boxed{R_n}\\ -1 & R_1 & \dots & R_{n-2} & R_{n-1}\\ 0 & -2 & \dots & R_{n-3} & R_{n-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & -n+1 & R_1\end{vmatrix}=\begin{vmatrix} R_1 & R_2 & \dots & R_{n-1} & \boxed{R_n}\\ -1 & R_1 & \dots & R_{n-2} & R_{n-1}\\ 0 & -1 & \dots & \frac{1}{2}R_{n-3} & \frac{1}{2}R_{n-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & -1 & \frac{1}{n+1}R_1\end{vmatrix}. \qquad \square$$

Hence, for any $J=(j_1,\dots,j_n)\in(\mathbb{N}_+)^*$, by denoting $R^J=R_{j_1}\dots R_{j_n}$, one obtains

$$y_n=\sum_{w(J)=n}\frac{R^J}{\pi(J)}=\frac{R_n}{n}+\sum_{w(J)=n,l(J)>1}\frac{R^J}{\pi(J)}. \tag{65}$$

It means that y_n is triangular and homogeneous in weight w.r.t. $\{R_k\}_{k\geq 1}$. Conversely, R_n is also triangular and homogeneous in weight in $\{y_k\}_{k\geq 1}$. The R_k's are linearly independent and generating and then constitute a new alphabet. In the same way, the L_k's are primitive, generating and linearly independent. The expected results follow.

Definition 1. For $S=L$ or R, let

1. $\{\Pi_w^{(S)}\}_{w\in Y^*}$ be the families of $\mathcal{H}_{\text{⧢}}$ defined as follows
$$\begin{cases}\Pi_{y_n}^{(S)} &= L_n \text{ if } S=L\\ &\quad\text{or } R_n \text{ if } S=R, \qquad \text{for } y_n\in Y,\\ \Pi_l^{(S)} &= [\Pi_s^{(S)},\Pi_r^{(S)}], \qquad \text{for } l=(s,r)\in\mathcal{L}ynY,\\ \Pi_w^{(S)} &= (\Pi_{l_1}^{(S)})^{i_1}\dots(\Pi_{l_k}^{(S)})^{i_k}, \quad \text{for } w=l_1^{i_1}\dots l_k^{i_k}\end{cases}$$
and $l_1>\dots>l_k, l_1\dots,l_k\in\mathcal{L}ynY$.
2. $\{\Sigma_w^{(S)}\}_{w\in Y^*}$ be the families[17] obtained by duality with $\{\Pi_w^{(S)}\}_{w\in Y^*}$:
$$\forall u,v\in Y^*,\ \langle\Pi_u^{(S)}\mid\Sigma_v^{(S)}\rangle=\delta_{u,v}.$$

Theorem 3.1.

1. *The family* $\{\Pi_l^{(S)}\}_{l\in\mathcal{L}ynY}$ *forms a basis of the Lie algebra* $\mathrm{Prim}(\mathcal{H}_{\text{⧢}})$.
2. *The family* $\{\Pi_w^{(S)}\}_{w\in Y^*}$ *forms a basis of* $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\text{⧢}}))=\mathcal{H}_{\text{⧢}}$.
3. *The family* $\{\Sigma_w^{(S)}\}_{w\in Y^*}$ *is a basis of* $\mathbf{k}\langle Y\rangle$.
4. *The family* $\{\Sigma_l^{(S)}\}_{l\in\mathcal{L}ynY}$ *forms a transcendence basis of* $(\mathbf{k}\langle Y\rangle,\text{⧢},1_{Y^*})$.

[17] *A priori* they are series but, due to the grading in weight, they are in fact polynomials.

