Families of eulerian functions involved in regularizations of divergent polyzetas

V.C. Bùi[0], G.H.E. Duchamp[1,4],
V. Hoang Ngoc Minh[2,4], K.A. Penson[3], Q.H. Ngô[5]

[0]Hue University of Sciences, 77 - Nguyen Hue street - Hue city, Vietnam.
[1]Université Paris 13, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.
[2]Université Lille, 1 Place Déliot, 59024 Lille, France.
[3]Université Paris VI, 75252 Paris Cedex 05, France
[4]LIPN-UMR 7030, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.
[5]University of Hai Phong, 171, Phan Dang Luu, Kien An, Hai Phong, Viet Nam

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Outline

INTRODUCTION

Riemann zeta function and eulerian functions

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^t - 1} \quad \text{and} \quad \Gamma(s) = \int_0^\infty du u^{s-1} e^{-u}, \quad \Re(s) > 1.$$

The function Γ is meromorphic, with no zeroes and $-\mathbb{N}$ as set of (simple) poles. It satisfies $\Gamma(\bar{z}) = \overline{\Gamma(z)}$.

Γ^{-1} is entire and admits simple zeroes in $-\mathbb{N}$. For $|z| < 1$, we have

$$\begin{aligned} \frac{1}{\Gamma(z+1)} &= e^{\gamma z} \prod_{n \geq 1} \left(1 + \frac{z}{n}\right) e^{-z/n} \quad \text{(by Weierstrass factorization)} \\ &= e^{\gamma z - \sum_{k \geq 2} \zeta(k)(-z)^k / k} \quad \text{(by Newton-Girard formula).} \end{aligned}$$

For any $u, v \in \mathbb{C}$ such that $|u| < 1, |v| < 1, |u+v| < 1$, one has

$$\frac{\Gamma(1-u)\Gamma(1-v)}{\Gamma(1-u-v)} = \exp\left(\sum_{n \geq 2} \zeta(n) \frac{(u+v)^n - (u^n + v^n)}{n}\right).$$

For any $z, a, b \in \mathbb{C}$ such that $|z| < 1$ and $\Re a > 0, \Re b > 0$, letting

$$\mathrm{B}(z; a, b) := \int_0^z dt\, t^{a-1}(1-t)^{b-1} \text{ and } \mathrm{B}(1; a, b) =: \mathrm{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

and using $\Gamma(1+z) = z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z) = \pi / \sin(z\pi)$, one obtains

$$\begin{aligned} \exp\left(\sum_{n \geq 2} \zeta(n) \frac{(u+v)^n - (u^n + v^n)}{n}\right) &= \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)} \pi \frac{\sin((u+v)\pi)}{\sin(u\pi)\sin(v\pi)} \\ &= \frac{\pi}{\mathrm{B}(u, v)} (\cot(u\pi) + \cot(v\pi)). \end{aligned}$$

$\zeta(2k)$ and Weierstrass factorization

In particular, for $v = -u$ ($|u| < 1$), one gets

$$\exp\left(-\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k}\right) = \frac{1}{\Gamma(1-u)\Gamma(1+u)} = \frac{\sin(u\pi)}{u\pi}.$$

Taking the logarithms and then taking the Taylor expansions, one obtains

$$\begin{aligned}
-\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k} &= \log\left(1+\sum_{n\geq 1}\frac{(u\mathrm{i}\pi)^{2n}}{\Gamma(2n)}\right)\\
&= \sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{k\geq 1}(u\mathrm{i}\pi)^{2k}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i)}\\
&= \sum_{k\geq 1}(u\mathrm{i}\pi)^{2k}\sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i)}.
\end{aligned}$$

One can deduce then the following expression for $\zeta(2k)$:

$$\frac{\zeta(2k)}{\pi^{2k}} = k\sum_{l=1}^{k}\frac{(-1)^{k+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i)} \in \mathbb{Q}.$$

Euler gave an other explicit formula using Bernoulli numbers $\{b_k\}_{k\in\mathbb{N}}$:

$$\frac{\zeta(2k)}{(2\mathrm{i}\pi)^{2k}} = -\frac{b_{2k}}{2(2k)!} \in \mathbb{Q}.$$

Zeta functions in several variables

Let $\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}$, for $r \in \mathbb{N}_+$, the following zeta function converges for $(s_1, \ldots, s_r) \in \mathcal{H}_r$

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}.$$

From a theorem by Abel, for $N \in \mathbb{N}, z \in \mathbb{C}, |z| < 1$, it can be obtained as

$$\zeta(s_1, \ldots, s_r) = \lim_{n \to +\infty} \mathrm{H}_{s_1, \ldots, s_r}(n) = \lim_{z \to 1} \mathrm{Li}_{s_1, \ldots, s_r}(z),$$

where the following functions are well defined for $(s_1, \ldots, s_r) \in \mathbb{C}^r$

$$\mathrm{Li}_{s_1, \ldots, s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} \text{ and } \frac{\mathrm{Li}_{s_1, \ldots, s_r}(z)}{1 - z} = \sum_{n \geq 0} \mathrm{H}_{s_1, \ldots, s_r}(n) z^n.$$

Let X^* and Y^* be the free monoids[1] generated by $X = \{x_0, x_1\}$ and $Y = \{y_k\}_{k \geq 1}$, respectively. We will use the one-to-one correspondence

$$(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1 \in X^* x_1.$$

For $s_1, \ldots, s_r \in \mathbb{N}_+$, $\mathrm{Li}_{s_1, \ldots, s_r}(z) = \alpha_0^z(x_0^{s_1 - 1} x_1 \ldots x_o^{s_k - 1} x_1)$, where

$$\alpha_{z_0}^z(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k) \quad \text{and} \quad \alpha_{z_0}^z(1_{X^*}) = 1_\Omega,$$

$(z_0, z_1, \ldots, z_k, z)$ is a subdivision of the path $z_0 \rightsquigarrow z$ in Ω and

$$\omega_0(z) = z^{-1} dz \quad \text{and} \quad \omega_1(z) = (1 - z)^{-1} dz.$$

$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_{s_1, \ldots, s_r}(1)\}_{\substack{s_1, \ldots, s_r \in \mathbb{N}_+ \\ s_1 > 1}} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{H}_{s_1, \ldots, s_r}(+\infty)\}_{\substack{s_1, \ldots, s_r \in \mathbb{N}_+ \\ s_1 > 1}}.$$

[1] their units are denoted, respectivement, by 1_{X^*} and 1_{Y^*}.

