Symmetric Polynomials in Quantum Algebras

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Plan

Let V be a vector space over the field $\mathbb{C}$ and P be the usual flip acting in $V^{\otimes 2}$ or its matrix.
Also, let $M = (m_i^j)$ be a numerical $N \times N$ matrix. Consider the system

$$P\, M_1\, M_2 - M_1\, M_2\, P = 0, \;\; M_1 = M \otimes I, \;\; M_2 = I \otimes M.$$

Note that $M_2 = P\, M_1\, P$ and consequently, this system can be cast under the form

$$P\, M_1\, P\, M_1 - M_1\, P\, M_1\, P = 0.$$

This system written via the entries reads

$$m_i^j\, m_k^l = m_k^l\, m_i^j, \;\; \forall\, i, j, k, l,$$

i.e. the entries commute with each other.

Example $N = 2$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$M_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad M_2 = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

The corresponding system reads

$$ab = ba, \; ac = ca, \ldots$$

Let us introduce some symmetric polynomials of M (namely, elementary ones and power sums)

$$\det(M - t\,I) = \sum_0^N (-t)^{N-k} e_k(M), \quad p_k(M) = Tr\, M^k.$$

If M is a triangular matrix these elements are respectively elementary symmetric polynomials and power sums in the eigenvalues.

Also, note that these symmetric polynomials of M are related by the Newton identities

$$k\, e_k - p_1\, e_{k-1} + p_2\, e_{k-2} + \cdots + (-1)^k p_k\, e_0 = 0.$$

Together with the initial system $P\,M_1\,P\,M_1 - M_1\,P\,M_1\,P = 0$ consider its inhomogeneous analog

$$P\,M_1\,P\,M_1 - M_1\,P\,M_1\,P = P\,M_1 - M_1\,P.$$

In terms of the entries we have the relations

$$m_i^j\,m_k^l - m_k^l\,m_i^j = m_i^l\,\delta_k^j - m_k^j\,\delta_i^l,$$

which define the enveloping algebra $U(gl(N))$.

Note that if in the homogeneous (inhomogeneous) system we replace P by the super-flip $P_{m|n}$, we get the defining relations of the super-commutative algebra $Sym(gl(m|n))$ (resp., the enveloping algebra $U(gl(m|n))$).

Now, deform $P \to R$ and the corresponding systems–homogeneous and not. And do the same with the super-flip $P_{m|n}$. Namely, take R as follows (here $N = 2, \quad m = n = 1$)

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

Note that for $q \to 1$ we respectively recover the flip P and the super-flip $P_{1|1}$.

If we deform the system $P\, M_1\, P\, M_1 - M_1\, P\, M_1\, P = 0$ and its inhomogeneous analog, we get

$$R\, M_1\, R\, M_1 - M_1\, R\, M_1\, R = 0.$$

$$R\, M_1\, R\, M_1 - M_1\, R\, M_1\, R = R\, M_1 - M_1\, R.$$

The first one will be called Reflection Equation (RE) algebra. The second one–modified RE algebra.

If we deform P in the system $P\, M_1\, M_2 - M_1\, M_2\, P = 0$, we get

$$R\, M_1\, M_2 - M_1\, M_2\, R = 0.$$

This algebra will be called RTT (or Leningrad) algebra.

All these algebras make sense for other braidings R.

Moreover, for some R deforming P, namely, involutive or Hecke symmetries, these algebras have deformation property.

For the mentioned symmetries R analogs of the symmetric polynomials are well defined in the RE algebras and RTT ones.

However, if in the RTT algebras these analogs generate a commutative subalgebras called *Bethe*,in the RE algebras they are central.