Proof. The family $\{\Pi_l^{(S)}\}_{l\in\mathcal{L}ynY}$ of primitive upper triangular homogeneous in weight polynomials is generating and free and the first result follows. The second is a direct consequence of the PBW theorem. By duality with the second (and from the fact that the bases are homogeneous in weight), we get the third one and the last one is obtained as a consequence of the constructions of $\{\Sigma_l^{(S)}\}_{l\in\mathcal{L}ynY}$ and $\{\Sigma_w^{(S)}\}_{w\in Y^*}$. $\square$

Corollary 5. *We have, for $S = L$ or R,*

$$\mathcal{D}_{\uplus} = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\Sigma_l^{(S)} \otimes \Pi_l^{(S)}).$$

Example 2 *(of $\{\Pi_w^{(L)}\}_{w\in Y^*}$ and $\{\Sigma_w^{(L)}\}_{w\in Y^*}$).*

$$\begin{aligned}
\Pi_{y_1}^{(L)} &= y_1,\\
\Pi_{y_2}^{(L)} &= 2y_2 - y_1^2,\\
\Pi_{y_2y_1}^{(L)} &= 2y_2y_1 - 2y_1y_2,\\
\Pi_{y_3y_1y_2}^{(L)} &= 6y_3y_1y_2 - 3y_3y_1^3 - 3y_1^2y_2y_1^2 + 2y_1^2y_2^2\\
&\quad - 6y_2y_1^2y_2 + 2y_2y_1^4 + 2y_1y_2y_1y_2 - y_1y_2y_1^3\\
&\quad + 3y_1y_3y_1^2 - 6y_1y_3y_2 + y_1^3y_2y_1 + 3y_1^2y_3y_1\\
&\quad + y_1^4y_2 - 3y_1^3y_3 - 2y_2y_1y_2y_1 + 4y_2^2y_1^2 - 6y_2y_3y_1 + 6y_2y_1y_3,\\
\Pi_{y_3y_1y_2y_1}^{(L)} &= 6y_3y_1y_2y_1 + 2y_2y_1^3y_2 - 6y_2y_1^2y_2y_1 + 4y_1y_2y_1y_2y_1\\
&\quad - 6y_3y_1^2y_2 + 2y_1^2y_2^2y_1 - 2y_1^2y_2y_1y_2 - 6y_1y_3y_2y_1\\
&\quad + 6y_1y_3y_1y_2 + 4y_2y_1y_2y_1^2 - 6y_2y_1y_3y_1 + 6y_2y_1^2y_3\\
&\quad - 4y_1y_2^2y_1^2 + 6y_1y_2y_3y_1 - 6y_1y_2y_1y_3,\\
\Sigma_{y_1}^{(L)} &= y_1,\\
\Sigma_{y_2}^{(L)} &= \frac{1}{2}y_2,\\
\Sigma_{y_2y_1}^{(L)} &= \frac{1}{2}y_2y_1 + \frac{1}{3}y_3,\\
\Sigma_{y_3y_1y_2}^{(L)} &= \frac{1}{6}y_3y_1y_2 + \frac{1}{6}y_3y_2y_1 + \frac{1}{6}y_3^2 + \frac{1}{8}y_4y_2 + \frac{1}{10}y_5y_1 + \frac{1}{12}y_6,\\
\Sigma_{y_3y_1y_2y_1}^{(L)} &= \frac{1}{6}y_3y_1y_2y_1 + \frac{1}{9}y_3y_1y_3 + \frac{1}{3}y_3y_2y_1^2 + \frac{1}{6}y_3y_2^2 + \frac{5}{18}y_3^2y_1 + \frac{1}{9}y_3y_4\\
&\quad + \frac{1}{8}y_4y_2y_1 + \frac{1}{12}y_4y_3 + \frac{1}{5}y_5y_1^2 + \frac{1}{10}y_5y_2 + \frac{5}{36}y_6y_1 + \frac{1}{21}y_7.
\end{aligned}$$