STRUCTURES OF POLYLOGARITHMS AND HARMONIC SUMS

Indexing polylogarithms and harmonic sums by polynomials

Putting $\mathrm{Li}_{x_0}(z) := \log(z)$, the following morphisms are injective

$$\begin{array}{rcl} \mathrm{Li}_\bullet : (\mathbb{Q}\langle X\rangle, ⧢, 1_{X^*}) & \longrightarrow & (\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1), \\ \mathrm{H}_\bullet : (\mathbb{Q}\langle Y\rangle, \text{⧢}\!\!\!-, 1_{Y^*}) & \longrightarrow & (\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1), \end{array}$$

mapping $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$ to $\mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} = \mathrm{Li}_{s_1,\ldots,s_r}$ and $y_{s_1}\ldots y_{s_r}$ to $\mathrm{H}_{y_{s_1}\ldots y_{s_r}} = \mathrm{H}_{s_1,\ldots,s_r}$, respectively.

Hence[2], $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ and $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ are algebraically independent.

Let $\{P_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$) be a basis of Lie algebra of primitive elements and $\{S_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$) is a pure transcendence basis[3] of $(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})$ (resp. $(\mathbb{C}\langle Y\rangle, \text{⧢}\!\!\!-, 1_{Y^*})$).

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w\, w = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} P_l}, \quad \mathrm{L}_{\mathtt{reg}} := \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\mathrm{Li}_{S_l} P_l},$$

$$\mathrm{H} := \sum_{w\in Y^*} \mathrm{H}_w w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}, \quad \mathrm{H}_{\mathtt{reg}} := \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}.$$

In all the sequel, let $\mathcal{X}$ denotes X or Y and $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ denotes the set formal power series, over $\mathcal{X}$ and with coefficients in $\mathbb{C}$.

[2] $\mathcal{L}ynX, \mathcal{L}ynY$ denote the sets of Lyndon words over X, Y, respectively.

[3] In duality with $\{P_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$).

Indexing polylogarithms by noncommutative rational series

Let $\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ and $\mathbb{C}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ denote the sets of noncommutative rational[4] and exchangeable[5], respectively, series.

Noncommutative multivariate exponential transforms $(x_0x_1 \neq x_1x_0)$:

$$\begin{array}{cccccc} x_0^n & \longmapsto & \log^n(z)/n!, & x_1^n & \longmapsto & \log^n((1-z)^{-1})/n!, \\ (tx_0)^* & \longmapsto & z^t, & (tx_1)^* & \longmapsto & (1-z)^{-t}. \end{array}$$

Example (polylogarithms indexed by rational series)

$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}, \quad \mathrm{Li}_{(ax_0+bx_1)^*}(z) = z^a(1-z)^{-b}.$$

$\mathrm{Li}_{-s_1,\ldots,-s_r} = \mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}$, where $R_{y_{s_1}\ldots y_{s_r}} \in (\mathbb{Z}[x_1^*], ш, 1_{X^*})$ and

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1} \ldots \sum_{k_r=0}^{(s_1+\ldots+s_r)-(k_1+\ldots+k_{r-1})} \binom{s_1}{k_1} \ldots \binom{\sum_{i=1}^r s_i - \sum_{i=1}^{r-1} k_i}{k_r} \rho_{k_1} ш \ldots ш \rho_{k_r},$$

where, for any $i = 1, \ldots, r$, if $k_i = 0$ then $\rho_{k_i} = x_1^* - 1_{X^*}$ else

$$\rho_{k_i} = x_1^* ш \sum_{j=1}^{k_i} S_2(k_i, j) j! (x_1^* - 1_{X^*})^{ш j}$$

and the $S_2(k_i, j)$'s are the Stirling numbers of second kind.

[4] $\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is the closure by $\{\mathrm{conc}, +, *\}$ of $\mathbb{C}\langle\mathcal{X}\rangle$ in $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ and is closed under ш $\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed under ⧢. If $S \in \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ s.t. $\langle S|1_{\mathcal{X}^*}\rangle = 0$ then $S^* = \sum_{n\geq 0} S^n$.

[5] *i.e.* if $S \in \mathbb{C}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u, v \in \mathcal{X}^*)((\forall x \in \mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle)$.

Representative series and $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ ($\mathcal{X} = X$ or Y)

1. $(\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, ш, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e}) = (\mathbb{C}\langle\mathcal{X}\rangle, \mathtt{conc}, \Delta_{ш}, 1_{\mathcal{X}^*}, \mathrm{e})^{\circ}$.
2. The x^*'s, $x \in \mathcal{X}$, are group-like, for $\Delta_{\mathtt{conc}}$, and are algebraically independent over $(\mathbb{C}\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*})$ within $(\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, ш, 1_{\mathcal{X}^*})$. So are y^*'s, $y \in Y^*$, over $(\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*})$ within $(\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, ⧢, 1_{Y^*})$.
3. $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle := \mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \bigcap \mathbb{C}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle = ш\{\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle\}_{x\in\mathcal{X}}$ and $\forall x \in \mathcal{X}, \mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbb{C}}\{(ax)^* ш \mathbb{C}\langle x\rangle | a \in \mathbb{C}\}$.
4. $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ iff it admits a representation of dimension n, (ν, μ, η) : $\nu \in M_{1,n}(\mathbb{C})$, $\eta \in M_{n,1}(\mathbb{C}), \mu : \mathcal{X}^* \to M_{n,n}(\mathbb{C})$ s.t.
$$R = \sum_{w\in\mathcal{X}^*} (\nu\mu(w)\eta)\, w = \nu\Big(\sum_{w\in\mathcal{X}} \mu(x)x\Big)^* \eta.$$

Now, for $i = 1, 2$, let $R_i \in \mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ and (ν_i, μ_i, η_i) be, respectively, representations of dimension n_i. Then the linear representation of

$R_1 + R_2$ is $\left(\begin{pmatrix}\nu_1 & \nu_2\end{pmatrix}, \left\{\begin{pmatrix}\mu_1(x) & \mathbf{0}\\ \mathbf{0} & \mu_2(x)\end{pmatrix}\right\}_{x\in\mathcal{X}}, \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right)$,

$R_1 ш R_2$ is $(\nu_1 \otimes \nu_2, \{\mu_1(x) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x)\}_{x\in\mathcal{X}}, \eta_1 \otimes \eta_2)$,

$R_1 ⧢ R_2$ is $(\nu_1 \otimes \nu_2, \{\mu_1(y_k) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(y_k) + \sum_{i+j=k} \mu_1(y_i) \otimes \mu_2(y_j)\}_{k\geq 1}, \eta_1 \otimes \eta_2)$.