We call an invertible linear operator $R: V^{\otimes 2} \to V^{\otimes 2}$ *braiding* if it satisfies the so-called *braid relation*

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R \otimes I, \;\; R_{23} = I \otimes R.$$

Then the operator $\mathcal{R} = R\, P$ is subject to the QYBE

$$\mathcal{R}_{12}\, \mathcal{R}_{13}\, \mathcal{R}_{23} = \mathcal{R}_{23}\, \mathcal{R}_{13}\, \mathcal{R}_{12}.$$

A braiding R is called *involutive symmetry* if $R^2 = I$.
A braiding is called *Hecke symmetry* if it is subject to the Hecke condition

$$(q\,I - R)(q^{-1}\,I + R) = 0, \;\; q \in \mathbb{K}, \;\; q \neq \pm 1.$$

In particular, such a symmetry comes from the QG $U_q(sl(N))$ and $U_q(sl(m|n))$.

As for the braidings coming from the QG of other series B_n, C_n, D_n, each of them has 3 eigenvalues and it is called BMW symmetry.

Let us introduce "R-symmetric" and "R-skew-symmetric" algebras

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \ \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle$$

and the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \ P_-(t) = \sum_k dim \bigwedge\nolimits_R^{(k)}(V)t^k,$$

where the upper index (k) labels homogenous components of these quadratic algebras.

Remark.

Let us compare two symmetries. The first one is that above, the second one is involutive:

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the first (resp., second) symmetry we have

$$Sym_R = T(V)/ < xy - qyx >, \bigwedge_R = T(V)/ < x^2, y^2, qxy + yx > .$$

$$Sym_R = T(V)/ < xy - qyx >, \bigwedge_R = T(V)/ < x^2, y^2, xy + qyx > .$$

Observe that the algebras $Sym_R(V)$ are similar, but $\bigwedge_R(V)$ are not.

One example more. Consider an involutive symmetry

$$\begin{pmatrix} 1 & a & -a & ab \\ 0 & 0 & 1 & -b \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$Sym_R(V) = T(V)/ < xy - yx + b\,y^2 >,$$

$$\bigwedge\nolimits_R(V) = T(V)/ < x^2 + \frac{a}{2}(xy - yx), xy + yx, y^2 > .$$

If $b = 0,\ a \neq 0$, the algebra $Sym_R(V)$ is usual but $\bigwedge_R(V)$ is not.

The following holds $P_-(-t)P_+(t) = 1$.

Proposition. (Phung Ho Hai)

The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function:

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \ldots + a_r\, t^r}{1 - b_1\, t + \ldots + (-1)^s\, b_s\, t^s} = \frac{\prod_{i=1}^r (1 + x_i t)}{\prod_{j=1}^s (1 - y_j t)},$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

We call the couple $(r|s)$ bi-rank. In this sense all Hecke symmetries are similar to super-flips, for which the role of the bi-rank is played by the super-dimension $(m|n)$.

Examples. If R comes from the QG $U_q(sl(m))$, then

$$P_-(t) = (1+t)^m.$$

If R is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $N \geq 2$ there exits a Hecke symmetry such that

$$P_-(t) = 1 + Nt + t^2.$$

Here $dimV = N$, the bi-rank is $(2|0)$.
If $P_-(t)$ is a polynomial, R is called *even*.

Given an even Hecke symmetry R, we want to extend it to the dual space V^* in order to get an extended braiding

$$V \otimes V^* \Leftrightarrow V^* \otimes V, \;\; V^* \otimes V^* \to V^* \otimes V^*$$

with a "correct" paring $\langle \, , \, \rangle : V \otimes V^* \to \mathbb{C}$.
Such an extension exists iff there exists an operator $\Psi = (\Psi_{i\,j}^{k\,l})$ such that

$$R_{i\,j}^{k\,l} \, \Psi_{l\,m}^{j\,n} = \delta_i^n \, \delta_m^k.$$

Note that if such an extension exists it is unique.

This extension exists for $R = P_{m|n}$ and its deformations but it does not if $R = I$.