Example 3 *(of $\{\Pi_w^{(R)}\}_{w\in Y^*}$ and $\{\Sigma_w^{(R)}\}_{w\in Y^*}$).*

$$\begin{aligned}
\Pi_{y_1}^{(R)} &= y_1,\\
\Pi_{y_2}^{(R)} &= 2y_2 - y_1^2,\\
\Pi_{y_2y_1}^{(R)} &= 2y_2y_1 - 2y_1y_2,\\
\Pi_{y_3y_1y_2}^{(R)} &= 6y_3y_1y_2 + y_2y_1^4 - 6y_2y_1^2y_2 + y_1y_2y_1^3 - 2y_1y_2y_1y_2 - 3y_3y_1^3
\end{aligned}$$

$$
\begin{aligned}
&- 3y_1^2y_2y_1^2 + 4y_1^2y_2^2 + 3y_1y_3y_1^2 - 6y_1y_3y_2 + 2y_2y_1y_2y_1 - 6y_2y_3y_1\\
&- y_1^3y_2y_1 + 3y_1^2y_3y_1 + 2y_1^4y_2 - 3y_1^3y_3 + 2y_2^2y_1^2 + 6y_2y_1y_3,\\
\Pi^{(R)}_{y_3y_1y_2y_1} &= 6y_3y_1y_2y_1 - 2y_2y_1^3y_2 - 4y_1y_2y_1y_2y_1 + 6y_1y_2y_1^2y_2 - 6y_3y_1^2y_2\\
&+ 4y_1^2y_2^2y_1 - 4y_1^2y_2y_1y_2 - 6y_1y_3y_2y_1 + 6y_1y_3y_1y_2 + 6y_1y_2y_3y_1\\
&+ 2y_2y_1y_2y_1^2 - 6y_2y_1y_3y_1 + 6y_2y_1^2y_3 - 2y_1y_2^2y_1^2 - 6y_1y_2y_1y_3,\\
\Sigma^{(R)}_{y_1} &= y_1,\\
\Sigma^{(R)}_{y_2} &= \frac{1}{2}y_2,\\
\Sigma^{(R)}_{y_2y_1} &= \frac{1}{2}y_2y_1 + \frac{1}{6}y_3,\\
\Sigma^{(R)}_{y_3y_1y_2} &= \frac{1}{6}y_3y_1y_2 + \frac{1}{40}y_6 + \frac{1}{15}y_5y_1 + \frac{1}{24}y_4y_2 + \frac{1}{6}y_3^2 + \frac{1}{6}y_3y_2y_1,\\
\Sigma^{(R)}_{y_3y_1y_2y_1} &= \frac{1}{6}y_3y_1y_2y_1 + \frac{1}{18}y_3y_1y_3 + \frac{13}{2520}y_7 + \frac{13}{360}y_6y_1 + \frac{1}{15}y_5y_2\\
&+ \frac{1}{72}y_4y_3 + \frac{1}{24}y_4y_2y_1 + \frac{1}{18}y_3y_4 + \frac{2}{9}y_3^2y_1 + \frac{1}{6}y_3y_2^2\\
&+ \frac{2}{15}y_5y_1^2 + \frac{1}{3}y_3y_2y_1^2.
\end{aligned}
$$

Note that the $u = y_{i_1} \dots y_{i_k} \in Y^*$ are in one-to-one correspondence with the $I(u) = (i_1, \dots, i_k) \in (\mathbb{N}_+)^*$ (1_{Y^*} corresponding to [], the empty list). Note also that noncommutative symmetric and quasi-symmetric functions can be indexed by words in Y^* instead of compositions in $(\mathbb{N}_+)^*$. Indeed, let J be a composition, finer than I, associated to the word v and let $J = (J_1, \dots, J_k)$ be the decomposition of J such that, for any $p = 1, \dots, k$, $w(J_p) = i_p$ and J_p is associated to u_p with $w(u_p) = i_p$. Then, if $(s_1, \dots, s_k) = I(v) \preceq I(u)$, u has the unique factorization $u = u_1 \dots u_k$ such that the weight of u_j is s_j and this will be denoted as a bracketing of the word u.