Representations of $(-t^2x_0x_1)^* ⧢ (t^2x_0x_1)^*$

$$
\begin{aligned}
(t^2x_0x_1)^* &\leftrightarrow (\nu_1,\{\mu_1(x_0),\mu_1(x_1)\},\eta_1),\\
(-t^2x_0x_1)^* &\leftrightarrow (\nu_2,\{\mu_2(x_0),\mu_2(x_1)\},\eta_2)\\
(\nu,\{\mu(x_0),\mu(x_1)\},\eta) &= (\nu_1\otimes\nu_2,\{\mu_1(x_0)\otimes \mathrm{I}_{n_2}+\mathrm{I}_{n_1}\otimes\mu_2(x_0),\\
&\qquad \mu_1(x_1)\otimes \mathrm{I}_{n_2}+\mathrm{I}_{n_1}\otimes\mu_2(x_1),\eta_1\otimes\eta_2).
\end{aligned}
$$

$$
\nu_1=\begin{pmatrix}1 & 0\end{pmatrix},\quad \eta_1=\begin{pmatrix}1\\0\end{pmatrix},\quad \mu_1(x_0)=\begin{pmatrix}0 & t\\0 & 0\end{pmatrix},\quad \mu_1(x_1)=\begin{pmatrix}0 & 0\\t & 0\end{pmatrix},
$$

$$
\nu_2=\begin{pmatrix}1 & 0\end{pmatrix},\quad \eta_2=\begin{pmatrix}1\\0\end{pmatrix},\quad \mu_2(x_0)=\begin{pmatrix}0 & \mathrm{i}t\\0 & 0\end{pmatrix},\quad \mu_2(x_1)=\begin{pmatrix}0 & 0\\\mathrm{i}t & 0\end{pmatrix}
$$

$$
\nu=\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix},\quad \eta=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},
$$

$$
\mu(x_0)=\begin{pmatrix}0&0&t&0\\0&0&0&t\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&\mathrm{i}t&0&0\\0&0&0&0\\0&0&0&\mathrm{i}t\\0&0&0&0\end{pmatrix}=\begin{pmatrix}0&\mathrm{i}t&t&0\\0&0&0&t\\0&0&0&\mathrm{i}t\\0&0&0&0\end{pmatrix},
$$

$$
\mu(x_1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\t&0&0&0\\0&t&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\\mathrm{i}t&0&0&0\\0&0&0&0\\0&0&\mathrm{i}t&0\end{pmatrix}=\begin{pmatrix}0&0&0&0\\\mathrm{i}t&0&0&0\\t&0&0&0\\0&t&\mathrm{i}t&0\end{pmatrix}.
$$

Sub bialgebras of $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle, ⧢, 1_{X^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$

In all the sequel, let (ν,μ,η) be of minimal dimension of $R\in\mathbb{C}\langle\langle X\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$.

1. $R\in\mathbb{C}\langle X\rangle$ if $\{\mu(P)\}_{P\in\mathcal{L}ie_{\mathbb{C}}\langle X\rangle}=\mathcal{L}$.
2. $R\in\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$ iff $\mathcal{L}$ is commutative.
3. The modules generated by the following families are closed by $\mathtt{conc}$, ⧢ and coproducts :

$$\begin{array}{llll}(F_0) & E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1}, & \text{where} & x_{i_k}\in X, E_k\in\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle,\\ (F_1) & E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1}, & \text{where} & x_{i_k}\in X, E_k\in\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle,\\ (F_2) & E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1}, & \text{where} & x_{i_k}\in X, E_k\in\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle.\end{array}$$

Letting $M(x):=\mu(x)x$, for $x\in X$, one has[6] $R=\nu M(X^*)\eta$ and

$$M(X^*)=(M(x_1^*)M(x_0))^*M(x_1^*)=(M(x_0^*)M(x_1))^*M(x_0^*).$$

Hence,

3.1 R is linear combination of expressions in the form (F_0) (resp. (F_1)) iff $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent,

3.2 R is linear combination of expressions in the form (F_2) iff $\mathcal{L}$ is solvable. Hence, if $R\in\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle ⧢ \mathbb{C}\langle X\rangle$ then $\mathcal{L}$ is solvable.

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[6] If $\{\mu(x)\}_{x\in X}$ are triangular then $M(X^*)=((D(X^*)T(X))^*D(X^*))$, where $D(X)$ (resp. $N(X)$) is diagonal (resp. nilpotent) letter matrix s.t. $M(X)=D(X)+N(X)$.

Extension[7] of $\mathrm{Li}_\bullet$

Theorem

Let $\mathcal{C}_{\mathbb{C}} := \mathbb{C}[z^a, (1-z)^b]_{a,b\in\mathbb{C}}$.

1. *The family* $\{\mathrm{Li}_w\}_{w\in X^*}$ *is* $\mathcal{C}_{\mathbb{C}}$*-linearly independent.*
2. *The algebra* $\mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$ *is closed*
   - *under the differential operators* $\theta_0 := z\partial_z, \theta_1 := (1-z)\partial_z$,
   - *and under their sections* ι_0, ι_1 $(\theta_0\iota_0 = \theta_1\iota_1 = \mathrm{Id})$.
3. *The bi-integro differential algebra* $(\mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}, \theta_0, \theta_1, \iota_0, \iota_1)$ *is closed under the action of the group of transformations,* $\mathcal{G}$, *generated by* $\{z \mapsto 1-z, z \mapsto z^{-1}\}$, *permuting* $\{0, 1, +\infty\}$:
   $$\forall h \in \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}, \quad \forall g \in \mathcal{G}, \quad h(g) \in \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}.$$
4. *The following map is surjective*
   $$\begin{array}{rrcl}\mathrm{Li}_\bullet : (\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\!\langle X\rangle\!\rangle \sqcup\!\sqcup \mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) & \longrightarrow\!\!\!\!\rightarrow & (\mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1_\Omega), \\ R & \longmapsto & \mathrm{Li}_R\end{array}$$
   and $\ker \mathrm{Li}_\bullet$ *is the* $\sqcup\!\sqcup$ *-ideal generated by* $x_0^* \sqcup\!\sqcup x_1^* - x_1^* + 1$.
5. *If* $R \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\!\langle X\rangle\!\rangle \sqcup\!\sqcup \mathbb{C}\langle X\rangle$ *(resp.* $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\!\langle X\rangle\!\rangle$*) then* $\mathrm{Li}_R \in \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$ *(resp.* $\mathcal{C}_{\mathbb{C}}[\log(z), \log(1-z)]$*).*

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[7] See also the talk of G.H.E Duchamp.