If it exists, then R is called *skew-invertible*. If it is so, it is possible to construct a quasi-tensor rigid category $SW(V)$ similar to that $Rep - U_q(sl(N))$ or its super-analog.

It is also possible to define the so-called *R-trace* $Tr_R : End(U) \to \mathbb{C}$ where $End(U) \cong U \otimes U^*$ for any object U.
It has the form

$$Tr_R\, X = Tr\, C\, X, \;\; X \in End(U),$$

where C is uniquely defined by any skew-invertible braiding R.
Then, we define R-dimension of U via the usual formula

$$dim_R\, U = Tr_R I_U.$$

Also, for any $N \times N$ matrix A we define its R-trace $Tr_R\, A = Tr\, C\, A$.

Example. If $P_-(t) = 1 + Nt + t^2$ then $dim V = N$ but $dim_R\, V = q^{-1} + q^{-3}$ for a Hecke R and $dim\, V = 2$ for an involutive R.

In general, if R is a Hecke symmetry of bi-rank $(m|n)$, then

$$dim\, V = m + n,$$

$$dim_R\, V = q^{-(m-n)}\,(m-n)_q.$$

Let us introduce the following notation

$$L_{\overline{1}} = L_1,\ \ L_{\overline{2}} = R_{12}\, L_{\overline{1}}\, R_{12}^{-1},\ \ L_{\overline{3}} = R_{23}\, L_{\overline{2}}\, R_{23}^{-1} = R_{23}\, R_{12}\, L_{\overline{1}}\, R_{12}^{-1}\, R_{23}^{-1}, \dots$$

Recall that

$$L_2 = P_{12}\, L_1\, P_{12},\ \ L_3 = P_{23}\, L_2\, P_{23} = P_{23}\, P_{12}\, L_1\, P_{12}\, P_{23}, \dots$$

In this notation the defining relations of the RE algebra become similar to these in the RTT algebras

$$RL_{\overline{1}}L_{\overline{2}} = L_{\overline{1}}L_{\overline{2}}R.$$

Remark.

How to find relations in QMA? Let R be the Hecke symmetry coming from QG the $U_q(sl(m))$. Then the relations in $Sym_R(V)$ are $x_i\, x_j - q\, x_j\, x_i = 0$. Apply the coproduct $x_i \to \sum_k t_i^k \otimes x_k$ to this relation. We have

$$(\sum_k t_i^k \otimes x_k)(\sum_l t_j^l \otimes x_l) - q(\sum_l t_j^l \otimes x_l)(\sum_k t_i^k \otimes x_k) = 0.$$

Now, we have to take away the terms t_j^l from the second factors by transposing them with x_k. If it is done via the usual flip, we get the RTT algebra. If we identify $t_i^j \cong x_i \otimes x^j$ where $x_i \in V$ and $x^j \in V^*$ we get the RE one.

Now, consider a more general construction giving rise to more general QMAs.

In [IOP] the notion of compatible braidings is introduced. Two braidings R and F are called so, if

$$R_{12}\, F_{23}\, F_{12} = F_{23}\, F_{12}\, R_{23}, \;\; R_{23}\, F_{12}\, F_{23} = F_{12}\, F_{23}\, R_{12}.$$

Let us introduce the following notations

$$L_{\overline{1}} = L_1,\; L_{\overline{2}} = F_{12}\, L_{\overline{1}}\, F_{12}^{-1},\; L_{\overline{3}} = F_{23}\, L_{\overline{2}}\, F_{23}^{-1} = F_{23}\, F_{12}\, L_{\overline{1}}\, F_{12}^{-1}\, F_{23}^{-1}, \dots$$

Let us denote $A(R, F)$ the algebra defined by

$$RL_{\overline{1}}L_{\overline{2}} = L_{\overline{1}}L_{\overline{2}}R.$$

It is clear that the couples (R, P) and (R, R) are compatible.
The algebra $A(R, P)$ is just a usual RTT algebra.
The algebra $A(R, R)$ is an RE algebra.