Example 4. One has

- $(1, 2, 2) \preceq (1, (1, 1), 2) = (1, 1, 1, 2) \longleftrightarrow y_1y_2y_2 \preceq y_1(y_1y_1)y_2 = y_1y_1y_1y_2$.
- $(1, 2, 2) \preceq (1, 2, (1, 1)) = (1, 2, 1, 1) \longleftrightarrow y_1y_2y_2 \preceq y_1y_2(y_1y_1) = y_1y_2y_1y_1$.
- $(1, 2, 2) \preceq (1, (1, 1), (1, 1)) = (1, 1, 1, 1, 1) \longleftrightarrow y_1y_2y_2 \preceq y_1(y_1y_1)(y_1y_1) = y_1y_1y_1y_1y_1$.

Hence, we can state the following

Definition 2. Let $\mathcal{S}$ and $\mathcal{M}$ be the following linear maps

$$
\begin{aligned}
\mathcal{S} : (\mathbf{k}\langle Y\rangle, \bullet, 1, \Delta_{\text{⧢}}, \mathrm{e}) &\longrightarrow (\mathbf{k}\langle S_1, S_2, \dots\rangle, \bullet, 1, \Delta_\star, \epsilon),\\
u = y_{i_1} \dots y_{i_k} &\longmapsto \mathcal{S}(u) = S^{(i_1,\dots,i_k)},\\
\mathcal{M} : (\mathbf{k}\langle Y\rangle, \text{⧢}, 1, \Delta_\bullet, \mathrm{e}) &\longrightarrow (\mathbf{k}\langle M_1, M_2, \dots\rangle, \star, 1, \Delta_\bullet, \varepsilon),\\
u = y_{i_1} \dots y_{i_k} &\longmapsto \mathcal{M}(u) = M_{(i_1,\dots,i_k)}.
\end{aligned}
$$

Theorem 3.2. *The maps $\mathcal{S}$ and $\mathcal{M}$ are isomorphisms of Hopf algebras.*

Corollary 6. *Let $\mathcal{G}$ be the Lie algebra generated by $\{\Pi_y\}_{y\in Y}$. Then, we have $\mathbf{Sym_k} \cong \mathcal{U}(\mathcal{G})$.*

Corollary 7. *The families $\{\mathcal{M}(l)\}_{l\in\mathcal{L}ynY}$ and $\{\mathcal{M}(\Sigma_l)\}_{l\in\mathcal{L}ynY}$ are pure transcendence bases of the free commutative* $\mathbf{k}$*-algebra* $\mathbf{QSym_k}$.

Corollary 8. *Let $w = y_{i_1} \dots y_{i_k} \in Y^*$ be associated to $I(w) = (i_1, \dots, i_k) \in (\mathbb{N}_+)^*$. Then, we have*

$$S^I = \mathcal{S}(w), \quad \frac{\Phi^I}{\pi(I)} = \mathcal{S}(\pi_1(y_{i_1}) \dots \pi_1(y_{i_k})), \quad \Psi^I = \mathcal{S}(R_w).$$

Proof. On the one hand, the power series $\mathcal{Y}, \log\mathcal{Y}$ and $L, R \in \mathbf{k}\langle Y\rangle[\![t]\!]$ are summable. On the other hand, by (9) and (10), since $\mathcal{S}$ is continuous and commutes with log, one can deduce

$$\begin{aligned}
\sigma(t) = \mathcal{S}(\mathcal{Y}(t)) &= 1 + \sum_{k\ge1} \mathcal{S}(y_k)t^k, \\
\sum_{k\ge1} \frac{\Phi_k}{k} t^k = \log\sigma(t) &= \mathcal{S}(\log\mathcal{Y}(t)) = \sum_{k\ge1} \mathcal{S}(\pi_1(y_k))t^k, \\
\sum_{k\ge1} \Psi_k t^{k-1} = \psi(t) &= \mathcal{S}(R(t)) = \sum_{k\ge1} \mathcal{S}(R_k)t^{k-1}, \\
\sum_{k\ge1} t^{k-1}\Psi_k^* = \psi^*(t) &= \mathcal{S}(L(t)) = \sum_{k\ge1} \mathcal{S}(L_k)t^{k-1}.
\end{aligned}$$