FIRST STRUCTURES OF POLYZETAS

Chen series over ω_0, ω_1 and along a path on Ω

$$C_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \alpha_{z_0}^z(w) w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(\alpha_{z_0}^z(S_l) P_l).$$

One has $C_{z_0 \rightsquigarrow z_0} = 1_{X^*}$ and $C_{z_0 \rightsquigarrow z}$ is solution of

$$(DE) \qquad dG = (\omega_0 x_0 + \omega_1 x_1) G.$$

Now, let $\gamma_0(\varepsilon), \gamma_1(\varepsilon)$ be the circular paths of radius ε encircling 0, 1 clockwise, respectively. In particular, for $\beta = \beta_1 - \beta_0$,

$$\begin{aligned} \gamma_0(\varepsilon, \beta) &= \varepsilon e^{i\beta_0} \rightsquigarrow \varepsilon e^{i\beta_1} &\subset \gamma_0(\varepsilon), \\ \gamma_1(\varepsilon, \beta) &= 1 - \varepsilon e^{i\beta_0} \rightsquigarrow 1 - \varepsilon e^{i\beta_1} &\subset \gamma_1(\varepsilon). \end{aligned}$$

On one hand, one has, for any $i = 0$ or 1 and $w \in X^+$,

$$|\langle C_{\gamma_i(\varepsilon,\beta)} | w \rangle| \leq \varepsilon^{|w|_{x_i}} \beta^{|w|} |w|!^{-1}.$$

It follows then $C_{\gamma_i(\varepsilon,\beta)} = e^{i\beta x_i} + o(\varepsilon)$ and $C_{\gamma_i(\varepsilon)} = e^{2i\pi x_i} + o(\varepsilon)$.

On the other hand, let $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X \rangle\rangle$ of minimal representation (ν, μ, η) of dimension n. One has, for any $w \in X^*$,

$$|\langle R | w \rangle| \leq \|\nu\|_\infty^{1,n} \|\mu(w)\|_\infty^{n,n} \|\eta\|_\infty^{n,1}.$$

Hence,

$$\alpha_{z_0}^z(R) := \langle R \| C_{z_0 \rightsquigarrow z} \rangle = \sum_{w \in X^*} (\nu \mu(w) \eta) \alpha_{z_0}^z(w).$$

Note that the map $\alpha_{z_0}^z : \mathbb{C}^{\mathrm{rat}}\langle\langle X \rangle\rangle \longrightarrow \mathcal{H}(\Omega)$ is not injective. For

Three characters of regularization

The following poly-morphism is surjective

$$\zeta : \begin{array}{c}(\mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1, ш, 1_{X^*})\\(\mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle, ⧢, 1_{Y^*})\end{array} \twoheadrightarrow (\mathcal{Z}, ., 1),$$

mapping, both, $x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1$ and $y_{s_1} \ldots y_{s_r}$ to $\zeta(s_1, \ldots, s_r)$.

It can be extended as characters as follows

$$\zeta_{ш} : (\mathbb{R}\langle X\rangle, ш, 1_{X^*}) \longrightarrow (\mathbb{R}, ., 1),$$
$$\zeta_{⧢}, \gamma_\bullet : (\mathbb{R}\langle Y\rangle, ⧢, 1_{Y^*}) \longrightarrow (\mathbb{R}, ., 1),$$

s.t., for any $l \in \mathcal{L}ynX - X$, $\zeta_{⧢}(l) = \zeta_{⧢}(\pi_Y l) = \gamma_{\pi_Y l} = \zeta(l)$, and

$$\begin{array}{rclll}
\zeta_{ш}(x_0) = & 0 & = \log(1), & \\
\zeta_{ш}(x_1) = & 0 & = \mathrm{f.p.}_{z\to1}\log(1-z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\
\zeta_{⧢}(y_1) = & 0 & = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\
\gamma_{y_1} = & \gamma & = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.
\end{array}$$

$$Z_{ш} := \mathrm{L}_{\mathrm{reg}}(1), \quad Z_{⧢} := \mathrm{H}_{\mathrm{reg}}(+\infty), \quad Z_\gamma := \sum_{w\in Y^*}\gamma_w w = e^{\gamma y_1} Z_{⧢}.$$

Then[8] $Z_\gamma = B(y_1)\pi_Y Z_{ш} \Leftrightarrow Z_{⧢} = B'(y_1)\pi_Y Z_{ш}$, where $B(y_1) = e^{\gamma y_1 - \sum_{k\geq2}\zeta(k)(-y_1)^k/k}$ and $B'(y_1) = e^{-\sum_{k\geq2}\zeta(k)(-y_1)^k/k}$.

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[8] $Z_{ш}$ corresponds to the associator Φ_{KZ} (see CAP'2017).