In [IOP] there is constructed a Bethe subalgebra in this algebra (for R Hecke) via quantum versions of some symmetric polynomials.

Thus, *elementary symmetric polynomials* in an algebra $A(R,F)$ are defined as follows

$$e_k(L) = Tr_{F(12...k)} A_R^{(k)} L_{\overline{1}} ... L_{\overline{k}}.$$

F is assumed to be skew-invertible. And $A_R^{(k)}$ are skew-symmetrizers corresponding to the symmetry (involutive or Hecke) R. For a Hecke symmetry they are defined by

$$A^{(1)} = I, \qquad A^{(k+1)}_{1...k+1} = \frac{k_q}{(k+1)_q} A^{(k)}_{1...k} \Big(\frac{q^k}{k_q} I - R_k \Big) A^{(k)}_{1...k}, \quad k \geq 1.$$

Also, there are defined quantum analogs of power sums

$$p_k(L) = Tr_{F(12...k)} R_{k-1\,k} ..., R_{23} R_{12} L_{\overline{1}} ... L_{\overline{k}}.$$

As shown in [IOP], they are related by the quantum version of the Newton identities

$$p_k - q p_{k-1}\, e_1 + (-q)^2 p_{k-2}\, e_2 + \dots + (-q)^{k-1} p_1\, e_k + (-1)^k k_q e_k = 0$$

and commute with each other.

The algebra generated by these quantum symmetric polynomials is called *Bethe*.

Observe that quantum analogs of the full symmetric and Schur polynomials can be also defined.

As noticed above in the RE algebras $A(R, R)$ the Bethe subalgebras are central.

Also, in this case the power sums can be reduced to the form similar to the classical one:

$$p_k = Tr_R\, L^k.$$

Moreover, in this case there exists a quantum analog of the Cayley-Hamilton identity similar to the classical one

$$L^m - q\, L^{m-1}\, e_1 + (-q)^2 L^{m-2}\, e_2 + \ldots + (-q)^{m-1} L\, e_{m-1} + (-q)^m\, I\, e_m = 0,$$

provided R is a symmetry of bi-rank $(m|0)$.
The last coefficient e_m in this formula is called "quantum determinant" and is usually denoted $det_R(L)$.

In an RE algebra $A(R,R)$ it is possible to define the so-called quantum characteristic polynomial

$$ch(t) = t^m - q\, t^{m-1}\, e_1 + (-q)^2 t^{m-2}\, e_2 + \dots$$

$$+(-q)^{m-1} t\, e_{m-1} + (-q)^m\, 1\, e_m = 0,$$

such that $ch(L) = 0$.

Observe that the polynomial $det_R(L - tI)$ is well defined but it is not equal to $ch(t)$.

Isaev-Ogievetsky introduced the notion of so-called half quantum algebras (HQA), which are more general than these $A(R,F)$ above.

Each of them is defined by a system

$$A_R^{(2)}\, L_{\overline{1}}\, L_{\overline{2}}\, S_R^{(2)} = 0,$$

where

$$S_R^{(2)} = \frac{q^{-1}\, I + R}{q+q^{-1}} : V^{\otimes 2} \to V^{\otimes 2}$$

is the symmetrizer acting in $V^{\otimes 2}$.

The so-called Manin and q-Manin matrices introduced represent particular cases of these HQA.

In HQAs analogs of the symmetric polynomials can be defined by the same formulae.

Also, in these algebras there are analogs of the Newton and CH identities.

However, in general, it is not possible to show that these symmetric polynomials commute with each other. So, it is not possible to construct Bethe subalgebras.

Now, pass to algebras associated with "current braidings".
First, describe the Baxterization procedure.

Proposition.