Thus, the expected result follows immediately. □

3.3. Monoidal factorization and dual bases for noncommutative symmetric and quasi-symmetric functions

Definition 3. With the notations of (64), let us consider the following noncommutative generating series $\{\mathcal{M}(w)\}_{w\in Y^*}$ and $\{\mathcal{S}(w)\}_{w\in Y^*}$

$$\begin{aligned}
\mathrm{M} &= \sum_{w\in Y^*} \mathcal{M}(w)\, w \in \mathbf{QSym_k}\langle\langle Y\rangle\rangle, \\
\mathrm{S} &= \sum_{w\in Y^*} \mathcal{S}(w)\, w \in \mathbf{Sym_k}\langle\langle Y\rangle\rangle.
\end{aligned}$$

Proposition 5. *For the coproduct $\Delta_{\uplus}$, using (64), the generating series* M *is group-like (and then the generating series* $\log \mathrm{M}$ *is primitive).*

Proof. The first result follows from Friedrichs' criterion (Hoang Ngoc Minh, 2013a, 2013b). By using the previous result and by applying the log map on the power series M, we get the second result. □

Corollary 9.

$$\begin{aligned}
\mathrm{M} &= \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\mathcal{M}(\Sigma_l)\, \Pi_l) \in \mathbf{QSym_k}\langle\langle Y\rangle\rangle, \\
\log\mathrm{M} &= \sum_{w\in Y^*} \mathcal{M}(w)\, \pi_1(w) \quad \in \mathbf{QSym_k}\langle\langle Y\rangle\rangle.
\end{aligned}$$

Proof. The first identity is equivalent to the image of the diagonal series $\mathcal{D}_{\uplus}$ by the tensor $\mathcal{M}\otimes\mathrm{Id}$. The second one is then equivalent to the image of

$$\log\mathcal{D}_{\uplus} = \mathrm{Id}\otimes\pi_1(\mathcal{D}_{\uplus})$$

by the tensor $\mathcal{M}\otimes\mathrm{Id}$. □

Finally, using (64) we deduce the following property which completes formula (120) given in Gelfand et al. (1995):

Corollary 10. *We have, for $S = L$ or R,*

$$\sum_{w\in Y^*} \mathcal{M}(w)\mathcal{S}(w) = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\mathcal{M}(\Sigma_l)\,\mathcal{S}(\Pi_l)) = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\mathcal{M}(\Sigma_l^{(S)})\,\mathcal{S}(\Pi_l^{(S)})).$$

Or equivalently,

$$\sum_{w\in Y^*} M_w S_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(M_{\Sigma_l}\, S_{\Pi_l}) = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(M_{\Sigma_l^{(S)}}\, S_{\Pi_l^{(S)}})$$

Proof. By Theorem 3.2, the first double identity is obtained as the image, by the tensor $\mathcal{M}\otimes\mathcal{S}$, of the diagonal series $\mathcal{D}_{\text{ш}}$ and the second as the image, by the tensor $\mathrm{Id}\otimes\pi_1$, of the series M. □

Note that these formulas are universal for any pair of bases in duality one of them being of PBW type and they do not depend on the specific (infinite) alphabets, usually denoted by A and X, used to define $S(A)\in\mathbf{Sym_k}(A)$ and $M(X)\in\mathbf{QSym_k}(X)$.