Homogenous polynomials relations among local coordinates

$$Z_{\gamma} = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$$

	Polynomial relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Polynomial relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

Homogenous polynomials generating $\ker \zeta$

	$\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	$\{Q_l\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1} - \frac{3}{2}\Sigma_{y_3}) = 0$	$\zeta(S_{x_0x_1^2} - S_{x_0^2x_1}) = 0$
4	$\zeta(\Sigma_{y_4} - \frac{2}{5}\Sigma_{y_2}^{\uplus 2}) = 0$	$\zeta(S_{x_0^3x_1} - \frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2}) = 0$
	$\zeta(\Sigma_{y_3y_1} - \frac{3}{10}\Sigma_{y_2}^{\uplus 2}) = 0$	$\zeta(S_{x_0^2x_1^2} - \frac{1}{10}S_{x_0x_1}^{\sqcup\!\sqcup 2}) = 0$
	$\zeta(\Sigma_{y_2y_1^2} - \frac{2}{3}\Sigma_{y_2}^{\uplus 2}) = 0$	$\zeta(S_{x_0x_1^3} - \frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2}) = 0$
5	$\zeta(\Sigma_{y_3y_2} - 3\Sigma_{y_3} \uplus \Sigma_{y_2} - 5\Sigma_{y_5}) = 0$	$\zeta(S_{x_0^3x_1^2} - S_{x_0^2x_1} \sqcup\!\sqcup S_{x_0x_1} + 2S_{x_0^4x_1}) = 0$
	$\zeta(\Sigma_{y_4y_1} - \Sigma_{y_3} \uplus \Sigma_{y_2}) + \frac{5}{2}\Sigma_{y_5}) = 0$	$\zeta(S_{x_0^2x_1x_0x_1} - \frac{3}{2}S_{x_0^4x_1} + S_{x_0^2x_1} \sqcup\!\sqcup S_{x_0x_1}) = 0$
	$\zeta(\Sigma_{y_2^2y_1} - \frac{3}{2}\Sigma_{y_3} \uplus \Sigma_{y_2} - \frac{25}{12}\Sigma_{y_5}) = 0$	$\zeta(S_{x_0^2x_1^3} - S_{x_0^2x_1} \sqcup\!\sqcup S_{x_0x_1} + 2S_{x_0^4x_1}) = 0$
	$\zeta(\Sigma_{y_3y_1^2} - \frac{5}{12}\Sigma_{y_5}) = 0$	$\zeta(S_{x_0x_1x_0x_1^2} - \frac{1}{2}S_{x_0^4x_1}) = 0$
	$\zeta(\Sigma_{y_2y_1^3} - \frac{1}{4}\Sigma_{y_3} \uplus \Sigma_{y_2}) + \frac{5}{4}\Sigma_{y_5}) = 0$	$\zeta(S_{x_0x_1^4} - S_{x_0^4x_1}) = 0$
6	$\zeta(\Sigma_{y_6} - \frac{8}{35}\Sigma_{y_2}^{\uplus 3}) = 0$	$\zeta(S_{x_0^5x_1} - \frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}) = 0$
	$\zeta(\Sigma_{y_4y_2} - \Sigma_{y_3}^{\uplus 2} - \frac{4}{21}\Sigma_{y_2}^{\uplus 3}) = 0$	$\zeta(S_{x_0^4x_1^2} - \frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3} - \frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2}) = 0$
	$\zeta(\Sigma_{y_5y_1} - \frac{2}{7}\Sigma_{y_2}^{\uplus 3} - \frac{1}{2}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0^3x_1x_0x_1} - \frac{4}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3}) = 0$
	$\zeta(\Sigma_{y_3y_1y_2} - \frac{17}{30}\Sigma_{y_2}^{\uplus 3} + \frac{9}{4}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0^3x_1^3} - \frac{23}{70}S_{x_0x_1}^{\sqcup\!\sqcup 3} - S_{x_0^2x_1}^{\sqcup\!\sqcup 2}) = 0$
	$\zeta(\Sigma_{y_3y_2y_1} - 3\Sigma_{y_3}^{\uplus 2} - \frac{9}{10}\Sigma_{y_2}^{\uplus 3}) = 0$	$\zeta(S_{x_0^2x_1x_0x_1^2} - \frac{2}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3}) = 0$
	$\zeta(\Sigma_{y_4y_1^2} - \frac{3}{10}\Sigma_{y_2}^{\uplus 2} - \frac{3}{4}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0^2x_1^2x_0x_1} - \frac{89}{210}S_{x_0x_1}^{\sqcup\!\sqcup 3} + \frac{3}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2}) = 0$
	$\zeta(\Sigma_{y_2^2y_1^2} - \frac{11}{63}\Sigma_{y_2}^{\uplus 2} - \frac{1}{4}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0^2x_1^4} - \frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3} - \frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2}) = 0$
	$\zeta(\Sigma_{y_3y_1^3} - \frac{1}{21}\Sigma_{y_2}^{\uplus 3}) = 0$	$\zeta(S_{x_0x_1x_0x_1^3} - \frac{8}{21}S_{x_0x_1}^{\sqcup\!\sqcup 3} - S_{x_0^2x_1}^{\sqcup\!\sqcup 2}) = 0$
	$\zeta(\Sigma_{y_2y_1^4} - \frac{17}{50}\Sigma_{y_2}^{\uplus 3} + \frac{3}{16}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0x_1^5} - \frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}) = 0$

$\mathcal{R}_Y := (\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}, \uplus, 1_{Y^*})$ and $\mathcal{R}_X := (\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynX-X}, \sqcup\!\sqcup, 1_{X^*})$

Noetherian rewriting system & irreducible coordinates[9]

$$Z_{\gamma} = B(y_1)\pi_Y Z_{⧢}$$

	Rewriting among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) \to \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) \to \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) \to \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) \to \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) \to 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) \to -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) \to -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) \to \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) \to \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) \to \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) \to \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) \to \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) \to \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) \to \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) \to \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) \to -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) \to \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) \to 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) \to \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) \to -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) \to \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) \to \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) \to \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) \to \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$

[9] The set of irreducible coordinates forms algebraic generator system for $\mathcal{Z}$.