1. If R is an involutive symmetry, then

$$R(u,v) = R - \frac{a\, I}{u - v}$$

is an R-matrix, i.e. it meets the quantum Yang-Baxter equation

$$R_{12}(u,v)\, R_{23}(u,w)\, R_{12}(v,w) = R_{23}(v,w)\, R_{12}(u,w)\, R_{23}(u,v).$$

2. If $R = R(q)$ is a Hecke symmetry, then the same is valid for

$$R(u,v) = R(q) - \frac{(q - q^{-1})u\, I}{u - v}.$$

The Yang R-matrix is

$$R(u,v) = P - \frac{I}{u-v}.$$

The corresponding Yangian $\mathbf{Y}(gl(N))$ introduced by Drinfeld is in fact an RTT algebra defined by

$$R(u,v)\, T_1(u)\, T_2(v) = T_1(v)\, T_2(u)\, R(u,v).$$

Here, one assumes that $T(u)$ is a series

$$T(u) = \sum_{k\geq 0} T[k]\, u^{-k}$$

and $T[0] = I$.

Why do HQA appear in the study of the Yangians?
If in the relations of the trigonometric Yangians

$$R(u,v)\,T_1(u)\,T_2(v) = T_1(v)\,T_2(u)\,R(u,v),$$

we put $u/v = q^2$, then the current R-matrix becomes

$$R(u,q^{-2}u) = R(q) - \frac{(q-q^{-1})q^2\,I}{u(q^2-1)} = R(q) - q\,I$$

i.e. it equals the skew-symmetrizer up to the factor $q+q^{-1}$.
After multiplying by $S_R^{(2)}$ on the right hand side, we have

$$A_R^{(2)}\,T_1(u)\,T_2(q^{-2}u)\,S_R^{(2)} = 0 \Leftrightarrow$$

$$A_R^{(2)}\,(q^{-2u\partial_u}\,T_1(u))\,(q^{-2u\partial_u}\,T_2(u))\,S_R^{(2)} = 0,\ \partial_u = \frac{d}{d\,u}$$

which looks like a HQA.

Consider two types of "generalized Yangians".

1. Yangians of RTT type

$$R(u,v)\,T_1(u)\,T_2(v) = T_1(u)\,T_2(v)\,R(u,v),$$

where $R(u,v)$ is one of the above R-matrices.

2. Braided Yangians (here $L_{\overline{2}} = F\,L_{\overline{1}}\,F^{-1}$).

$$R(u,v)L_{\overline{1}}(u)\,L_{\overline{2}}(v) = L_{\overline{1}}(v)\,L_{\overline{2}}(u)\,R(u,v).$$

If $F = R$, these relations can be presented as follows

$$R(u,v)L_1(u)RL_1(v) = L_1(v)RL_1(u)R(u,v).$$

In this case we say that this braided Yangian is of RE type.

Now, define analogs of elementary symmetric polynomials $e_k(u)$ in the braided Yangians of RE type by

$$e_k(u) = Tr_{R(1\dots k)} \left(A_R^{(k)} L_{\overline{1}}(u) L_{\overline{2}}(q^{-2}u) \dots L_{\overline{k}}(q^{-2(k-1)}u) \right), \; k \geq 1,$$

provided the corresponding R-matrix is trigonometric (i.e. it arises from a Hecke R).

Let us define "quantum powers" of the generating matrices in braided Yangians of RE type by

$$L^{[k]}(u) = L(q^{-2(k-1)}u)L(q^{-2(k-2)}u)...L(u), \; k \geq 1.$$

Then the quantum power sums are defined by

$$p_k(u) = Tr_R\, L^{[k]}(u) = Tr_R\, L(q^{-2(k-1)}u)L(q^{-2(k-2)}u)...L(u).$$

Let us exhibit the quantum Newton relations and Cayley-Hamilton identities in braided Yangians of RE type

Proposition.

$$p_k(u) - q p_{k-1}(q^{-2}u)e_1(u) + (-q)^2 p_{k-2}(q^{-4}u)e_2(u) + \dots$$

$$+(-q)^{k-1} p_1(q^{-2(k-1)}u)e_k(u) + (-1)^k k_q e_k(u).$$

Proposition.