Example 5 *(Cauchy type identity).* Let A be a noncommutative alphabet and X a totally ordered commutative alphabet. The symmetric functions of the noncommutative alphabet XA are defined by means of

$$\sigma(XA;t) = \sum_{n\ge 0} S_n(XA)t^n := \prod_{x\in X}^{\leftarrow} \sigma(A;xt).$$

Let $\{U_I\}_{I\in(\mathbb{N}_+)^*}$ and $\{V_I\}_{I\in(\mathbb{N}_+)^*}$ be linear bases of respectively $\mathbf{Sym_k}(A)$ and $\mathbf{QSym_k}(X)$. The duality of these means that[18]

$$\sigma(XA;1) = \sum_{I\in(\mathbb{N}_+)^*} M_I(X)S^I(A) = \sum_{I\in(\mathbb{N}_+)^*} V_I(X)\,U_I(A).$$

Typically, the linear basis $\{U_I\}_{I\in(\mathbb{N}_+)^*}$ is the basis of *ribbon* Schur functions $\{\mathrm{R}_I\}_{I\in(\mathbb{N}_+)^*}$, and, by duality, $\{V_I\}_{I\in(\mathbb{N}_+)^*}$ is the basis of *quasi-ribbon* Schur functions $\{\mathrm{F}_I\}_{I\in(\mathbb{N}_+)^*}$:

$$\begin{aligned}
\sigma(XA;1) &= \sum_{I\in(\mathbb{N}_+)^*} M_I(X)\Bigg[\sum_{\substack{I,J\in(\mathbb{N}_+)^*\\ J\preceq I}} \mathrm{R}_I(A)\Bigg]\\
&= \sum_{J\in(\mathbb{N}_+)^*}\Bigg[\sum_{\substack{I,J\in(\mathbb{N}_+)^*\\ I\succeq J}} M_I(X)\Bigg]\mathrm{R}_J(A)\\
&= \sum_{J\in(\mathbb{N}_+)^*} \mathrm{F}_J(X)\mathrm{R}_J(A).
\end{aligned}$$

Also, if one specializes the alphabets of the quasi-symmetric functions $\{M_I\}_{I\in(\mathbb{N}_+)^*}$ and $\{F_I\}_{I\in(\mathbb{N}_+)^*}$ to the commutative alphabet $X_q = \{1, q, q^2, \ldots\}$ then the generating series $\sigma(X_qA;t)$ can be viewed as the image of the diagonal series $\mathcal{D}_{\text{ш}}$ by the tensor $f\otimes\mathcal{S}$:

$$\sigma(X_qA;1) = (f\otimes\mathcal{S})\mathcal{D}_{\text{ш}}, \tag{66}$$

[18] *I.e.* the formula (120) given in Gelfand et al. (1995).

where f is the map defined as follows

$$f : \mathbb{Q}[Y] \longrightarrow \mathbb{Q}[X_q][t], \qquad y_i \longmapsto q^i t. \tag{67}$$

Hence, one has

Example 6 *(Generating series of the analog Hall–Littlewood functions).* Let $X_q = 1/(1-q)$ denote the totally ordered commutative alphabet $X_q = \{\cdots < q^n < \cdots < q < 1\}$. The complete symmetric functions of the noncommutative alphabet $A/(1-q)$ are given by the following ordinary generating series

$$\sigma(\frac{A}{1-q}; t) = \sum_{n\geq 0} S_n(\frac{A}{1-q})t^n := \overleftarrow{\prod_{n\geq 0}} \sigma(A; q^n t).$$

Hence, by specializing each letter $x_i \in X$ to q^i in $M_I(X)$:

$$\begin{aligned}\sigma(\frac{A}{1-q}; 1) &= \overleftarrow{\prod_{n\geq 0}} \sum_{i\geq 0} S_i\, q^{ni} \\ &= \sum_{\substack{I=(i_1,\ldots,i_r)\\ I\in(\mathbb{N}_+)^*}} \left[\sum_{n_1>\ldots>n_r\geq 1} q^{\sum_{k=1}^r n_k i_k} \right] S^I(A) \\ &= \sum_{I\in(\mathbb{N}_+)^*} M_I(X) S^I(A).\end{aligned}$$