Irreducible polyzetas ($\mathcal{X} = X$ or Y)

By $Z_{\gamma} = B(y_1)\pi_Y Z_{ш}$, identification of locale coordinates yields families of homogenous in weight polynomials $\{Q_l\}_{l \in \mathcal{L}yn\mathcal{X}}$ s.t.

$$\mathcal{R}_Y := \quad (\mathbb{Q}\{Q_l\}_{l \in \mathcal{L}ynY - \{y_1\}}, \,⧢\, , 1_{Y^*}) \quad = \ker \zeta,$$

$$(\text{resp. } \mathcal{R}_X := \quad (\mathbb{Q}\{Q_l\}_{l \in \mathcal{L}ynX - X}, \,ш\, , 1_{X^*}) \quad = \ker \zeta),$$

and families of algebraic generators, as irreducible coordinates, s.t. restricted on $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, the poly-morphism ζ is injective, where

$$\mathcal{L}_{irr}^{\infty}(\mathcal{X}) := \lim_{p \to +\infty} \mathcal{L}_{irr}^{\leq p}(\mathcal{X}) \text{ with } \emptyset \subset \mathcal{L}_{irr}^{\leq 2}(\mathcal{X}) \subset \ldots \mathcal{L}_{irr}^{\leq p}(\mathcal{X}) \subset \ldots$$

Example (of irreducible coordinates, see CAP'2017)

$$\mathcal{L}_{irr}^{\leq 12}(Y) = \{\Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_5}, \Sigma_{y_7}, \Sigma_{y_3 y_1^5}, \Sigma_{y_9}, \Sigma_{y_3 y_1^7}, \Sigma_{y_{11}}, \Sigma_{y_2 y_1^9}, \Sigma_{y_3 y_1^9}, \Sigma_{y_2^2 y_1^8}\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(X) = \{S_{x_0 x_1}, S_{x_0^2 x_1}, S_{x_0^4 x_1}, S_{x_0^6 x_1}, S_{x_0 x_1^2 x_0 x_1^4}, S_{x_0^8 x_1}, S_{x_0 x_1^2 x_0 x_1^6}, S_{x_0^{10} x_1}, S_{x_0 x_1^3 x_0 x_1^7}, S_{x_0 x_1^2 x_0 x_1^8}, S_{x_0 x_1^4 x_0 x_1^6}\}.$$

$$\Sigma_{y_2} = y_2, \Sigma_{y_3} = y_3, \Sigma_{y_5} = y_5, \Sigma_{y_7} = y_7, \Sigma_{y_9} = y_9, \Sigma_{y_{11} = y_{11}}.$$

$$S_{x_0 x_1} = x_0 x_1, S_{x_0^2 x_1} = x_0^2 x_1, S_{x_0^4 x_1} = x_0^4 x_1, S_{x_0^6 x_1} = x_0^6 x_1, S_{x_0^8 x_1} = x_0^8 x_1, S_{x_0^{10} x_1} = x_0^{10} x_1.$$

Note that, for any $n \geq 1$,

1. Let $l \in \mathcal{L}yn\mathcal{X}$ s.t. $(l) = n$. Then $y_n \preceq l$ (resp. $x_0^{n-1} x_1 \preceq l$).
2. $\Sigma_{y_n} = y_n \in \mathcal{L}ynY, S_{x_0^{n-1} x_1} = x_0^{n-1} x_1 \in \mathcal{L}ynX$.
3. $\zeta(2) = \zeta(\Sigma_{y_2}) = \zeta(S_{x_0 x_1})$ is irreducible but by Euler's formula, $\Sigma_{y_{2n}} = y_{2n} \notin \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2n-1} x_1} = x_0^{2n-1} x_1 \notin \mathcal{L}_{irr}^{\infty}(X)$.
4. $\Sigma_{y_{2n+1}} = y_{2n+1} \in \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2n} x_1} = x_0^{2n} x_1 \in \mathcal{L}_{irr}^{\infty}(X)$.

MORE ABOUT STRUCTURES OF POLYZETAS

Families of eulerian functions

For $r \geq 2$ and $|t| < 1$, let

$$f_1(t) := \gamma t - \sum_{k\geq 2} \zeta(k)\frac{(-t)^k}{k} \text{ and } f_r(t) := \sum_{k\geq 1} \zeta(kr)\frac{(-t^r)^{k-1}}{k}.$$

Proposition

The family $\{f_r\}_{r\geq 1}$ is linearly independent and the family $\{\exp(f_r)\}_{r\geq 1}$ is linearly independent.

For any $r \geq 1$ and $|t| < 1$, one put[10] $\Gamma_{y_r}(1+t) := e^{-f_r(t)}$ s.t.

$$\frac{1}{\Gamma_{y_1}(1+t)} = \exp\left(\gamma t - \sum_{k\geq 2} \zeta(k)\frac{(-t)^k}{k}\right) = e^{\gamma t}\prod_{n\geq 1}\left(1+\frac{t}{n}\right)e^{-\frac{t}{n}},$$

$$\frac{1}{\Gamma_{y_r}(1+t)} = \exp\left(\sum_{k\geq 1} \zeta(kr)\frac{(-t^r)^{k-1}}{k}\right) = \prod_{n\geq 1}\left(1+\frac{t^r}{n^r}\right),$$

and

$$\mathrm{B}_{y_r}(a,b) := \frac{\Gamma_{y_r}(a)\Gamma_{y_r}(b)}{\Gamma_{y_r}(a+b)}.$$

[10] Note that $\Gamma_{y_1}(t) = \Gamma(t)$ and $\mathrm{B}_{y_1}(a,b) = \mathrm{B}(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

Extended double regularization by Newton-Girard formula

Lemma

For any $r \geq 1$, the arithmetic function $\mathrm{H}_{y_r^*}$ is transcendent and

$$\forall t \in \mathbb{C}, |t| < 1, \quad \mathrm{H}_{(t^r y_r)^*} = \sum_{k\geq 0} \mathrm{H}_{y_r^k} t^{kr} = \exp\left(\sum_{k\geq 1} \mathrm{H}_{y_{kr}} \frac{(-t^r)^{k-1}}{k}\right).$$

By identification the coefficients of t^k and by injectivity, one gets

$$y_r^k = \frac{(-1)^k}{k!} \sum_{\substack{s_1,\ldots,s_k>0 \\ s_1+\ldots+ks_k=k}} \frac{(-y_r)^{\uplus s_1}}{1^{s_1}} \uplus \ldots \uplus \frac{(-y_{kr})^{\uplus s_k}}{k^{s_k}}.$$