$$\sum_{p=0}^{m} (-q)^p L^{[m-p]}(q^{-2p}u)e_p(u) = 0.$$

According to a Frobenius theorem any numerical matrix M can be reduced by transformations

$$M \mapsto g\, M\, g^{-1}$$

to the so-called second canonical form

$$M_{can} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_m & a_{m-2} & a_{m-1} & \dots & a_1 \end{pmatrix}.$$

It is clear that

$$a_1 = Tr\, M, \dots, a_m = (-1)^{N-1}\, det\, M.$$

Consider a connection operator

$$\partial_u - M(u),$$

where $M(u)$ is a numerical matrix smoothly depending on a parameter u.
According to the famous DS theorem this operator can be reduced to that

$$\partial_u - M_{can}(u),$$

where $M_{can}(u)$ has the forme above, by gauge transformations

$$\partial_u - M(u) \mapsto g(u)\,(\partial_u - M(u))\,g(u)^{-1}.$$

Also, Drinfeld-Sokolov identified the reduced Poisson structure corresponding to $\widehat{gl(N)}$ with the second Gelfand-Dikey structure.

The question: what is q-analog of this reduction?

Frenkel-Reshetikhin-STS suggested to replace ∂_u by that $D_q\, f(u) = f(qu)$ and the above gauge transformations by

$$D_q - M(u) \to g(qu)\,(D_q - M(u))\,g(u)^{-1}.$$

Then similar claims are valid.
Observe that F-R-STS are dealing with numerical group-valued currents $A(u)$.

We are dealing with matrices whose entries are not commutative. For instance consider the above matrix $M = (m_i^j)$ whose entries generate $U(gl(N))$. For it there exists the characteristic polynomial

$$Q(t) = Tr_{(1...N)}\left(A_-^{(N)}(tI - M_1)((t-1)I - M_2)...((t-N+1)I - M_N)\right),$$

such that $Q(M) = 0$. This is a sort of the CH identity. In virtue of that property this matrix can be reduced in the sense of the Chervov-Talalaev approach.

A similar claim is valid for the RE and mRE algebras.

Proposition.

For the generating matrix M of any of these algebras there exists a matrix

$$M_{can} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_m & a_{m-2} & a_{m-1} & \dots & a_1 \end{pmatrix}.$$

where the entries a_k are expressed via the elementary symmetric polynomials and such that the matrices M and M_{can} are similar in the following sense: there exists a nontrivial matrix C with entries from the the same algebra such that

$$C\,M = M_{can}\,C.$$

Now, pass to the braided Yangians of RE type. Consider the operator

$$L(u)q^{2\,u\,\partial_u},\;\; \partial_u=\frac{d}{du}$$

and the following matrix $L_{can}(u)$

$$L_{can}(u)=\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_m(u)\\ 1 & 0 & \dots & 0 & 0 & a_{m-1}(u)\\ 0 & 1 & \dots & 0 & 0 & a_{m-2}(u)\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1 & 0 & a_2(u)\\ 0 & 0 & \dots & 0 & 1 & a_1(u) \end{pmatrix}.$$

Proposition.

There exists a nontrivial matrix $C(u)$ such that the following holds

$$L(u)q^{2\,u\,\partial_u}C(u) = C(u)L_{can}(u)q^{2\,u\,\partial_u},$$

where

$$a_k(u) = -(-q)^k\, e_k(q^{2(m-1)}u).$$

Observe that in the braided Yangians the quantum symmetric elements are not central (except the quantum determinant in some braided Yangians). But they commute with each other and generate a commutative subalgebras, also called Bethe.

If a given braided Yangian is of RE type and $R(u, v)$ is rational, it is possible by passing to a proper limit to construct analogs of rational Gaudin models.

Many thanks