4. Conclusion

Once again, Schützenberger's monoidal factorization plays a central rôle in the construction of pairs of bases in duality, as exemplified for the (mutually dual) Hopf algebras of quasi-symmetric functions ($\mathbf{QSym_k}$) and of noncommutative symmetric functions ($\mathbf{Sym_k}$), obtained as isomorphic images of the quasi-shuffle Hopf algebra ($\mathcal{H}_{\uplus}$) and its dual ($\mathcal{H}_{\uplus}^{\vee}$), by $\mathcal{M}$ and $\mathcal{S}$ respectively.

References

Bui, V.C., Duchamp, G.H.E., Hoang Ngoc Minh, V., 2013a. Schützenberger's factorization on the (completed) Hopf algebra of q-stuffle product. JP J. Algebra Number Theory Appl. 30 (2), 191–215.
Bui, V.C., Duchamp, G.H.E., Hoang, N., Hoang Ngoc Minh, V., Tollu, C., 2013b. Combinatorics of ϕ-deformed stuffle Hopf algebras. arXiv:1302.5391 [math.CO].
Chen, K.T., Fox, R.H., Lyndon, R.C., 1958. Free differential calculus, IV. The quotient groups of the lower central series. Ann. Math. 68 (1), 81–95.
Duchamp, G.H.E., Klyachko, A., Krob, D., Thibon, J.-Y., 1997. Noncommutative symmetric functions III: deformations of Cauchy and convolution algebras. Discret. Math. Theor. Comput. Sci. 1, 159–216.
Enjalbert, J.-Y., Hoang Ngoc Minh, V., 2012. Combinatorial study of Hurwitz colored polyzêtas. Discrete Math. 312 (24), 3489–3497.
Gelfand, I.M., Krob, D., Lascoux, A., Leclerc, B., Retakh, V.S., Thibon, J.Y., 1995. Noncommutative symmetric functions. Adv. Math. 112 (2), 218–348.
Gelfand, I.M., Retakh, V.S., 1991. Determinants of matrices over noncommutative rings. Funct. Anal. Appl. 25, 91–102.
Gelfand, I.M., Retakh, V.S., 1992. A theory of noncommutative determinants and characteristic functions of graphs. Funct. Anal. Appl. 26, 1–20. Publ. LACIM UQAM, Montreal 14 (1993) 1–26.
Hoang Ngoc Minh, V., 2013a. On a conjecture by Pierre Cartier about a group of associators. Acta Math. Vietnam. 38 (3), 339–398.
Hoang Ngoc Minh, V., 2013b. Structure of polyzetas and Lyndon words. Vietnam J. Math. 41 (4), 409–450.
Hoffman, M., 2000. Quasi-shuffle products. J. Algebr. Comb. 11, 49–68.
Krob, D., Leclerc, B., Thibon, J.Y., 1997. Noncommutative symmetric functions II: transformations of alphabets. Int. J. Algebra Comput. 7, 181–264.
Krob, D., Thibon, J.Y., 1997. Noncommutative symmetric functions IV: quantum linear groups and Hecke algebras at $q = 0$. J. Algebr. Comb. 6, 339–376.
Krob, D., Thibon, J.Y., 1999. Noncommutative symmetric functions V: a degenerate version of $U_q(gl_N)$. Int. J. Algebra Comput. 9, 405–430.

Macdonald, I.G., 1979. Symmetric Functions and Hall Polynomials. Oxf. Math. Monogr. Oxford Science Publications, second edition. 1995.
Reutenauer, C., 1993. Free Lie Algebras. Lond. Math. Soc. Monogr. (N.S.), vol. 7. Oxford Science Publications.