Theorem

The characters $\zeta_{\sqcup\!\sqcup}$ and $\gamma_\bullet$ are extended algebraically as follows

$$\begin{array}{rcl}
\zeta_{\sqcup\!\sqcup} : (\mathbb{C}\langle X\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle, \sqcup\!\sqcup, 1_{X^*}) & \longrightarrow & (\mathbb{C}, ., 1), \\
\forall t \in \mathbb{C}, |t| < 1, (tx_0)^*, (tx_1)^* & \longmapsto & 1_{\mathbb{C}}. \\
\gamma_\bullet : (\mathbb{C}\langle Y\rangle \uplus \{\mathbb{C}^{\mathrm{rat}}\langle\langle y_r\rangle\rangle\}_{r\geq 1}, \uplus, 1_{Y^*}) & \longrightarrow & (\mathbb{C}, ., 1), \\
\forall t \in \mathbb{C}, |t| < 1, \forall r \geq 1, (t^r y_r)^* & \longmapsto & \Gamma_{y_r}^{-1}(1+t).
\end{array}$$

Moreover, the following morphism

$$\begin{array}{rcl}
g : (\mathbb{C}[\{(y_r)^*\}_{r\geq 1}], \uplus, 1_{Y^*}) & \longrightarrow & (\mathbb{C}[\{\exp(f_r)\}_{r\geq 1}], \times, 1), \\
y_r^* & \longmapsto & \Gamma_{y_r}^{-1}
\end{array}$$

is injective and, for $|t| < 1$, $\Gamma_{y_{2r}}(1-t) = \Gamma_{y_r}(1+t)\Gamma_{y_r}(1-t)$.

Comparison formula

Corollary

For any $z, a, b \in \mathbb{C}$ such that $|z| < 1$ and $\Re a > 0, \Re b > 0$, one has
$\mathrm{Li}_{x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1)^*]}(z) = \mathrm{Li}_{x_1[((a-1)x_0)^* \sqcup\!\sqcup (-bx_1)^*]}(z) = \mathrm{B}(z; a, b)$ *(partial Beta function) and* $\mathrm{B}(1; a, b) = \mathrm{B}(a, b)$. *Hence,*

$$\mathrm{B}(a,b) = \frac{\gamma_{((a+b-1)y_1)^*}}{\gamma_{((a-1)y_1)^* \uplus ((b-1)y_1)^*}} = \zeta_{\sqcup\!\sqcup}(x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1)^*]) = \zeta_{\sqcup\!\sqcup}(x_1[((a-1)x_0)^* \sqcup\!\sqcup (-bx_1)^*]).$$

Example (of $\{\gamma_{-s_1,\ldots,-s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}^r, r\in\mathbb{N}_+}$, see CAP'2017)

$\mathrm{Li}_{-1,-1} = \mathrm{Li}_{-x_1^*+5(2x_1)^*-7(3x_1)^*+3(4x_1)^*}$,
$\mathrm{Li}_{-2,-1} = \mathrm{Li}_{x_1^*-11(2x_1)^*+31(3x_1)^*-33(4x_1)^*+12(5x_1)^*}$,
$\mathrm{Li}_{-1,-2} = \mathrm{Li}_{x_1^*-9(2x_1)^*+23(3x_1)^*-23(4x_1)^*+8(5x_1)^*}$,
$\mathrm{H}_{-1,-1} = \mathrm{H}_{-y_1^*+5(2y_1)^*-7(3y_1)^*+3(4y_1)^*}$,
$\mathrm{H}_{-2,-1} = \mathrm{H}_{y_1^*-11(2y_1)^*+31(3y_1)^*-33(4y_1)^*+12(5y_1)^*}$,
$\mathrm{H}_{-1,-2} = \mathrm{H}_{y_1^*-9(2y_1)^*+23(3y_1)^*-23(4y_1)^*+8(5y_1)^*}$.

Hence, $\zeta_{\sqcup\!\sqcup}(-1,-1) = 0, \zeta_{\sqcup\!\sqcup}(-2,-1) = -1, \zeta_{\sqcup\!\sqcup}(-1,-2) = 0$ and
$\gamma_{-1,-1} = -\Gamma^{-1}(2) + 5\Gamma^{-1}(3) - 7\Gamma^{-1}(4) + 3\Gamma^{-1}(5) = 11/24$,
$\gamma_{-2,-1} = \Gamma^{-1}(2) - 11\Gamma^{-1}(3) + 31\Gamma^{-1}(4) - 33\Gamma^{-1}(5) + 12\Gamma^{-1}(6) = -73/120$,
$\gamma_{-1,-2} = \Gamma^{-1}(2) - 9\Gamma^{-1}(3) + 23\Gamma^{-1}(4) - 23\Gamma^{-1}(5) + 8\Gamma^{-1}(6) = -67/120$.

Kleene stars of the plane

Proposition

For any $s \geq 1$, *let* $a_s, b_s \in \mathbb{C}$. *Then*

$$\Big(\sum_{s\geq 1} a_s y_s\Big)^* \stuffle \Big(\sum_{s\geq 1} b_s y_s\Big)^* = \Big(\sum_{s\geq 1}(a_s+b_s)y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r}\Big)^*.$$

Hence[11], *for* $|a_s| < 1$, $|b_s| < 1$, $|a_s + b_s| < 1$,

$$\mathrm{H}_{(\sum_{s\geq 1}(a_s+b_s)y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r})^*} = \mathrm{H}_{(\sum_{s\geq 1} a_s y_s)^*} \mathrm{H}_{(\sum_{s\geq 1} b_s y_s)^*},$$

$$\gamma_{(\sum_{s\geq 1}(a_s+b_s)y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r})^*} = \gamma_{(\sum_{s\geq 1} a_s y_s)^*} \gamma_{(\sum_{s\geq 1} b_s y_s)^*}.$$

Example

$$\gamma_{(-t^2 y_2)^*} = \gamma_{(ty_1)^*}\gamma_{(-ty_1)^*}$$

$$\Leftrightarrow \quad \Gamma_{y_2}^{-1}(1-t) = \Gamma_{y_1}^{-1}(1+t)\Gamma_{y_1}^{-1}(1-t)$$

$$\Leftrightarrow \quad e^{-\sum_{k\geq 2}\zeta(2k)t^{2k}/k} = \frac{\sin(t\pi)}{t\pi} = \sum_{k\geq 1}\frac{(ti\pi)^{2k}}{(2k)!}.$$

$$\gamma_{(-t^4 y_4)^*} = \gamma_{(t^2 y_2)^*}\gamma_{(-t^2 y_2)^*}$$

$$\Leftrightarrow \quad \Gamma_{y_4}^{-1}(1-t) = \Gamma_{y_2}^{-1}(1+t)\Gamma_{y_2}^{-1}(1-t)$$

$$\Leftrightarrow \quad e^{-\sum_{k\geq 1}\zeta(4k)t^{4k}/k} = \frac{\sin(\mathrm{i}t\pi)}{\mathrm{i}t\pi}\frac{\sin(t\pi)}{t\pi} = \sum_{k\geq 1}\frac{2(-4t\pi)^{4k}}{(4k+2)!}.$$

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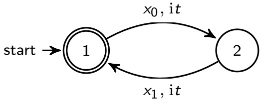
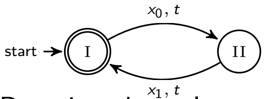
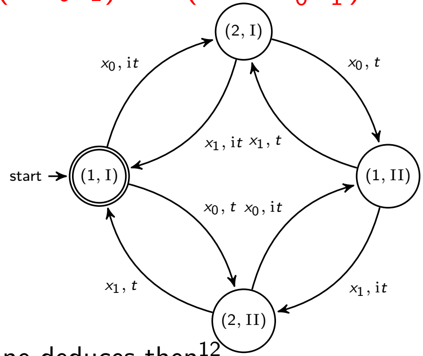
[11] In particular, since $(a_s y_s)^* \stuffle (a_r y_r)^* = (a_s y_s + a_r y_r + a_s a_r y_{s+r})^*$, $((a_s y_s)^*)^{\stuffle n} = (n a_s y_s + a_s^n y_{ns})^*$ and $(a_s y_s)^* \stuffle (-a_s y_s)^* = (-a_s^2 y_{2s})^*$ then $\gamma_{(a_s y_s + a_r y_r + a_s a_r y_{s+r})^*} = \gamma_{(a_s y_s)^*}\gamma_{(a_r y_r)^*}$, $\gamma_{(n a_s y_s + a_s^n y_{ns})^*} = \gamma^n_{(a_s y_s)^*}$, $\gamma_{(-a_s^2 y_{2s})^*} = \gamma_{(a_s y_s)^*}\gamma_{(-a_s y_s)^*}$.

Application of $(-t^2x_0x_1)^* ш (t^2x_0x_1)^* = (-4t^4x_0^2x_1^2)^*$







By using the poly-morphism ζ, one deduces then[12]

$$\zeta((-t^4y_4)^*) = \zeta((-t^2y_2)^*)\zeta(((t^2y_2)^*) = \zeta((-t^2x_0x_1)^*)\zeta(((t^2x_0x_1)^*))$$
$$= \zeta((-t^2x_0x_1)^* ш (t^2x_0x_1)^*) = \zeta((-4t^4x_0^2x_1^2)^*).$$

It follows then, by identification the coeffients of t^{2k} and t^{4k} :

$$\zeta(\overbrace{2,\ldots,2}^{k\texttt{times}})/\pi^{2k} = 1/(2k+1)! \in \mathbb{Q},$$

$$\zeta(\overbrace{3,1,\ldots,3,1}^{k\texttt{times}})/\pi^{4k} = 4^k\zeta(\overbrace{4,\ldots,4}^{k\texttt{times}})/\pi^{4k} = 2/(4k+2)! \in \mathbb{Q}.$$

[12] $\gamma_{(-t^4y_4)^*} = \zeta((-t^4y_4)^*), \gamma_{(-t^2y_2)^*} = \zeta((-t^2y_2)^*), \gamma_{(t^2y_2)^*} = \zeta((t^2y_2)^*).$

CONCLUDING REMARKS

Structure of polyzetas ($\mathcal{X} = X$ or Y)

Generated by homogenous in weight polynomials, $\ker\zeta$ is graded.

Since $\mathcal{Z} = \mathrm{Im}\,\zeta$ and

$$\mathrm{Im}\,\zeta \cong \mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1/\ker\zeta \cong \mathrm{span}_{\mathbb{Q}}\{\zeta(l)\}_{l\in\mathcal{L}^{\infty}_{irr}(X)}$$

$$(\text{resp. } \mathrm{Im}\,\zeta \cong \mathbb{Q}1_{Y^*} \oplus (Y-\{y_1\})\mathbb{Q}\langle Y\rangle/\ker\zeta \cong \mathrm{span}_{\mathbb{Q}}\{\zeta(l)\}_{l\in\mathcal{L}^{\infty}_{irr}(Y)})$$

then, as being quotient of graded algebras, $\mathcal{Z}$ is graded. For $p \geq 2$, let

$$\begin{aligned}\mathcal{Z}_p &:= \mathrm{span}_{\mathbb{Q}}\{\zeta(w) \mid w \in x_0X^*x_1, |w| = p\}\\ &= \mathrm{span}_{\mathbb{Q}}\{\zeta(w) \mid w \in (Y-\{y_1\})Y^*, (w) = p\}.\end{aligned}$$

Let $P \in \mathbb{Q}\langle X\rangle$ homogenous of degree n. Suppose $\xi = \zeta(P)$ satisfies $\xi^n + a_{n-1}\xi^{n-1} + \ldots = 0$, in which each monomial is of different weight (because $\mathcal{Z}_p\mathcal{Z}_q \subset \mathcal{Z}_{p+q}$). Then ξ is a transcendent over $\mathbb{Q}$.

Since $\{S_l\}_{l\in\mathcal{L}ynX}$ and $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ are homogenous in weight then $\{\zeta(l)\}_{l\in\mathcal{L}^{\infty}_{irr}(\mathcal{X})}$ are transcendent over $\mathbb{Q}$.

In particular, let ξ_{2p+1} and $\xi_{2q+1} \in \{\zeta(l)\}_{l\in\mathcal{L}^{\infty}_{irr}(\mathcal{X})}$ of weights equal to $2p+1$ and $2q+1$, respectively (for $p, q \geq 1, p \neq q$). Then

$$\xi_{2p+1}/\pi^{2p}, \xi_{2q+1}/\pi^{2q} \notin \mathbb{Q} \text{ and } \xi_{2p+1}/\xi_{2q+1} \notin \mathbb{Q}.$$

Thus, letting $\mathcal{Q}$ be the $\mathbb{Q}$-algebra generated by $\{\xi_{2p+1}\}_{p\geq 1}$, one has π^2 is transcendent over $\mathcal{Q}$. Or equivalently, π is transcendent over $\mathcal{Q}$.

THANK YOU FOR YOUR ATTENTION